RESEARCH ARTICLE • OPEN ACCESS

The Orthogonal Matrices of $\mathrm{O}(2)$ under A Transitive Standard Action of $\mathbb{S}^1$

Edi Kurniadi, Putri Nisa Pratiwi, Aurillya Queency and Kankan Parmikanti

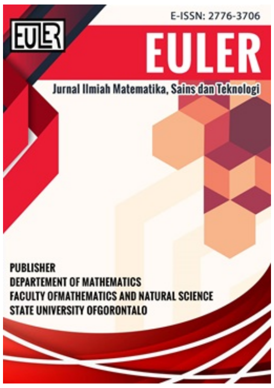
Volume 12, Issue 2, Pages 145–148, Dec 2024

Received 1 October 2024, Revised 4 November 2024, Accepted 10 November 2024, Published 1 December 2024

To Cite this Article : E. Kurniadi, P. N. Pratiwi, A. Queency and K. Parmikanti,"The Orthogonal Matrices of $\mathrm{O}(2)$ under A Transitive Standard Action of $\mathbb{S}^1$", *Euler J. Ilm. Mat. Sains dan Teknol.*, vol. 12, no. 2, pp. 145–148, 2024, *https://doi.org/10.37905/euler.v12i2.27752*

© 2024 by author(s)

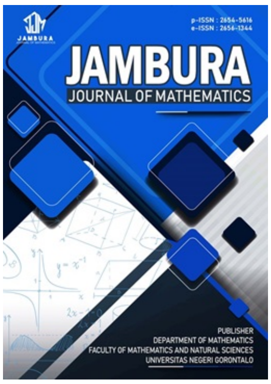
JOURNAL INFO • EULER : JURNAL ILMIAH MATEMATIKA, SAINS DAN TEKNOLOGI



Homepage	:	http://ejurnal.ung.ac.id/index.php/euler/index
Journal Abbreviation	:	Euler J. Ilm. Mat. Sains dan Teknol.
Frequency	:	Biannual (June and December)
Publication Language	:	English (preferable), Indonesia
DOI	:	https://doi.org/10.37905/euler
Online ISSN	:	2776-3706
Editor-in-Chief	:	Resmawan
Publisher	:	Department of Mathematics, Universitas Negeri Gorontalo
Country	:	Indonesia
OAI Address	:	http://ejurnal.ung.ac.id/index.php/euler/oai
Google Scholar ID	:	QF_r-_gAAAAJ
Email	:	euler@ung.ac.id

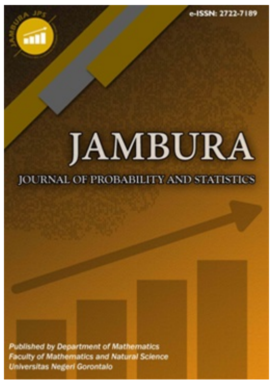
JAMBURA JOURNAL • FIND OUR OTHER JOURNALS

Jambura Journal of Biomathematics



Jambura Journal of Mathematics

Jambura Journal of Mathematics Education



Jambura Journal of Probability and Statistics

Euler : Jurnal Ilmiah Matematika, Sains dan Teknologi, Volume 12, Issue 2, Pages 145–148, Dec 2024 *https://doi.org/10.37905/euler.v12i2.27752*

Research Article

Check for updates

The Orthogonal Matrices of O (2) under A Transitive Standard Action of $\mathbb{S}^1$

Edi Kurniadi[1,*], **Putri Nisa Pratiwi**[1], **Aurillya Queency**[1] **and Kankan Parmikanti**[1]

[1]*Department of Mathematics, Universitas Padjadjaran, Sumedang, Indonesia*

ARTICLE HISTORY
Received 1 October 2024
Revised 4 November 2024
Accepted 10 November 2024
Published 1 December 2024

