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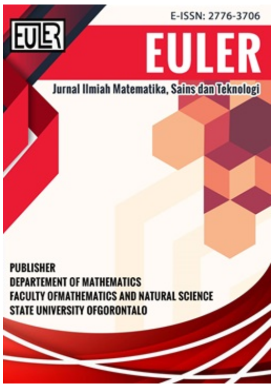

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Research Article

Numerical Solution of Fractional Order Differential Equations by Chebyshev Least Squares Approximation Method

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ABSTRACT. *In this paper, Fractional Differential Equations (FDE) is solved numerically using Least Squares Method (LSM). The Shifted Chebychev polynomials is used as the basis functions and the results is compared with the exact solutions. Some numerical examples are presented to illustrate the theoretical results and compared with the results obtained by other numerical methods. It was found that the results of the proposed approximate method converged rapidly to the exact solutions.*

1. Introduction

Most scientific and engineering problems are modeled and solved in differential or integro-differential equation forms for better and easy results. The literature on the applications and solution of differential and integro differential equations abounds and are still increasing by the day due to their importance in human life. Ahmed and Elzaki [1] said that ever since introduction of the concept of differential and or integro differential equations, they have played very important role in all facets of human life. The paper enumerated some areas of human life where these type of equations are applicable to include ice-shaping operation, heat transfer, neutron diffusion etc.

It is agreed among numerical analysts that fractional calculus is an extension of ordinary calculus with more than 300 years of history. Since then, lots of researchers have deployed different methods to solve fractional differential equations. Recently, Dabwan and Hasan [2] solved fractional order differential equations using modified Adomian decomposition method. The authors applied the method to singular and non-singular fractional order differential equations. In the opinion of the authors also, Adomian Decomposition Method is one of the most frequently used method for solving linear and non-linear ordinary, partial, fractional differential equations and also described the method as a powerful tool for the solution of fractional order differential equations linear and non-linear alike. Using a new method to solve fractional differential equations, Khalouta and Kadem [3] introduced Inverse Fractional Shehu Transform Method. The relatively new method was used to solve homogeneous and non-homogenous linear fractional differential equations with the derivatives described in Caputo and Riemann-Liouville sense. In the end, the author concluded that the method was a powerful and efficient technique for finding the exact solution of linear fractional differential equations. In the same vain, Bulut et al. [4] used another new form of transform called Sumudu Transform Method (STM) to solve fractional ordinary differential equations. The author noted that the method while compared to other methods turned out to be pragmatic in getting analytical solution of ordinary fractional differential equations fast adding that it can also be applied to initial-value and boundary-value problems.

Application of a semi-analytical numerical technique, Fractional Differential Transform Method (FDTM) was implemented by Ibis et al. [5] to find solution of fractional-algebric equations. The authors compared the results with Homotopy Analysis Method and the exact solutions and found that the proposed method is very effective and simple and that their results highly agreed. The authors added that the method is computational reliable, straightforward and accurate. Kehaili et al. [6] and Mohammed et al. [7] used Homotopy perturbation method to solve fractional differential equations. While Kehaili et al. [6] applied Homotopy Perturbation Transform method for solving partial and time-fractional differential equations with variable coefficients, Mohammed et al. [7] however used Homotopy Analysis Transform method to solve fractional integro-differential equations. The results obtained using the method converges to the exact solution rapidly and the algorithm was found to be suitable and very user friendly. A Least Squares Differential Quadrature Method for a class of nonlinear partial differential equations of fractional order was implemented by Bota et al. [8]. The method was ap-

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plied to nonlinear partial differential equations with fractional time derivatives and the result converged rapidly to the exact solution.

The introduction of Bernstein Least-Squares Technique (BLST) via Bernstein polynomials as basis functions was done by Oyedepo et al. [9]. The method was used to solved fractional Integro-Differential Equations (FIDEs) and the results show that the method is easy to implement and accurate when applied to FIDEs as it reduces the type of problem to the solution of a system of linear algebraic equations and then solved using MAPLE 18. The authors concluded that the calculations showed that (BLST) is a powerful and efficient technique in finding a very good solution for this type of equation. In order to further improve on the result of solutions of fractional differential and integro differential equations, Taiwo and Fesojaye [10] and Uwaheren et al. [11] introduced Perturbation. Perturbation method, which adds some terms to the method under consideration, produces a more accurate approximate solution. Other researchers that have applied Least squares method successfully includes [12, 13].

