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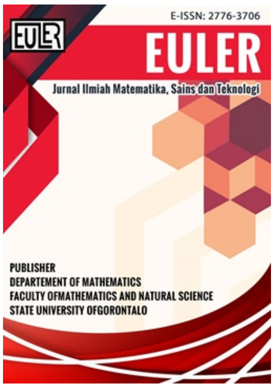

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Research Article

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Meir Keeler's Fixed-Point Theorem in Complex-Valued Modular Metric Space

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ABSTRACT. *In this paper, we introduce the notion of Meir-Keeler contraction mapping, which is defined in complex-valued modular metric space. Some properties of sequences in this space, which are convergence, Cauchyness and completeness, are used to prove the fixed-point theorem under this mapping. Additionally, the Δ_2-type condition is also defined as the sufficient condition in order to have a unique fixed-point.*

1. Introduction

A fixed-point is defined as a point within the domain of a function that is equal to the value of the function at that point. Stefan Banach introduced the concept of fixed-points and subsequently established a theorem concerning the existence and uniqueness of such fixed-points within certain metric spaces. Since then, this theory has evolved by defining new types of contractions through the generalization of the mapping itself [1–7], as well as the generalization of the space in which this mapping is defined [8–12].

One of the most interesting generalizations of contractions is Meir Keeler contraction in a complete metric space [13]. In 2013, Kiftiah [14] proposed the concept of fixed-points from several contraction mappings developed from metric spaces to modular spaces. One of these mappings is the Meir Keeler ρ-contraction. Then, the existence and uniqueness of fixed-points under this mapping were proved. Following that, in 2018, Aksoya [15] additionally defined Meir Keeler type contraction mappings on modular metric space and succesfully established its fixed-points theorem.

The notion of complex valued modular metric spaces, which is more general than well-known modular metric spaces, was first introduced by Ozkan [16] in 2021. In addition, they showed the generalization of the Banach Fixed-Point Theorem, one of the most important and simple tools for the existence and uniqueness of solutions for problems arising for complex-valued modular metric spaces in the fields of engineering and mathematics.

The idea of the existence and uniqueness of fixed-points has always been an interesting topic to explore. However, no work has generalized the fixed-point problem through the Meir Keeler contraction in metric space to complex-valued modular metric space. Inspired by the work of Ozkan in [16], we first introduce a Meir Keeler contraction defined in a complex-valued modular metric space. Our main goal is to investigate the existence and uniqueness of fixed-point of Meir Keeler type mapping in the context of complex-valued modular metric spaces.

2. Methods

The first step involves studying the concept of complex-valued modular metric spaces, as defined by Ozkan in [16], including the definitions, topology, convergent sequences, and fixed-points. Based on these concepts, the notion of Meir-Keeler ω-contraction mappings is constructed in complex-valued modular metric spaces as previously defined in metric spaces [13], modular spaces [14], and modular metric spaces [15]. Subsequently, the sufficient conditions that the Meir-Keeler ω-contraction mappings must satisfy to ensure the existence and uniqueness of their fixed-points are investigated. A fixed-point theorem is formulated from sufficient conditions for Meir-Keeler ω-contraction mappings in complex-valued modular metric spaces. Additionally, the proof of this theorem is presented.

3. Results and Discussion

Before investigating the main topic, let us first review some notations and definitions introduced by Azam [11], who studied the concepts of complex-valued metric spaces. These will serve as the foundation for our later discussion.

Definition 1. [11] *Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$, satisfies: $z_1 \precsim z_2$ if and only if $Re\,(z_1) \leq Re(z_2)$ and $Im\,(z_1) \leq Im\,(z_2)$.*

It implies that if $z_1 \precsim z_2$ then one of the following conditions is satisfied:

(i) $Re\,(z_1) = Re(z_2)$ and $Im\,(z_1) < Im\,(z_2)$,
(ii) $Re\,(z_1) < Re(z_2)$ and $Im\,(z_1) = Im\,(z_2)$,
(iii) $Re\,(z_1) < Re(z_2)$ and $Im\,(z_1) < Im\,(z_2)$,
(iv) $Re\,(z_1) = Re(z_2)$ and $Im\,(z_1) = Im\,(z_2)$.