KEYWORDS
Homogeneous Space
Lie Group Action
Orthogonal Matrix
Unit Sphere

ABSTRACT. *In this paper we study a Lie group action of the matrix Lie group* $\mathrm{O}(2)$ *on the unit sphere* $\mathbb{S}^1$*. The research aims to establish the explicit formulas for all entries of* $\mathrm{O}(2)$ *whose action on* $\mathbb{S}^1$ *is transitive. All possibilities matrices of* $\mathrm{O}(2)$ *are given in which the space* $\mathbb{S}^1$ *is homogeneous. We prove that there are exactly two matrices in* $\mathrm{O}(2)$ *such that* $\mathbb{S}^1$ *is the homogeneous space. Moreover, the homogeneous spaces* $\mathbb{S}^{n-1}$ *of* $\mathrm{O(n)}$ *for* $n \geq 3$ *are also discussed.*

This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution-NonComercial 4.0 International License. Editorial of EULER: Department of Mathematics, Universitas Negeri Gorontalo, Jln. Prof. Dr. Ing. B. J. Habibie, Bone Bolango 96554, Indonesia.

1. Introduction

It is well known that a smooth manifold equipped with a transitive smooth action by a Lie group is called a homogeneous space. Many topics relate to homogeneous space. As instance we can see this realtion in a quantum weyl algebra which talks functions on homogeneous space [1], virtual constraints on Riemannian homogeneous space [2], invariant Koszul-Vinberg structures [3], holomorphically trivial cannonical bundle [4], and high-order integrators for Lagrangian system [5]. These show that homogeneous space has significance position in many topics and it is important to study more comprehensive.

The notion of the orthogonal group $\mathrm{O}(n)$ appears in many areas of representation theory of matrix Lie groups. The matrix Lie group $\mathrm{O}(n)$ consists of all $n \times n$ real matrices whose determinants are ± 1. Many researchers study this group such as syncronization problem over $A \subseteq \mathrm{O}(n)$ [6], a higher dimensional Chevalley restriction theorem [7]. On the other hand, the notion of the unit sphere $\mathbb{S}^n = \{x = (x_1, x_2, \ldots, x_{n+1}) \mid \|x\| = 1\} \subseteq \mathbb{R}^{n+1}$ appears as the Lie group. Furthermore, this space can be as a carrier space. As instance in the intertwining operator associated to summability of $\mathbb{S}^n$ [8], linear stability [9], and a python package [10]. In our research we study the action of the orthogonal group $\mathrm{O}(2)$ on the Lie group $\mathbb{S}^1$.

Different from the previous results, we give the explicits formulas for all matrices in the orthogonal matrix Lie group $\mathrm{O}(2)$ which the action of $\mathbb{S}^1$ realized on $\mathrm{O}(2)$ is transitive. In other words, the unit sphere $\mathbb{S}^1$ is homogeneous space. We have some reasons why this result is important. Firstly, it relates to homogeneous spaces. It is well known that the theory homogeneous space is very important in many fields of mathematics. One of them is the theory of relativity. Secondly, the Lie group actions can be considered in representation theory of Lie groups using the orbit method. Therefore by knowing the explicit formulas of all matrices in the orthogonal matrix Lie group $\mathrm{O}(2)$, one can apply the results of this paper to theory of homogeneous space and to theory of Lie group representations by the orbit method.

2. Methods

We involve the concept of a Lie group action of the matrix Lie group $\mathrm{O}(n)$ on the unit sphere $\mathbb{S}^1$. The action of $\mathrm{O}(n)$ on $\mathbb{S}^1$ is given by $A \cdot x = Ax$ as a usual matrix multiplication. We claim that this action is transitive which means for each $x, y \in \mathbb{S}^1$, there exists $A_0 \in \mathrm{O}(n)$ such that Ax. From this equation, we shall find the explicit formulas of $A_0 \in \mathrm{O}(n)$ such that $A_0 x = y$. This implies that the unit sphere $\mathbb{S}^1$ is homogeneous space.