Work on the solution of linear and nonlinear multi-term fractional differential equations (MFDEs) was carried out by Uwaheren and Taiowo [14] using Chebyshev polynomial based on operational matrix method for initial and boundary value problems. Their works were described in Caputo sense which allows initial and boundary conditions to be included in the formulation of the problem. The research articles [15–20] investigate several numerical approaches to solving Lane-Emden type equations and fractional order differential equations, including neural networks, efficient algorithms, and decomposition methods. Other approaches can be found in [21–26]. The aim of this work is to develop and demonstrate a numerical method for solving Fractional Differential Equations (FDEs) using the Least Squares Method (LSM) with Shifted Chebyshev polynomials as basis functions, and to evaluate the accuracy and convergence of the method by comparing it with exact solutions and other numerical methods.

The rest of the paper is structured as follows: Section 2, deals with preliminaries, i.e. definition of terms, the results and discussion, consisting of the approximation model used and several examples, presented in Section 3, and conclusion of the study are presented in Section 4.

2. Preliminaries

2.1. Fractional Differential Equation

A differential equation is known as a fractional differential equation if it contains at least one fractional order derivatives, D^α of the unknown function $y(x)$. The general form of a fractional differential equation is given as

$$D^\alpha y(x) = f(x, y(x)), \tag{1}$$

subject to the conditions:

$$D^\alpha y(0) = \omega_k, \quad k = 0, 1, ..., n,$$

where D^α is the fractional order derivative in the Caputo sense and α is a non-integer value, $n = \lceil \alpha \rceil$, is called the ceiling α, $\alpha > 0$, the highest order of the equation.

So, a fractional differential equation is one whose order of derivative is a non-integer and the order is commonly denoted, α, where α is between any two integers. There are two major fractional operators; the Caputo's fractional differential operator and the Riemann-Liouville's fractional differential operator.

2.1.1. Caputo's Fractional Differential Operator

The Caputo's fractional derivative of order $\alpha > 0$ is written as:

$$D_*^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (x-t)^{n-\alpha-1} \frac{d^n}{dt^n} f(t) dt & ; n-1 < \alpha < n, \\ \dfrac{d^n}{dt^n} f(t) & ; \alpha = n. \end{cases} \tag{2}$$

2.1.2. Riemann-Liouville's fractional differential operator

The Caputo's fractional derivative of order $\alpha > 0$ is written as:

$$D^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \dfrac{d^n}{dt^n} \displaystyle\int_0^t (x-t)^{n-\alpha-1} f(t) dt & ; n-1 < \alpha < n \\ \dfrac{d^n}{dt^n} f(t) & ; \alpha = n. \end{cases} \tag{3}$$

The generalized factorial form (property) of non-integer order derivatives in Euler's Gamma function $f(x) = x^m$ is given as

$$D^\alpha x^m = \frac{d^\alpha}{dx^\alpha} x^m = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{n-m}. \tag{4}$$

Some other basic properties of fractional derivatives and integrals are:

1. $D^\alpha(k) = 0$, k is a constant,
2. $D^\alpha x^n = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} x^{n-\alpha} \quad x > 0, \alpha > 0,$
3. $D^\alpha f(x^n) = 0,$ if $n \in N_0$: n $< \lceil \alpha \rceil$,
4. $D^\alpha(k_1 f(t)) + D^\alpha(k_2 f(t)) = k_1 D^\alpha f(t) + k_2 D^\alpha f(t) = (K_1 + K_2) D^\alpha f(t)$, for k_1, k_2 are constants,
5. $J^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} x^{\beta+\alpha} \quad x > 0, \beta > -1, \alpha > 0$, where $\lceil \alpha \rceil$ denoted the smallest integer greater than α and $N_0 = (0, 1, 2 \ldots)$.
6. $J^\alpha D^\alpha f(x) = f(x)$.