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If $z_1 \neq z_2$ and one of (i), (ii), or (iii) is satisfied, then we can write $z_1 \precsim z_2$. Particularly, if only (iii) is satisfied, then we can write $z_1 \prec z_2$.

For every $z_1, z_2 \in \mathbb{C}$, the partial order on $\mathbb{C}$ has the following properties:

(i) $0 \precsim z_1 \prec z_2 \Leftrightarrow |z_1| < |z_2|$,
(ii) $z_2 \precsim z_2$ and $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$,
(iii) $z \in \mathbb{C}, a, b \in \mathbb{R}, a \leq b \Rightarrow az \precsim bz$

Next, we recall some basic definitions and fundamental results on complex-valued modular metric space, which was proposed by Ozkan [16].

Let $X \neq \emptyset$, $\lambda > 0$ and a function $\omega : (0, \infty) \times X \times X \to \mathbb{C}$. In this article, for every $\lambda > 0$ and $x, y \in X$, then the function $\omega(\lambda, x, y)$ is denoted with $\omega_\lambda(x, y)$.

Definition 2. [16] *Let $X \neq \emptyset$. A function $\omega : (0, \infty) \times X \times X \to \mathbb{C}$ is said to be complex valued modular metric space on X, if it satisfies:*
(M1) $\omega_\lambda(z_1, z_2) \succcurlyeq 0$ and $\omega_\lambda(z_1, z_2) = 0 \Leftrightarrow z_1 = z_2$.
(M2) $\omega_\lambda(z_1, z_2) = \omega_\lambda(z_2, z_1)$.
(M3) $\omega_{\lambda+\mu}(z_1, z_2) \preccurlyeq \omega_\lambda(z_1, z_3) + \omega_\mu(z_3, z_2)$,
for all $\lambda, \mu > 0$ and $z_1, z_2, z_3 \in X$.

Definition 3. [16] *Let $X \neq \emptyset$ and $\omega : (0, \infty) \times X \times X \to \mathbb{C}$ be a complex modular metric on X. For all $x_0 \in X$, the set*

$$X_\omega = \left\{ x \in X \;\middle|\; \lim_{\lambda \to \infty} \omega_\lambda(x, x_0) = 0 \right\}$$

is said to be modular metric space (around x_0).

Definition 4. *Let X_ω be a complex valued modular metric space and a sequence x_n in X_ω.*
(i) *A sequence $x_n \subseteq X_\omega$ is said to be ω-complex convergent to $x \in X_\omega$ if for every $\varepsilon \in \mathbb{C}$ with $\varepsilon \succ 0$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and some $\lambda > 0$, we have $\omega_\lambda(x_n, x) \prec \varepsilon$. Further, x is called a ω- limit of x_n, and we write $\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$.*
(ii) *A sequence $x_n \subseteq X_\omega$ is said to be ω-complex Cauchy sequence, if for every $\varepsilon \in \mathbb{C}$ with $\varepsilon \succ 0$ there exists $n_0 \in \mathbb{N}$ such that for every $m, n \geq n_0$ and some $\lambda > 0$, we have $\omega_\lambda(x_n, x_m) \prec \varepsilon$. This is denoted with $\lim_{m,n\to\infty} \omega_\lambda(x_n, x_m) = 0$.*
(iii) *Complex modular metric space X_ω is said to be ω-complex complete if every ω-complex Cauchy sequence in X_ω is ω-complex convergent.*

Furthermore, we give some basic properties of ω-complex convergent.