We describe our research method as follows: The idea of Lie group actions arise in many types of Lie groups (see for detail: [11–14]). A homogeneous space is constructed by a Lie group action. Roughly speaking, the idea of homogeneous space construction arises from the notion of quotient of Lie groups by closed subgroups [15]. To see this, let G be a Lie group and $H \subseteq G$ be a Lie subgroup. The left coset space of H in G, denoted by G/H, consists of all left cosets of H and the elements of G/H form a partition of the Lie group G. A left action of Lie group G on G/H by $x_1 (x_2 H) = (x_1 x_2) H$ gives G/H into a homogeneous space in G or a homogeneous G-space. Moreover, let $\mathfrak{X}$ be a homogeneous G-space and a be any element in $\mathfrak{X}$. Let G_p be an isotropy group of G. Then the map given by $\psi : G/G_p \ni gG_p \mapsto gp \in \mathfrak{X}$ is an equivariant diffeomorphism.

Let $\mathrm{O}(2)$ be the orthogonal group consisting all 2×2 real matrices whose determinants are ± 1 and $\mathbb{S}^1$ be the unit sphere contained in $\mathbb{R}^2$. The main goal of this research is to consider the explicit formulas of all matrices in $\mathrm{O}(2)$, which preserve a transitive action of $\mathrm{O}(2)$ on $\mathbb{S}^1$. We show that there are only two possibilities of general forms of a matrix $A \in \mathrm{O}(2)$ whose imply the sphere $\mathbb{S}^1$ is homogeneous space. This result can motivate to give a generalization of general formulas of matrices in $\mathrm{O}(n)$, which admit a transitive smooth action of $\mathrm{O}(n)$ on the unit n-

*Corresponding Author.

Email : *edi.kurniadi@unpad.ac.id* (E. Kurniadi)
Homepage : http://ejurnal.ung.ac.id/index.php/euler/index / E-ISSN : 2776-3706
© 2024 by the Author(s).

sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. Furthermore, this result can be considered in homogeneous spaces and Lie group representation theories.

3. Results and Discussion

The main result in this paper stated in the following proposition .

Proposition 1. *Let* $\mathrm{O}(2)$ *be the orthogonal group consisting all* 2×2 *real matrices whose determinants are* ± 1 *and* $\mathbb{S}^1$ *be the unit sphere contained in* $\mathbb{R}^2$. *The action of* $\mathrm{O}(2)$ *on* $\mathbb{S}^1$ *is given by the standard action as follows:*

$$Ax = y$$

for $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{O}(2)$ *and* $x = (x_1, x_2)^t, y = (y_1, y_2)^t \in \mathbb{S}^1$. *Then* $\mathbb{S}^1$ *is a homogeneous space if for each and* $x = (x_1, x_2)^t, y = (y_1, y_2)^t \in \mathbb{S}^1$, *there are exist exactly two matrices in* $\mathrm{O}(2)$ *of the form*

$$A_1 = \begin{pmatrix} x_1y_1 - x_2y_2 & x_2y_1 + x_1y_2 \\ x_1y_2 + x_2y_1 & x_2y_2 - x_1y_1 \end{pmatrix}$$

and the form

$$A_2 = \begin{pmatrix} x_1y_1 + x_2y_2 & x_2y_1 - x_1y_2 \\ x_1y_2 - x_2y_1 & x_2y_2 + x_1y_1 \end{pmatrix}$$

with $x_1 \neq 0,\ x_2 \neq 0,$ *and* $y_1 \neq 0$ *such that* $A_1x = y$ *and* $A_2x = y$.

Proof. In order to show that $\mathbb{S}^1$ is a homogeneous space in $\mathrm{O}(2)$ we shall show that for all $x = (x_1, x_2)^t, y = (y_1, y_2)^t \in \mathbb{S}^1$ with $x_1 \neq 0,\ x_2 \neq 0,$ and $y_1 \neq 0$, there exist matrices $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{O}(2)$ such that $Ax = y$. The latter equation and the orthogonality elements in $\mathrm{O}(2)$ give the equations belows:

$$x_1a_{11} + x_2a_{12} = y_1, \tag{1}$$
$$x_1a_{21} + x_2a_{22} = y_2, \tag{2}$$
$$a_{11}^2 + a_{21}^2 = 1, \tag{3}$$
$$a_{12}^2 + a_{22}^2 = 1, \tag{4}$$
$$a_{11}a_{12} + a_{21}a_{22} = 0, \tag{5}$$
$$x_1^2 + x_2^2 = 1, \tag{6}$$
$$y_1^2 + y_2^2 = 1. \tag{7}$$

