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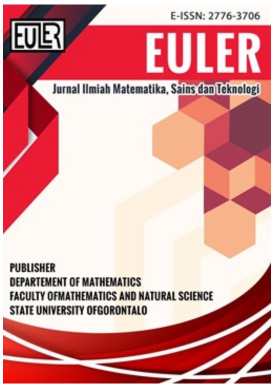

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Green's Equivalence Classes in Full Transformation Semigroups: A Height-based Approach

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ABSTRACT. *Green's relations constitute five distinct equivalence classes that serve to define the elements of a semigroup. In this work, the equivalence classes within the Full Transformation Semigroup (T_κ) were systematically enumerated according to their respective heights, and the findings derived were subsequently generalized through the application of a combinatorial assertion and their graphical representations included to substantiate the theoretical results.*

1. Introduction

A semigroup is simply a set S with an associative binary operation. Transformation semigroups are among the most fundamental mathematical objects and hold significant importance in semigroup theory, as every semigroup is isomorphic to a transformation semigroup. Consider a finite chain, say $X_\kappa = \{1, 2, \ldots, \kappa\}$ under the natural ordering. Define T_κ, P_κ, and I_κ as the full transformation semigroup, the partial transformation semigroup, and the partial one-to-one transformation semigroup on X_κ respectively. These three semigroups are fundamental structures in transformation semigroup theory.

For any transformation $\alpha : \text{Dom}(\alpha) \subseteq X_\kappa \to \text{Im}(\alpha) \subseteq X_\kappa$. The transformation α is classified as a full transformation if $\text{Im}(\alpha) \subseteq X_\kappa = X_\kappa$. If $\text{Im}(\alpha) \subset X_\kappa$, it is referred to as a partial transformation, and if α is not defined on all elements of X_κ it is strictly partial.

The notion of ideals naturally leads to the consideration of certain equivalence relations on a semigroup. These relations are named after James Alexander Green, who introduced them in [1]. Howie, a prominent semigroup theorist, described this work as foundational in semigroup theory.

If a is an element of a semigroup S, then Sa is called a left ideal, aS a right ideal, and SaS a two-sided ideal if it is both a left and a right ideal. The principal left/right ideal generated by a will also be written as Sa/aS, and the principal two-sided ideal as SaS for every $a \in S$.

Green's relations provide a fundamental classification of elements in a semigroup, organizing them into distinct equivalence classes based on the principal ideals they generate. These equivalence relations are: L-relation, R-relation, J-relation, D-relation, and H-relation. For elements $a, b \in S$, Green's relations L, R, and J are defined as follows:

$$aLb \iff Sa = Sb, \quad aRb \iff aS = bS, \quad \text{and}$$
$$aJb \iff SaS = SbS.$$

These five equivalence relations partition the elements of T_n into equivalence classes. The L-class of a is denoted by L_a (and similarly for the other relations). Generally, the intersection of any L-class with any R-class is either an H-class or the empty set.

Furthermore, D_a is the smallest equivalence class containing both L_a and R_a. Thus,

$$aDb \Rightarrow aJb,$$

so J contains D, and they are equivalent in a finite semigroup. Note that

$$aDb \iff R_a \cap L_a \neq \emptyset.$$

A D-class can be conveniently visualized as an "egg box," as illustrated in [2] . Each row of eggs represents an R-class, each column represents an L-class, and the egg box itself represents a D-class.

These equivalence classes play a fundamental role in the development of semigroup theory, as they address mutual divisibility of various kinds. An alternative characterization, emphasizing mutual divisibility is given in Proposition 2.

Despite many results on transformation semigroups (see [3–7]) and Green's relations in semigroup theory (see [5, 8–10]), the existing literature lacks a combinatorial approach that explicitly characterizes the structural properties of these classes in terms of their height. Prior work has investigated the R-height of semigroups and their bi-ideals. Recently, [11] obtained results on the connections between Green's relations in a Γ-semigroup and its operator semigroups.

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Although these relations have been extensively studied, particularly in the full transformation semigroup, existing approaches have primarily focused on their algebraic properties. In contrast, this work introduces a combinatorial framework to systematically enumerate and analyze Green's equivalence classes from a height-based perspective. A combinatorial function $\Upsilon(\kappa, p)$ is introduced to analyze the size and structure of these classes. The symbolic computation software MATLAB is used to visualize graphical representations of the classes. For more results on combinatorics, see [12, 13].