Lemma 1. *[17] Let X_ω be a complex valued modular metric space and a sequence x_n in X_ω. A sequence $x_n \subseteq X_\omega$ is ω-complex convergent to $x \in X_\omega$ if and only if $\lim_{n\to\infty} |\omega_\lambda(x_n, x)| = 0$.*

Lemma 2. *[17] Let X_ω be a complex valued modular metric space and a sequence x_n in X_ω. A sequence $x_n \subseteq X_\omega$ is ω-complex Cauchy sequence in X_ω if and only if $\lim_{m,n\to\infty} |\omega_\lambda(x_n, x_m)| = 0$.*

Lemma 3. *[16] Let $\omega, z \in \mathbb{C}$. If $\omega \succcurlyeq 0$, $|z| < 1$ and $\omega \preccurlyeq z\omega$ then $\omega = 0 \in \mathbb{C}$.*

The following is the definition of Δ_2-type condition in a complex-valued modular metric space by adopting the description of the Δ_2-type condition in modular metric space case in Abdou [18].

Definition 5. *Let X_ω be a complex valued modular metric space and a sequence x_n in X_ω.*
(i) *A function ω satisfies Δ_2-condition if $\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$, for some $\lambda > 0$ implies $\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$, for all $\lambda > 0$.*
(ii) *A function ω satisfies Δ_2-type condition if for any $\alpha > 0$ there exist $C > 0$ such that*

$$\omega_{\frac{\lambda}{\alpha}}(z_1, z_2) \preccurlyeq C \cdot \omega_\lambda(z_1, z_2)$$

for all $\lambda > 0$, $z_1, z_2 \in X_\omega$, and $z_1 \neq z_2$.

It is clear that if ω satisfies the Δ_2-type condition then ω satisfies the Δ_2-condition.

Inspired from the defnitions of Meir-Keeler contractions in modular metric space, we define the following complex-valued modular space versions of such type of mapping.

Definition 6. *Let X_ω be a complete complex valued modular metric space and $T : X_\omega \to X_\omega$ is a mapping. A mapping T is said Meir Keeler ω-complex contraction if and only if for every $\varepsilon \in \mathbb{C}$ with $\varepsilon \succ 0$, there exists $\delta \in \mathbb{C}$ with $\delta \succ 0$, such that for any*

$$x, y \in X_\omega \text{ and } \lambda > 0 \text{ with } \varepsilon \preccurlyeq \omega_\lambda(x, y) \prec \varepsilon + \delta,$$

we have $\omega_\lambda(Tx, Ty) \prec \varepsilon$.

Since T is a Meir Keeler ω-complex contraction mapping, we can derive the following equivalence based on Definition 6.

Definition 7. *Let X_ω be a complete complex valued modular metric space and $T : X_\omega \to X_\omega$ is a mapping. A mapping T is said Meir Keeler ω-complex contraction if and only if For every $\varepsilon \in \mathbb{C}$ with $\varepsilon \succ 0$ there exists $\delta \in \mathbb{C}$ with $\delta \succ 0$ such that*

$$\omega_\lambda(Tx, Ty) \prec \omega_\lambda(x, y),$$

for any $x, y \in X_\omega$ and $\lambda > 0$ with $\varepsilon \preccurlyeq \omega_\lambda(x, y) \prec \varepsilon + \delta$.

Before stating and proving our fixed-point result for the contraction defined above, we first prove some auxiliary results to be used in our further discussion on complex-valued modular metric spaces.

Lemma 4. *Let X_ω be a complete complex valued modular metric space and $T : X_\omega \to X_\omega$ is ω-complex contraction mapping. A mapping T is a Meir Keeler ω-complex contraction if and only if for every $\varepsilon_{\mathbb{R}} > 0$ there exists $\delta_{\mathbb{R}} > 0$ such that for any $x, y \in X_\omega$ and $\lambda > 0$ with $\varepsilon_{\mathbb{R}} \leq |\omega_\lambda(x, y)| < \varepsilon_{\mathbb{R}} + \delta_{\mathbb{R}}$, we have $|\omega_\lambda(Tx, Ty)| < \varepsilon_{\mathbb{R}}$.*

Proof.