The eq. (1) and eq. (2) give

$$a_{11} = \frac{y_1}{x_1} - \frac{x_2}{x_1}a_{12}, \quad x_1 \neq 0, \tag{8}$$
$$a_{21} = \frac{y_2}{x_1} - \frac{x_2}{x_1}a_{22}, \quad x_1 \neq 0. \tag{9}$$

By substituting the eq. (8) and eq. (9) to the eq. (5), then we have

$$\left(\frac{y_1}{x_1} - \frac{x_2}{x_1}a_{12}\right)a_{12} + \left(\frac{y_2}{x_1} - \frac{x_2}{x_1}a_{22}\right)a_{22} = 0,$$
$$\frac{y_1}{x_1}a_{12} - \frac{x_2}{x_1}a_{12}^2 + \frac{y_2}{x_1}a_{22} - \frac{x_2}{x_1}a_{22}^2 = 0,$$
$$\frac{y_1}{x_1}a_{12} + \frac{y_2}{x_1}a_{22} - \frac{x_2}{x_1}\left(a_{12}^2 + a_{22}^2\right) = 0,$$
$$\frac{y_1}{x_1}a_{12} + \frac{y_2}{x_1}a_{22} - \frac{x_2}{x_1} = 0,$$
$$\frac{y_1}{x_1}a_{12} + \frac{y_2}{x_1}a_{22} = \frac{x_2}{x_1}.$$

In other words, we have

$$y_1a_{12} + y_2a_{22} = x_2. \tag{10}$$

Eliminating the eq. (10) and eq. (1), then we obtain,

$$\begin{array}{c|c|c} y_1a_{12} + y_2a_{22} = x_2 & \times x_2 & x_2y_1a_{12} + x_2y_2a_{22} = x_2^2 \\ x_1a_{11} + x_2a_{12} = y_1 & \times -y_1 & \underline{-x_1y_1a_{11} - x_2y_1a_{12} = -y_1^2} \\ & & x_2y_2a_{22} - x_1y_1a_{11} = x_2^2 - y_1^2. \end{array} \tag{11}$$

We again eliminate the eq. (11) and eq. (2), then we get,

$$\begin{array}{rcl|c|} x_1a_{21} + x_2a_{22} & = & y_2 & \times y_2 \\ x_2y_2a_{22} - x_1y_1a_{11} & = & x_2^2 - y_1^2 & \times -1 \end{array}$$

$$\begin{array}{rcl} x_1y_2a_{21} + x_2y_2a_{22} & = & y_2^2 \\ \underline{-x_2y_2a_{22} + x_1y_1a_{11}} & = & \underline{-x_2^2 + y_1^2} \\ x_1y_1a_{11} + x_1y_2a_{21} & = & y_1^2 + y_2^2 - x_2^2 \\ x_1y_1a_{11} + x_1y_2a_{21} & = & 1 - x_2^2 \\ x_1y_1a_{11} + x_1y_2a_{21} & = & x_1^2. \end{array}$$

In the simple formula we have,

$$y_1a_{11} + y_2a_{21} = x_1 \tag{12}$$

From the eq. (12), we obtain

$$a_{11} = \frac{x_1}{y_1} - \frac{y_2}{y_1}a_{21}, \quad y_1 \neq 0. \tag{13}$$

Substituting the eq. (13) to the eq. (3), then we have

$$\left(\frac{x_1}{y_1} - \frac{y_2}{y_1}a_{21}\right)^2 + a_{21}^2 = 1,$$
$$\frac{x_1^2}{y_1^2} + \frac{y_2^2}{y_1^2}a_{21}^2 - \frac{2x_1y_2}{y_1^2}a_{21} + a_{21}^2 = 1,$$
$$\left(1 + \frac{y_2^2}{y_1^2}\right)a_{21}^2 - \frac{2x_1y_2}{y_1^2}a_{21} + \frac{x_1^2}{y_1^2} - 1 = 0,$$
$$\left(\frac{y_1^2 + y_2^2}{y_1^2}\right)a_{21}^2 - \frac{2x_1y_2}{y_1^2}a_{21} + \frac{x_1^2}{y_1^2} - 1 = 0.$$