2. Preliminaries

Enumerative issues of a fundamentally combinatorial nature arise naturally within the analysis of transformation semigroups. Various sequences and triangular arrangements of integers, regarded as combinatorial gems, prominently feature in these enumeration challenges. Notable examples include the Stirling numbers [14], factorial numbers [3, 15], binomial numbers [16, 17], Fibonacci numbers [18], Catalan numbers [19], Lah numbers [12, 20]. These sequences have played a significant role in enumeration problems, as documented in [21].

We now present some definitions and relevant results used in this paper.

Definition 1. Let $S = T_\kappa$ and $\alpha \in T_\kappa$, then the height of α is defined as

$$h(\alpha) = |\mathrm{Im}(\alpha)|. \tag{1}$$

The fix of α is defined as

$$f(\alpha) = |F(\alpha)| = |\{x \in \mathrm{Dom}(\alpha) : x\alpha = x\}|. \tag{2}$$

Definition 2. For natural numbers $\kappa \geq p \geq 1$,

$$\Upsilon(\kappa; p) = |\{\alpha \in S : h(\alpha) = |\mathrm{Im}(\alpha)| = p\}|.$$

Definition 3. [22]: A semigroup S is regular if, for any element $a \in S$, there exists an element a^* such that:
(i) $aa^*a = a$,
(ii) $aa^* = a^*a$, and
(iii) a^* is an inverse of a.

Theorem 1. *[1] If e is an idempotent in a semigroup S, then H_e is a subgroup of S. Moreover, no H_a in S can contain more than one idempotent.*

Proposition 1. *[2] If a is a regular element of a semigroup S, then every element of D_a is regular.*

Stirling Numbers of the Second Kind: The Stirling number of the second kind counts the ways to partition a set of κ elements into p non-empty subsets, denoted by $S(\kappa; p)$ and given as

$$S(\kappa; p) = \frac{1}{p!}\sum_{j=1}^{p}(-1)^{p-j}\binom{p}{j}j^{\kappa}.$$

Bell Numbers: The Bell number is the sum of Stirling numbers of the second kind, denoted by B_κ and given as

$$B_\kappa = \sum_{p=1}^{n} S(\kappa; p).$$

Factorial Numbers: A factorial number is the product of all positive integers less than or equal to κ, denoted by $\kappa!$.

Proposition 2. *[23] In the full transformation semigroup T_κ, the following hold:*
(i) $\alpha L \beta$ if and only if $\mathrm{Im}(\alpha) = \mathrm{Im}(\beta)$,
(ii) $\alpha R \beta$ if and only if $\ker\alpha = \ker\beta$,
(iii) $\alpha D \beta$ if and only if $|\mathrm{Im}(\alpha)| = |\mathrm{Im}(\beta)|$, and
(iv) $D = J$.

3. Results and Discussion

This section provides a comparative analysis of Proposition 2.6 in T_κ for $1 \leq \kappa \leq 5$.

Theorem 2. *Let $S = T_\kappa$ be the full transformation semigroup on κ elements, and let L_α be the $\mathcal{L}$-class of some transformation α in T_κ. Then, the number of elements in L_α is given by:*

$$|L_\alpha| = \sum_{p=1}^{\kappa}\binom{\kappa}{p}, \quad \forall \kappa \geq 1.$$

Proof. Let $\alpha \in L_\alpha$. Recall from Definition 2 that the function $\Upsilon(\kappa; p)$ represents the number of elements in an $\mathcal{L}$-class where the image of the transformation has exactly p elements.

In T_κ, the $\mathcal{L}$-class of a transformation α consists of all transformations with the same image size. That is, for each p in the range $1 \leq p \leq \kappa$, there exist transformations whose image has precisely p elements.

To construct such a transformation, we first choose a subset of p elements from the κ available elements to serve as the image. The number of ways to do this is given by the binomial coefficient:

$$\binom{\kappa}{p} = \frac{\kappa!}{p!(\kappa - p)!}.$$

Since this holds for all p from 1 to κ, summing over all possible image sizes gives:

$$|L_\alpha| = \sum_{p=1}^{\kappa}\binom{\kappa}{p}.$$

□

Corollary 1. *Let $S = T_\kappa$ and $L_a = \Upsilon(\kappa, p)$. Then, $\Upsilon(\kappa, p) = \Upsilon(\kappa, \kappa) = 1, \quad \forall \kappa$.*