- ($\Rightarrow$) Let $\varepsilon_{\mathbb{R}} \in \mathbb{R}$ with $\varepsilon_{\mathbb{R}} > 0$ be arbitrary. We choose

$$\varepsilon = \frac{\varepsilon_{\mathbb{R}}}{\sqrt{2}} + i\frac{\varepsilon_{\mathbb{R}}}{\sqrt{2}}$$

  Then $\varepsilon \in \mathbb{C}$ and $\varepsilon \succ 0$. Since T is a Meir Keeler ω-complex contraction mapping, by Definition 6, we have *for every $\varepsilon \succ 0$ there exists $\delta \in \mathbb{C}$ with $\delta = \frac{\delta_{\mathbb{R}}}{\sqrt{2}} + i\frac{\delta_{\mathbb{R}}}{\sqrt{2}} \succ 0$ such that for any $x, y \in X_\omega$ and $\lambda > 0$ with $\varepsilon \preccurlyeq \omega_\lambda(x, y) \prec \varepsilon + \delta$, we have $\omega_\lambda(Tx, Ty) \prec \varepsilon$.*
  Furthermore, using the property of a partial order, we obtain *for every $\varepsilon_{\mathbb{R}} = |\varepsilon| > 0$ there exists $\delta_{\mathbb{R}} = |\delta| > 0$ such that for any $x, y \in X_\omega$ and $\lambda > 0$ with $\varepsilon_{\mathbb{R}} \leq |\omega_\lambda(x, y)| < \varepsilon_{\mathbb{R}} + \delta_{\mathbb{R}}$, we have $|\omega_\lambda(Tx, Ty)| < \varepsilon_{\mathbb{R}}$.*
- ($\Leftarrow$) Let $\varepsilon \in \mathbb{C}$ with $\varepsilon \succ 0$ be arbitrary. Since *for every $\varepsilon_{\mathbb{R}} = |\varepsilon| > 0$ there exists $\delta_{\mathbb{R}} = |\delta| > 0$ such that for any $x, y \in X_\omega$ and $\lambda > 0$ with $\varepsilon_{\mathbb{R}} \leq |\omega_\lambda(x, y)| < \varepsilon_{\mathbb{R}} + \delta_{\mathbb{R}}$, we have $|\omega_\lambda(Tx, Ty)| < \varepsilon_{\mathbb{R}}$*
  then, using the property of a partial order, we have *for every $\varepsilon \in \mathbb{C}$ with $\varepsilon \succ 0$ there exists $\delta \in \mathbb{C}$ with $\delta \succ 0$ such that for any $x, y \in X_\omega$ and $\lambda > 0$ with $\varepsilon \preccurlyeq \omega_\lambda(x, y) \prec \varepsilon + \delta$, we have $\omega_\lambda(Tx, Ty) \prec \varepsilon$.*
  Hence, T is a Meir Keeler ω-complex contraction mapping. □

Lemma 5. *Let X_ω be a complete complex valued modular metric space and $T : X_\omega \to X_\omega$ is a Meir Keeler ω-complex contraction mapping. Define*

$$T^0 x_0 = x_0;$$
$$T^{n+1} x_0 = T(T^n x_0),$$

for $x_0 \in X_\omega$ and $n \in \{0, 1, 2, \cdots\}$, then

$$\lim_{n\to\infty} \left|\omega_\lambda\left(T^n x_0,\ T^{n+1} x_0\right)\right| = 0.$$

Proof. Let $n \in \{0,\ 1,\ 2,\ \cdots\}$ and $x_0 \in X_\omega$ be arbitrary. As T is a Meir Keeler ω-complex contraction mapping, using Definition 7, we have

$\omega_\lambda\left(T^n x_0,\ T^{n+1} x_0\right) \prec \omega_\lambda\left(T^{n-1} x_0,\ T^n x_0\right)$, for all $\lambda > 0$.
Taking modulus on both sides, we obtain

$$0 < \left|\omega_\lambda\left(T^n x_0,\ T^{n+1} x_0\right)\right| < \left|\omega_\lambda\left(T^{n-1} x_0,\ T^n x_0\right)\right|.$$