2.2. Fractional Integro-differential Equations

A differential equation is called a fractional integro - differential equation when the unknown function $y(x)$ appears under the integral sign and the equation also contains a fractional derivative D^α. The general form of fractional integro-differential equation is:

$$D^\alpha y(x) = f(t) + \lambda \int_{l(x)}^{p(x)} K(x,t) y(t) dt, \tag{5}$$

subject to the conditions:

$$D^\alpha y^k(0) = \phi_k,$$

where $k(x,t)$ and $y(t)$ are given smooth functions. Eq. (6) is called Volterra fractional integro-differential equation but it becomes Fredholm fractional integro-differential equation if the upper integral limit is an integer.

2.3. Chebyshev Polynomial

The Chebyshev polynomials of degree n defined on the interval [-1,1] is;

$$T_n(x) = cos^{-1}(ncos(x)) \quad -1 \leq 0 \leq 1, \tag{6}$$

with a recurrence relation:

$$T_{n+1}(x) = 2xT_n(x) - T_{n+1}(x), \quad n = 2, 3 \ldots, \tag{7}$$

and it is said to be shifted Chebyshev polynomials of degree k on the closed interval [0,1] defined as:

$$T_k(x) = cos[kcos^{-1}(2x-1)], \qquad k \geq 0, \tag{8}$$

with a the recurrence relation is given by:

$$T_{k+1}(x) = 2(2x-1)T_k(x) - T_{k-1}(x), \quad k \geq 1,$$

and few terms of the polynomial are:

$$\begin{aligned} T_0^*(x) &= 1 \\ T_1^*(x) &= 2x - 1 \\ T_2^*(x) &= 8x^2 - 8x + 1 \\ T_3^*(x) &= 32x^3 - 48x^2 + 18x - 1 \\ T_4^*(x) &= 128x^4 - 256x^3 + 160x^2 - 32x + 1 \\ &\vdots \end{aligned} \tag{9}$$

The orthogonality condition of Chebyshev polynomials is given by

$$\int_{-1}^{1} \left(\frac{T_i(x)T_j(x)}{w(x)} \right) dx \quad = \quad \begin{cases} 0 & \text{for } i \neq j, \\ \frac{\pi}{2} & \text{for } i = j \neq 0, \\ \pi & \text{for } i = j = 0, \end{cases} \tag{10}$$

where $w(x)$ is known as the weight function,is given as $w(t) = \sqrt{1-x^2}$. Generally, the Chebyshev polynomial valid in the interval $[a, b]$ in defined as

$$T_n(x) = cos\left\{ncos^{-1}\left(\frac{2x-a-b}{b-a}\right)\right\}, \tag{11}$$

and the recurrence relation is

$$T_{n+1}(x) = \frac{2x-a-b}{b-a}T_n(x) - T_{n-1}(x). \tag{12}$$

3. Results and Discussion

3.1. Approximation Method

In this session, the step by step procedure of the application of the proposed method is presented. Consider the general class of fractional order differential equation

$$D^{\alpha}(y(t)) + \sum_{i=0}^{n} c_i y^{(i)}(t) = f(t), \tag{13}$$

subject to conditions:

$$y^k(0) = \phi_k; \qquad k = 1, 2 \ldots n. \tag{14}$$

To solve Eq. (13) and Eq. (14), we assume an approximate solution

$$\begin{aligned} y_N(t) &= \sum_{j=0}^{N} c_j y_j(t) \\ &= a_0 + a_1(2x-1) + a_2(8x^2 - 8x + 1) \\ &\quad + a_3(32x^3 - 48x^2 + 18x - 1) \\ &\quad + a_4(128x^4 - 256x^3 + 160x^2 - 32x + 1). \end{aligned} \tag{15}$$

Substituting Eq. (15) into Eq. (13), gives:

$$D^{\alpha}\sum_{j=0}^{N} c_j y_j(t) + \sum_{j=0}^{N} c_j y_j(t) \sum_{i=0}^{n} c_i y^{(i)}(t) = f(t), \tag{16}$$

or

$$D^{\alpha}\sum_{j=0}^{N} c_j y_j(t) + \sum_{j=0}^{N}\sum_{i=0}^{n} a_{i,j} y_j(t) c_i y^{(i)}(t) = f(t). \tag{17}$$