Proof. Since $L_a \in T_\kappa$, consider two subsets $\alpha, \beta \subseteq L_a$. Define sequences $A_1 < A_2 < \cdots < A_i$ and $B_1 < B_2 < \cdots < B_i$. If $a_i = b_i$ and the sequences satisfy $a_1 \leq a_2 \leq \cdots \leq a_i, \quad b_1 \leq b_2 \leq \cdots \leq b_i$, then the cardinality of the image sets is given by $|\text{Im}(\alpha)| = |\text{Im}(\beta)| \geq 1$. Conversely, if $a_i \neq b_i$ but

$$a_1 = a_2 = \cdots = a_i, \quad b_1 = b_2 = \cdots = b_i,$$

then this shows that,

$$|\text{Im}(\alpha)| = |\text{Im}(\beta)| = 1.$$

By virtue of (1), it follows that

$$h(\alpha) = h(\beta) = p = 1, \quad \forall \kappa.$$

Hence, the result. □

Corollary 2. *If $S = T_\kappa$ and $L_a = \Upsilon(\kappa, p)$, then $\Upsilon(\kappa, p) = \Upsilon(\kappa, 1) = \kappa, \quad \forall \kappa$.*

Proof. For any $\alpha \in L_a$, it is evident that there exist exactly κ elements in the L-classes of height one. These elements correspond to the idempotent elements of the structure. □

Remark 1. Table 1 presents computed values of $\Upsilon(\kappa; p)$ in L_a.

Table 1. Structure of L-class on T_κ in Terms of Height

κ/p	1	2	3	4	5	$\sum \Upsilon(\kappa;p) = \lvert L_a \rvert$
1	1					1
2	2	1				3
3	3	3	1			7
4	4	6	4	1		15
5	5	10	10	5	1	31

Graphical illustration in Figure 1.

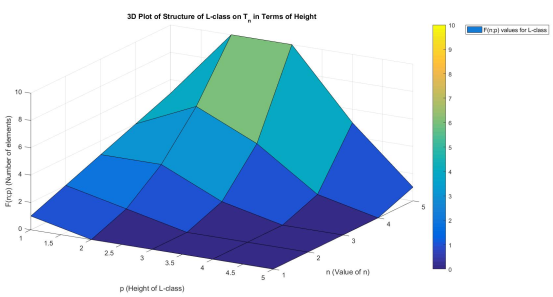

Figure 1. Graph of L-class

The κ values represent different values of κ and the p values correspond to different heights. The $\Upsilon(\kappa; p)$ contains the values from Table 1, where each entry corresponds to the number of elements in the L-class at the given height p for a specific κ.

Theorem 3. *Let $S = T_\kappa$ and R_a be the R-class in T_κ. Then, $|R_a| = B_\kappa$.*

Proof. It has been observed that the R-class on T_κ satisfies the Bell numbers with recurrence $B_0 = 1$. We have

$$B_\kappa = \sum_{p=1}^{\kappa} S(\kappa; p), \quad 1 \leq p \leq \kappa.$$

Let the number of partitions obtained for n elements be $S(\kappa; p)$ with exactly p parts. Then, the number $S(\kappa; p)$ is called Stirling numbers of the second kind. Hence, the result follow. □

Corollary 3. *If $S = T_\kappa$ and $R_a = \Upsilon(\kappa; p)$, then $\Upsilon(\kappa; p) = \Upsilon(\kappa; \kappa) = 1 \quad \forall \kappa$.*

Proof. The proof follows from Corollary 1. □

Corollary 4. *If $S = T_\kappa$ and $R_a = \Upsilon(\kappa; p)$, then $\Upsilon(\kappa; p) = \Upsilon(\kappa; 1) = 1 \quad \forall \kappa$.*

Proof. Suppose $S = T_\kappa$ and $R_a \subset T_\kappa$. Let $\alpha \in R_a$ such that $f(\alpha) = |p| = 1$. If $|p| > 1$, it implies that $f(\alpha) \geq 1$. Therefore, for all $|p| = 1$, it is clear that there exists exactly one R-class where only a single element is fixed. Hence, the proof. □

Remark 2. Table 2 presents computed values of $\Upsilon(\kappa; p)$ in R_a.

Table 2. Structure of R-class on T_κ in Terms of Height

κ/p	1	2	3	4	5	$\sum \Upsilon(\kappa;p) = \lvert R_a \rvert$
1	1					1
2	1	1				2
3	1	3	1			5
4	1	7	6	1		15
5	1	15	25	10	1	52

Graphical illustration in Figure 2.