Hence, sequence $\left\{\left|\omega_\lambda\left(T^n x_0,\ T^{n+1} x_0\right)\right|\right\}$ is a decreasing sequence on $\mathbb{R}$ and bounded by 0. This will imply this sequence converges to its infimum, that is, there exists $\varepsilon_{\mathbb{R}} \geq 0$ with $\varepsilon_{\mathbb{R}} = \inf\left\{\left|\omega_\lambda\left(T^n x_0,\ T^{n+1} x_0\right)\right| : n \in \{0,\ 1,\ 2,\ \cdots\}\right\}$ such that $\lim_{n\to\infty} \left|\omega_\lambda\left(T^n x_0,\ T^{n+1} x_0\right)\right| = \varepsilon_{\mathbb{R}}$.

We will prove $\varepsilon_{\mathbb{R}} = 0$.

If $\varepsilon_{\mathbb{R}} > 0$. Since T is a Meir Keeler ω-complex contraction mapping, using Lemma 4, we obtain that there exists $\delta_{\mathbb{R}} > 0$ such that $\varepsilon_{\mathbb{R}} \leq \left|\omega_\lambda\left(T^n x_0, T^{n+1} x_0\right)\right| < \varepsilon_{\mathbb{R}} + \delta_{\mathbb{R}}$. Furthermore, since $\lim_{n\to\infty} \left|\omega_\lambda\left(T^n x_0, T^{n+1} x_0\right)\right| = \varepsilon_{\mathbb{R}}$, then there exists $N \in \mathbb{N}$ such that $\varepsilon_{\mathbb{R}} \leq \left|\omega_\lambda\left(T^N x_0, T^{N+1} x_0\right)\right| < \varepsilon_{\mathbb{R}} + \delta_{\mathbb{R}}$. This implies $\left|\omega_\lambda\left(T^{N+1} x_0, T^{N+2} x_0\right)\right| < \varepsilon_{\mathbb{R}}$, which contradicts with $\varepsilon_{\mathbb{R}} = \inf\left\{\left|\omega_\lambda\left(T^n x_0,\ T^{n+1} x_0\right)\right| : n \in \{0,\ 1,\ 2,\ \cdots\}\right\}$. Then, $\varepsilon_{\mathbb{R}} = 0$. So, we conclude that $\lim_{n\to\infty} \left|\omega_\lambda\left(T^n x_0,\ T^{n+1} x_0\right)\right| = 0$. □

In the following, we present the Meir Keeler's fixed-point theorem based on the mapping given in Definition 6.

Theorem 1. *Let X_ω be a complete complex-valued modular metric space. Assume ω satisfies the Δ_2-type condition. If $T : X_\omega \to X_\omega$ is a Meir Keeler ω-complex contraction mapping, then T has a unique fixed-point on X_ω.*

Proof. Let $x_0 \in X_\omega$ be arbitrary. For any $n \in \{0,\ 1,\ 2,\ \cdots\}$, we define

$$T^0 x_0 = x_0;\ T^{n+1} x_0 = T(T^n x_0),\ \text{and}\ x_n = T^n x_0.$$

If $\omega_\lambda(x_n,\ x_{n+1}) = \omega_\lambda(x_n,\ Tx_n) = 0$, for every $\lambda > 0$ and some $n \in \{0,\ 1,\ 2,\ \cdots\}$ then T has a unique fixed-point, that is $x_n \in X_\omega$. If $\omega_\lambda(x_n,\ x_{n+1}) \succ 0$, we will prove that T has a unique fixed-point for any $n \in \{0,\ 1,\ 2,\ \cdots\}$. Let $n \in \{0,\ 1,\ 2,\ \cdots\}$ be arbitrary. Using Definition 7, we have

$$\begin{aligned}\omega_\lambda(x_n,\ x_{n+1}) &= \omega_\lambda\left(T^n x_0,\ T^{n+1} x_0\right)\\ &\prec \omega_\lambda\left(T^{n-1} x_0,\ T^n x_0\right)\\ &= \omega_\lambda(x_{n-1},\ x_n).\end{aligned}$$