We simplify the latter equation, then we get the nice formula as follows:

$$a_{21}^2 - 2x_1y_2a_{21} + x_1^2 - y_1^2 = 0 \tag{14}$$

By solving the quadratic eq. (14), then we have two different roots as follows:

$$(a_{21})_{1,2} = \frac{2x_1y_2 \pm \sqrt{(-2x_1y_2)^2 - 4(x_1^2 - y_1^2)}}{2}$$
$$= \frac{2x_1y_2 \pm \sqrt{4x_1^2y_2^2 - 4x_1^2 + 4y_1^2}}{2},$$

$$\begin{aligned}
(a_{21})_{1,2} &= \frac{2x_1y_2 \pm 2\sqrt{x_1^2\,(y_2^2-1)+y_1^2}}{2},\\
(a_{21})_{1,2} &= x_1y_2 \pm \sqrt{x_1^2\,(-y_1^2)+y_1^2}\\
&= x_1y_2 \pm \sqrt{y_1^2(1-x_1^2)}\\
&= x_1y_2 \pm \sqrt{y_1^2x_2^2}\\
&= x_1y_2 \pm x_2y_1.
\end{aligned}$$

For the first case $a_{21} = x_1y_2 + x_2y_1$, we have:

$$\begin{aligned}
a_{11} &= \frac{x_1}{y_1} - \frac{y_2}{y_1}a_{21} = \frac{x_1}{y_1} - \frac{y_2}{y_1}(x_1y_2 + x_2y_1)\\
&= \frac{x_1}{y_1} - \frac{x_1y_2^2}{y_1} - x_2y_2 = \frac{x_1}{y_1}\left(1-y_2^2\right) - x_2y_2\\
&= \frac{x_1}{y_1}y_1^2 - x_2y_2 = x_1y_1 - x_2y_2, \;\; y_1 \neq 0,\\
a_{12} &= \frac{y_1}{x_2} - \frac{x_1}{x_2}a_{11} = \frac{y_1}{x_2} - \frac{x_1}{x_2}(x_1y_1 - x_2y_2)\\
&= \frac{y_1}{x_2} - \frac{x_1^2y_1}{x_2} + x_1y_2 = \frac{y_1}{x_2}\left(1-x_1^2\right) + x_1y_2\\
&= \frac{y_1}{x_2}x_2^2 + x_1y_2 = x_2y_1 + x_1y_2, \; x_2 \neq 0,\\
a_{22} &= \frac{y_2}{x_2} - \frac{x_1}{x_2}a_{21} = \frac{y_2}{x_2} - \frac{x_1}{x_2}(x_1y_2 + x_2y_1)\\
&= \frac{y_2}{x_2} - \frac{x_1^2y_2}{x_2} - x_1y_1 = \frac{y_2}{x_2}\left(1-x_1^2\right) - x_1y_1\\
&= \frac{y_2}{x_2}x_2^2 - x_1y_1 = x_2y_2 - x_1y_1, \; x_2 \neq 0.
\end{aligned}$$

For the second case $a_{21} = x_1y_2 - x_2y_1$, we have:

$$\begin{aligned}
a_{11} &= \frac{x_1}{y_1} - \frac{y_2}{y_1}a_{21} = \frac{x_1}{y_1} - \frac{y_2}{y_1}(x_1y_2 - x_2y_1)\\
&= \frac{x_1}{y_1} - \frac{x_1y_2^2}{y_1} + x_2y_2 = \frac{x_1}{y_1}\left(1-y_2^2\right) + x_2y_2\\
&= \frac{x_1}{y_1}y_1^2 + x_2y_2 = x_1y_1 + x_2y_2, \;\; y_1 \neq 0,\\
a_{12} &= \frac{y_1}{x_2} - \frac{x_1}{x_2}a_{11} = \frac{y_1}{x_2} - \frac{x_1}{x_2}(x_1y_1 + x_2y_2)\\
&= \frac{y_1}{x_2} - \frac{x_1^2y_1}{x_2} - x_1y_2 = \frac{y_1}{x_2}\left(1-x_1^2\right) - x_1y_2\\
&= \frac{y_1}{x_2}x_2^2 - x_1y_2 = x_2y_1 - x_1y_2, \;\; x_2 \neq 0,\\
a_{22} &= \frac{y_2}{x_2} - \frac{x_1}{x_2}a_{21} = \frac{y_2}{x_2} - \frac{x_1}{x_2}(x_1y_2 - x_2y_1)\\
&= \frac{y_2}{x_2} - \frac{x_1^2y_2}{x_2} + x_1y_1 = \frac{y_2}{x_2}\left(1-x_1^2\right) + x_1y_1\\
&= \frac{y_2}{x_2}x_2^2 + x_1y_1 = x_2y_2 + x_1y_1, \;\; x_2 \neq 0.
\end{aligned}$$