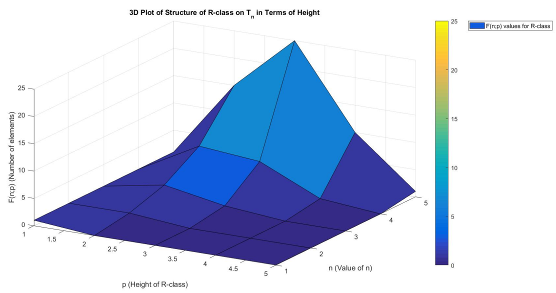

Figure 2. Graph of R-class

The $\Upsilon(\kappa; p)$ is based on the data from Table 2, which represents the structure of the R-class on T_κ in terms of height.Labels for the x-axis (p), y-axis (κ), and z-axis ($\Upsilon(\kappa;p)$) are added for clarity.

Theorem 4. *Let $S = T_\kappa$ and H_a be the H-class in T_κ. Then, $|H_a| = \sum_{p=1}^{\kappa} \binom{\kappa}{p} S(\kappa;p)$.*

Proof. Let $\alpha \in H_a$ and let $\Upsilon(\kappa;p)$ be as defined in Definition 2. It is easy to see that the H_a in T_κ is equivalent to the product of L_a and R_a, which can be visualized as a D-class. Hence, the result follows. □

Corollary 5. *If $S = T_\kappa$ and $H_a = \Upsilon(\kappa;p)$, then $\Upsilon(\kappa;p) = \Upsilon(\kappa;\kappa) = 1 \quad \forall \kappa$.*

Proof. The proof follows from Corollary 2. □

Corollary 6. *If $S = T_\kappa$ and $H_a = \Upsilon(\kappa;p)$, then $\Upsilon(\kappa;p) = \Upsilon(\kappa;1) = \kappa \quad \forall \kappa$.*

Proof. The proof follows from Corollary 3. □

Remark 3. Table 3 presents computed values of $\Upsilon(\kappa;p)$ in H_a

Table 3. Structure of H-class on T_κ in Terms of Height

κ/p	1	2	3	4	5	$\sum \Upsilon(\kappa;p) = \|H_a\|$
1	1					1
2	2	1				3
3	3	9	1			13
4	4	42	24	1		71
5	5	150	250	50	1	456

Graphical illustration in Figure 3.

Figure 3. Graph of H-class

The $\Upsilon(\kappa;p)$ is based on the data from Table 3, which represents the structure of the H-class on T_κ in terms of height.The `meshgrid` function creates a grid of n (rows) and p (columns) for plotting and the `legend` describes the surface plot as representing the $\Upsilon(\kappa;p)$ values for the H-class.

Theorem 5. *Let $S = T_\kappa$ and D_a be the D-class in T_κ.Then, $|D_a| = \kappa$.*

Proof. Let $\alpha \in D_a$ and let $\Upsilon(\kappa;p)$ be as defined in Definition 2. It is obvious to see that, for each value of p, there is exactly a D-class for all values of n, since the egg box itself represents the D-classes. □

Remark 4. Table 4 presents computed values of $\Upsilon(\kappa;p)$ in D_a.

Table 4. Structure of D-class on T_κ in Terms of Height

κ/p	1	2	3	4	5	$\sum \Upsilon(\kappa;p) = \|D_a\|$
1	1					1
2	1	1				2
3	1	1	1			3
4	1	1	1	1		4
5	1	1	1	1	1	5

4. Conclusion

In this paper, the combinatorial function $\Upsilon(\kappa;p)$ was used to derive some triangles of numbers for each equivalence class of Green's relation on full transformation semigroups, and the orders of their classes were also obtained. The graphical representations in Figure 1, 2, and 3 provide visual insights into the structure of Green's equivalence classes within the full transformation semigroup T_κ. These graphs illustrate the distribution of elements across different heights, helping to analyze how transformations are classified based on their image sizes. For any transformation α in the L-class on T_κ, there exists a partition (κ,p) for all values of κ, as discussed in Theorem 2, while the R-class on T_κ satisfies Bell numbers, as shown in Table 2. Moreover, within a D-class, all H-classes have the same size, as stated in Theorem 3, and contain at least one idempotent. This demonstrates that every D-class in T_κ is regular. Green's relations offers a promising area of study, with significant potential for further research on partial transformations, partial one-to-one transformations, and their subsemigroups. Future investigations could explore their combinatorial structures.

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