Taking modulus on both sides, we obtain

$$0 < |\omega_\lambda(x_n,\ x_{n+1})| < |\omega_\lambda(x_{n-1},\ x_n)|.$$

Thus, the sequence $\{|\omega_\lambda(x_n,\ x_{n+1})|\}$ is decreasing on $\mathbb{R}$ and bounded by 0. From Lemma 5, we derive $\lim_{n\to\infty} |\omega_\lambda(x_n,\ x_{n+1})| = 0$. Using the property of a partial order, we obtain

$$\lim_{n\to\infty} \omega_\lambda(x_n,\ x_{n+1}) = 0.$$

Further, we will prove that x_n is ω-complex Cauchy sequence. Let $\varepsilon \in \mathbb{C}(\varepsilon \succ 0)$, then there is $\delta \in \mathbb{C}\ (\delta \succ 0)$ such that for $x,\ y \in X_\omega$ with $\varepsilon \preccurlyeq \omega_\lambda(x,\ y) \prec \varepsilon + \delta$ implies $\omega_\lambda(Tx, Ty) \prec \varepsilon$.

Since $\omega_\lambda(Tx, Ty) \prec \omega_\lambda(x,\ y)$, for any $\varepsilon \in \mathbb{C}$ with $\varepsilon \succ 0$ implies the above Definition 6 is still satisfied if we choose $\delta \preccurlyeq \varepsilon$ such that when $\omega_\lambda(x,\ y) \prec \delta$ implies $\omega_\lambda(Tx, Ty) \prec \varepsilon$. Since $\lim_{n\to\infty} \omega_\lambda(x_n,\ x_{n+1}) = 0$, then there exists $\mathrm{K} \in \mathbb{N}$ such that $\omega_\lambda(x_{n-1},\ x_n) \prec \delta$ for any $n > \mathrm{K}$.

Let $m, n \in \mathbb{N}$ such that $m, n > \mathrm{K}$. Without loss of generality, we assume $m > n$, then $m = n + p$, for some $p \in \mathbb{N}$. In what follows, we prove that $\{x_n\}$ is ω-complex Cauchy sequence. That is,

$$\omega_\lambda\left(x_n,\ x_m\right) = \omega_\lambda\left(x_n,\ x_{n+p}\right) \prec \varepsilon.$$

We will use mathematical induction. For $p = 1$. By Definition 7, we have

$$\omega_\lambda\left(x_n,\ x_{n+1}\right) \prec \omega_\lambda\left(x_{n-1},\ x_n\right) \prec \delta \preccurlyeq \varepsilon.$$

We assume that the statement holds for some fixed $p \in \mathbb{N}$,

$$\omega_\lambda\left(x_n,\ x_{n+p}\right) \prec \varepsilon.$$

Now, we show that the statement also holds for $p+1$. Since ω satisfies the Δ_2-type condition, there exist $C = 1$ such that

1. $\omega_{\frac{\lambda}{2}}\left(x_{n-1},\ x_n\right) \prec \omega_\lambda\left(x_{n-1},\ x_n\right)$, which implies $\omega_{\frac{\lambda}{2}}\left(x_{n-1},\ x_n\right) \prec \delta$.
2. $\omega_{\frac{\lambda}{2}}\left(x_n,\ x_{n+p}\right) \prec \omega_\lambda\left(x_n,\ x_{n+p}\right)$, which implies $\omega_{\frac{\lambda}{2}}\left(x_n,\ x_{n+p}\right) \prec \varepsilon$.