Thus, Eq. (17) is rewritten as:

$$R(c_0, c_1, c_2, ...c_N) = D^{\alpha}\sum_{j=0}^{N} c_j y_j(t) + \sum_{j=0}^{N}\sum_{i=0}^{n} a_{i,j} y_j(t) c_i y^{(i)}(t) - f(t) \tag{18}$$

where, $R(c_0, c_1, c_2, ...c_N)$ is the residual. We minimize Eq. (18) by writing

$$S(c_0, c_1, c_2, ...c_N) = \int_0^1 \left[D^{\alpha}\sum_{j=0}^{N} c_j y_j(t) + \sum_{j=0}^{N}\sum_{i=0}^{n} a_{i,j} y_j(t) c_i y^{(i)}(t) - f(t)\right]^2 dt \tag{19}$$

Applying the D^{α} to Eq. (19), we have

$$\begin{aligned} S(c_0, c_1, c_2, ...c_N) = \int_0^1 \Bigg[&\sum_{j=0}^{N} c_j \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} t^{j-\alpha} \\ &+ \sum_{j=0}^{N}\sum_{i=0}^{n} a_{i,j} y_j(t) c_i y^{(i)}(t) - f(t)\Bigg]^2 dt. \end{aligned} \tag{20}$$

Eq. (20) is minimized by differentiating it partially with respect to $c_0, c_1, c_2 ....c_N$ and equating to zero to give the $(n+1)$

Table 1. Error of results for Example 1

x	Exact	$\alpha = 0.25$	Error	$\alpha = 0.5$	Error	$\alpha = 0.75$	Error
0.0	0.00000000	-0.005634	5.6343e-03	0.000003	3.0646e-06	0.000934	9.3374e-04
0.1	0.01000000	0.030690	2.0690e-02	0.009975	2.4674e-05	0.009164	8.3575e-04
0.2	0.04000000	0.087390	4.7390e-02	0.039907	9.3224e-05	0.035096	4.9042e-03
0.3	0.09000000	0.162886	7.2886e-02	0.089800	1.9969e-04	0.078910	1.1090e-02
0.4	0.16000000	0.255849	9.5849e-02	0.159658	3.4151e-04	0.140776	1.9224e-02
0.5	0.25000000	0.365206	1.1521e-01	0.249484	5.1640e-04	0.220844	2.9156e-02
0.6	0.36000000	0.490137	1.3014e-01	0.359278	7.2243e-04	0.319253	4.0747e-02
0.7	0.49000000	0.630079	1.4008e-01	0.489042	9.5796e-04	0.436126	5.3874e-02
0.8	0.64000000	0.784721	1.4472e-01	0.638778	1.2217e-03	0.571572	6.8428e-02
0.9	0.81000000	0.954007	1.4401e-01	0.808487	1.5126e-03	0.725684	8.4316e-02
1.0	1.00000000	1.138136	1.3814e-01	0.998170	1.8300e-03	0.898541	1.0146e-01

system of equations:

$$
\begin{aligned}
\frac{\partial S}{\partial c_0} &= -2\int_0^1 \left[c_0 \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} t^{j-\alpha} + \cdots + c_N \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} t^{j-\alpha} \right. \\
&\qquad \left. + \sum_{j=0}^{N} \sum_{i=0}^{n} a_{i,j} y_j^{(i)}(t) c_i - f(t) \right] dt = 0, \\
\frac{\partial S}{\partial c_1} &= -2\int_0^1 \left[c_0 \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} t^{j-\alpha} + \cdots + c_N \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} t^{j-\alpha} \right. \\
&\qquad \left. + \sum_{j=0}^{N} \sum_{i=0}^{n} a_{i,j} y_j^{(i)}(t) c_i - f(t) \right] t\, dt = 0, \\
\frac{\partial S}{\partial c_2} &= -2\int_0^1 \left[c_0 \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} t^{j-\alpha} + \cdots + c_N \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} t^{j-\alpha} \right. \\
&\qquad \left. + \sum_{j=0}^{N} \sum_{i=0}^{n} a_{i,j} y_j^{(i)}(t) c_i - f(t) \right] t^2\, dt = 0, \\
&\ \vdots \\
\frac{\partial S}{\partial c_N} &= -2\int_0^1 \left[c_0 \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} t^{j-\alpha} + \cdots + c_N \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} t^{j-\alpha} \right. \\
&\qquad \left. + \sum_{j=0}^{N} \sum_{i=0}^{n} a_{i,j} y_j^{(i)}(t) c_i - f(t) \right] t^n\, dt = 0.
\end{aligned} \tag{21}
$$

The $(n+1)$ system of linear equations obtained are solved using a computer softwares, maple 18 to obtain the constant coefficients. The values of the constants are substituted into Eq. (15) to obtain the required approximate solution.