We obtain all solution of the quadratic eq. (14) as follows:

(i) $a_{11} = x_1y_1 - x_2y_2$, $a_{12} = x_2y_1 + x_1y_2$, $a_{21} = x_1y_2 + x_2y_1$, $a_{22} = x_2y_2 - x_1y_1$

(ii) $a_{11} = x_1y_1 + x_2y_2$, $a_{12} = x_2y_1 - x_1y_2$, $a_{21} = x_1y_2 - x_2y_1$, $a_{22} = x_2y_2 + x_1y_1$

with $x_1 \neq 0,\ x_2 \neq 0,\ y_1 \neq 0$. In other words, we find for all $x = (x_1, x_2)^t, y = (y_1, y_2)^t \in \mathbb{S}^1$ with $x_1 \neq 0,\ x_2 \neq 0,$ and $y_1 \neq 0$, there are exactly two matrices in $\mathrm{O}(2)$ of the form

$$A_1 = \begin{pmatrix} x_1y_1 - x_2y_2 & x_2y_1 + x_1y_2 \\ x_1y_2 + x_2y_1 & x_2y_2 - x_1y_1 \end{pmatrix},$$

and the form

$$A_2 = \begin{pmatrix} x_1y_1 + x_2y_2 & x_2y_1 - x_1y_2 \\ x_1y_2 - x_2y_1 & x_2y_2 + x_1y_1 \end{pmatrix},$$

such that $A_1x = y$ and $A_2x = y$. Thus, $\mathbb{S}^1$ is a homogeneous space in $\mathrm{O}(2)$ as required. □

For further research, we can show that $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ is homogeneous space under the action of the orthogonal group $\mathrm{O}(n+1)$. The problem is how to find the explicit formulas of some matrices $A \in \mathrm{O}(n+1)$. The study can start from the problem $\mathrm{O}(3)$ which acts on the space $\mathbb{S}^2$.

4. Conclusion

The unit sphere $\mathbb{S}^1$ is homogeneous space since we find that for every $x = (x_1, x_2)^t, y = (y_1, y_2)^t \in \mathbb{S}^1$ with $x_1 \neq 0,\ x_2 \neq 0,$ and $y_1 \neq 0$ then there are exist exactly two matrices in $\mathrm{O}(2)$ of the forms:

$$A_1 = \begin{pmatrix} x_1y_1 - x_2y_2 & x_2y_1 + x_1y_2 \\ x_1y_2 + x_2y_1 & x_2y_2 - x_1y_1 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} x_1y_1 + x_2y_2 & x_2y_1 - x_1y_2 \\ x_1y_2 - x_2y_1 & x_2y_2 + x_1y_1 \end{pmatrix}$$

such that $A_1x = y$ and $A_2x = y$ are satisfied.

Author Contributions. **Edi Kurniadi:** Conceptualization, writing—review and editing, visualization, supervision. **Putri Nisa Pratiwi:** Writing—original draft preparation, software. **Aurillya Queency:** Conceptualization, methodology, software, validation. **Kankan Parmikanti:** Writing—original draft preparation. All authors discussed the results and contributed to the final manuscript.

Acknowledgement. The authors sincerely thank to editors and reviewers for their valuable comments and advice in this article.

Funding. This research received no external funding.