Furthermore, we have

$$\omega_\lambda\left(x_{n-1},\ x_{n+p}\right) \preccurlyeq \omega_{\frac{\lambda}{2}}\left(x_{n-1},\ x_n\right) + \omega_{\frac{\lambda}{2}}\left(x_n, x_{n+p}\right) \prec \delta + \varepsilon.$$

Now, we consider two cases.

1. If $\omega_\lambda\left(x_{n-1},\ x_{n+p}\right) \succcurlyeq \varepsilon$, then using Definition 6 we get
$$\omega_\lambda\left(x_n,\ x_{n+p+1}\right) \prec \varepsilon.$$
2. If $\omega_\lambda\left(x_{n-1},\ x_{n+p}\right) \prec \varepsilon$, then using Definition 7 we get
$$\omega_\lambda\left(x_n,\ x_{n+p+1}\right) \prec \omega_\lambda\left(x_{n-1},\ x_{n+p}\right) \prec \varepsilon.$$

Hence, we conclude that $\omega_\lambda\left(x_n,\ x_{n+p+1}\right) \prec \varepsilon$. So, x_n is ω-complex Cauchy sequence on X_ω. By completeness of X_ω, there exist $u \in X_\omega$ such that sequence x_n converges to u. Hence, $\lim_{n\to\infty} \omega_\lambda\left(\ x_n,\ u\right) = 0$.

Next, we show that u is a fixed-point of T. Since T is a Meir Keeler ω-complex contraction mapping, we obtain

$$\begin{aligned}\omega_\lambda\left(u, Tu\right) &\precsim \omega_{\lambda/2}\left(u, T^{n+1}u\right) + \omega_{\lambda/2}(T^{n+1}u, Tu)\\ &\prec \omega_{\lambda/2}\left(u, T^{n+1}u\right) + \omega_{\lambda/2}\left(\ T^n u,\ u\right).\end{aligned}$$

Taking modulus on both sides, we get

$$\left|\omega_\lambda\left(u, Tu\right)\right| \leq \left|\omega_{\lambda/2}\left(u, T^{n+1}u\right)\right| + \left|\omega_{\lambda/2}\left(\ T^n u, u\right)\right|.$$

Since $\lim_{n\to\infty} \omega_\lambda\left(x_n, u\right) = 0$, Lemma 1 implies $\lim_{n\to\infty} \left|\omega_\lambda\left(x_n,\ u\right)\right| = 0$. Since ω satisfies the Δ_2- condition, we have

$$\begin{aligned}0 &\leq \lim_{n\to\infty} \left|\omega_\lambda\left(u, Tu\right)\right|\\ &\leq \lim_{n\to\infty} \left|\omega_{\lambda/2}\left(u, T^{n+1}u\right)\right| + \lim_{n\to\infty} \left|\omega_{\lambda/2}\left(T^n u, u\right)\right|.\end{aligned}$$

Hence, $\lim_{n\to\infty} \left|\omega_\lambda\left(u, Tu\right)\right| = 0$, which implies $\lim_{n\to\infty} \omega_\lambda\left(u, Tu\right) = 0$. Therefore, $Tu = u$. So, u is a fixed-point of T.

Finally, we show the uniqueness of the fixed-point u of the mapping T. We assume there exists $u, v \in X_\omega$ such that $Tu = u$ and $Tv = v$. We deduce

$$\omega_\lambda\left(u, v\right) = \omega_\lambda\left(Tu, Tv\right) \prec \omega_\lambda\left(u, v\right),$$

as $\omega_\lambda\left(u, v\right) \in \mathbb{C}$, this leads to a contradiction. Then, u is a unique fixed-point of T. This completes the proof. □

4. Conclusion

Based on the discussion, we conclude that the fixed-point theorem for Meir-Keeler contraction mapping can be extended to complex-valued modular metric spaces by adding sufficient conditions for such a contraction mapping to have a unique fixed-point. To ensure the existence of a fixed-point for a Meir-Keeler contraction mapping in this space, then ω must satisfy the Δ_2-type condition.

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