3.2. Numerical Examples

In this section, the demonstration of the methodology is presented with illustrations on some examples.

Example 1. Consider the fractional differential equation

$$D^{\alpha} y(x) + y(x) = x^2 + \frac{2}{\Gamma(2.5)} x^{2-\alpha}.$$

Subject to the conditions $y(0) = 0, 0 \leq x \leq 1$.

The exact solution is $y(x) = x^2$. The example is solved for $\alpha = 0.25$, D^{α} for $\alpha = 0.5$ and D^{α} for $\alpha = 0.75$ following the algorithm above. Solving the problem at $\alpha = 0.25$, we have an approximate solution

$$
\begin{aligned}
y_4(x) = &- 0.1314722268e^{-3}x^4 + 0.5608850767e^{-3}x^3 \\
&+ 0.9978003823x^2 - 0.629045e^{-4}x + 0.30646e^{-5}.
\end{aligned}
$$

For $\alpha = 0.5$, we obtained an approximate solution

$$
\begin{aligned}
y_4(x) = &- 0.19143918e^{-5}x^4 + 0.45562443e^{-5}x^3 \\
&+ 1.0000223x^2 + 0.1177e^{-5}x - 1.30410^{-7}.
\end{aligned}
$$

For $\alpha = 0.75$, we obtained an approximate solution and

$$
\begin{aligned}
y_4(x) = &- 0.6135873344e^{-2}x^4 + 0.3402463791e^{-1}x^3 \\
&+ 0.8752745178x^2 - 0.55563752e^{-2}x + 0.9337354e^{-3}.
\end{aligned}
$$

The results of these are shown in Table 1 and Figure 1.