Conflict of interest. The authors declare that there are no conflicts of interest related to this article.

References

[1] G. Letzter, S. Sahi, and H. Salmasian, "Weyl algebras for quantum homogeneous spaces," *J Algebra*, vol. 655, pp. 651–721, Oct. 2024, doi: 10.1016/j.jalgebra.2023.12.023.

[2] E. Stratoglou, A. A. Simoes, A. Bloch, and L. Colombo, "Virtual constraints on Riemannian homogeneous spaces," *IFAC-PapersOnLine*, vol. 58, no. 6, pp. 77–82, 2024, doi: 10.1016/j.ifacol.2024.08.260.

[3] A. Abouqateb, M. Boucetta, and C. Bourzik, "Homogeneous spaces with invariant Koszul-Vinberg structures," *Journal of Geometry and Physics*, vol. 193, p. 104965, Nov. 2023, doi: 10.1016/j.geomphys.2023.104965.

[4] A. Otal and L. Ugarte, "Six dimensional homogeneous spaces with holomorphically trivial canonical bundle," *Journal of Geometry and Physics*, vol. 194, p. 105014, Dec. 2023, doi: 10.1016/j.geomphys.2023.105014.

[5] R. T. Sato Martín de Almagro, "High-order integrators for Lagrangian systems on homogeneous spaces via nonholonomic mechanics," *J Comput Appl Math*, vol. 421, p. 114837, Mar. 2023, doi: 10.1016/j.cam.2022.114837.

[6] H. Liu, M.-C. Yue, and A. M.-C. So, "A unified approach to synchronization problems over subgroups of the orthogonal group," *Appl Comput Harmon Anal*, vol. 66, pp. 320–372, Sep. 2023, doi: 10.1016/j.acha.2023.05.002.

[7] L. Song, X. Xia, and J. Xu, "A higher-dimensional Chevalley restriction theorem for orthogonal groups," *Adv Math (N Y)*, vol. 426, p. 109104, Aug. 2023, doi: 10.1016/j.aim.2023.109104.

[8] Y. Xu, "Intertwining operator associated to symmetric groups and summability on the unit sphere," *J Approx Theory*, vol. 272, p. 105649, Dec. 2021, doi: 10.1016/j.jat.2021.105649.

[9] T.-H. Miura, "Linear stability and enhanced dissipation for the two-jet Kolmogorov type flow on the unit sphere," *J Funct Anal*, vol. 283, no. 8, p. 109607, Oct. 2022, doi: 10.1016/j.jfa.2022.109607.

[10] H. Hacıhabiboğlu, "sphstat: A Python package for inferential statistics on vectorial data on the unit sphere," *SoftwareX*, vol. 24, p. 101514, Dec. 2023, doi: 10.1016/j.softx.2023.101514.

[11] T. Inaba, S. Matsumoto, and Y. Mitsumatsu, "Normally contracting Lie group actions," *Topol Appl*, vol. 159, no. 5, pp. 1334–1338, Mar. 2012, doi: 10.1016/j.topol.2011.12.012.

[12] O. Goertsches and S. Haghshenas Noshari, "Equivariant formality of isotropy actions on homogeneous spaces defined by Lie group automorphisms," *J Pure Appl Algebra*, vol. 220, no. 5, pp. 2017–2028, May 2016, doi: 10.1016/j.jpaa.2015.10.013.

[13] S. A. Antonyan, A. Mata-Romero, and E. Vargas-Betancourt, "Two homotopy characterizations of G-ANR spaces for proper actions of Lie groups," *Topol Appl*, vol. 340, p. 108719, Dec. 2023, doi: 10.1016/j.topol.2023.108719.

[14] F.-J. Turiel and A. Viruel, "Smooth actions of connected compact Lie groups with a free point are determined by two vector fields," *Journal of Geometry and Physics*, vol. 201, p. 105196, Jul. 2024, doi: 10.1016/j.geomphys.2024.105196.

[15] J. M. Lee, *Introduction to Smooth Manifolds*, vol. 218. New York, NY: Springer New York, 2012. doi: 10.1007/978-1-4419-9982-5.