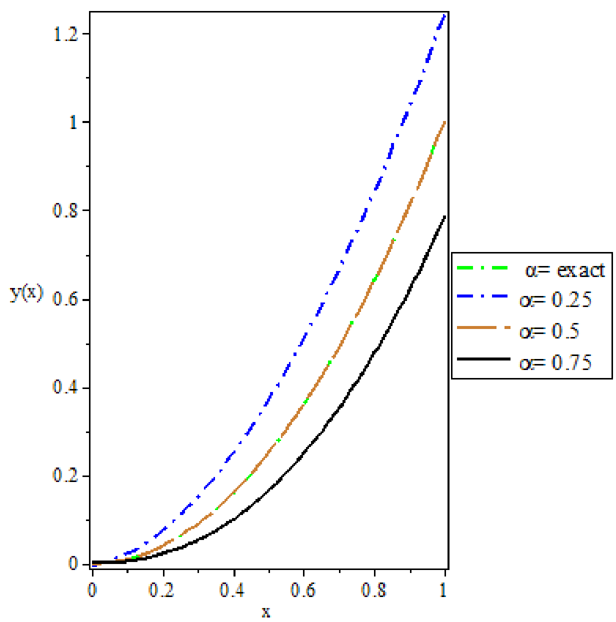

Figure 1. Graphical representation of Example 1

Table 2. Error of results for Example 2

x	Exact	$\alpha = 0.25$	Error	$\alpha = 0.5$	Error	$\alpha = 0.75$	Error
0.0	0.0000000	-0.00400510	4.0051e-03	-0.0000001	1.3040e-07	0.0000014	1.4479e-06
0.1	0.0100000	0.02207872	1.2079e-02	0.010000215	2.1499e-07	0.00468972	5.3103e-03
0.2	0.0400000	0.07447037	3.4470e-02	0.04000102	1.0317e-06	0.02123989	1.8760e-02
0.3	0.0900000	0.15103577	6.1036e-02	0.09000234	2.3402e-06	0.05262424	3.7376e-02
0.4	0.1600000	0.24992183	8.9922e-02	0.16000415	4.1563e-06	0.10022031	5.9780e-02
0.5	0.2500000	0.36955644	1.1956e-01	0.25000641	6.4912e-06	0.16525823	8.4742e-02
0.6	0.3600000	0.50864851	1.4865e-01	0.3600093	9.3517e-06	0.24882021	1.1118e-01
0.7	0.4900000	0.66618789	1.7619e-01	0.49001273	1.2740e-05	0.35184083	1.3816e-01
0.8	0.6400000	0.84144546	2.0145e-01	0.64001663	1.6653e-05	0.47510675	1.6489e-01
0.9	0.8100000	1.03397306	2.2397e-01	0.81002104	2.1084e-05	0.61925700	1.9074e-01
1.0	1.0000000	1.24360352	2.4360e-01	1.00002602	2.6021e-05	0.78478307	2.1522e-01

Example 2. Consider the fractional differential equation

$$D^{\alpha}y(x) = \frac{2}{\Gamma(2.5)}x^{1.5} \qquad 0 \leq \alpha \leq 1.$$

Subject to the conditions $y(0) = 0$.

The exact solution is $y(x) = x^2$. The example is solved for $\alpha = 0.25$, D^{α}, $\alpha = 0.5$ and $\alpha = 0.75$ following the algorithm above. Solving the problem at D^{α} for $\alpha = 0.25$, we have an approximate solution

$$y_4(x) = 0.1170779082x^4 - 0.4259256655x^3 + 1.434973200x^2 + 0.121483195x - 0.40051098e^{-2}.$$

For $\alpha = 0.5$, we obtained an approximate solution

$$y_4(x) = -0.1914391829e^{-5}x^4 + 0.4556244360e^{-5}x^3 + 1.000022333x^2 + 0.1177e^{-5}x - 1.30410^{-7}.$$

For $\alpha = 0.5$, we obtained an approximate solution and

$$y_4(x) = 0.1350122115e^{-5}x^4 - 0.3227413005x^3 + 0.5030498858x^2 - 1.288082683\,x - 1.246810^{-5}.$$

The results are shown in Table 2 and Figure 2.

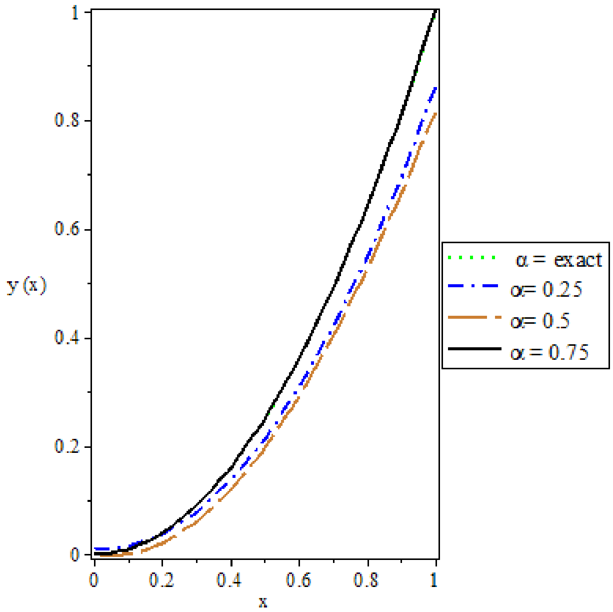

Figure 2. Graphical representation of Example 2

Example 3. Consider the fractional differential equation

$$y''(x) + D^{\frac{3}{2}}y(x) + y(x) = x + 1.$$

Subject to the conditions $y(0) = 0, 0 \leq x \leq 1$.

The exact solution is $y(x) = x + 1$. The example is solved for $\alpha = 0.25$, D^{α}, $\alpha = 0.5$ and $\alpha = 1.5$ following the algorithm above. Solving the problem at D^{α} for $\alpha = 0.25$, we have an approximate solution

$$y_4(x) = 0.80e^{-9}x^4 - 0.489e^{-7}x^3 + 1.02138019x^2 + 1.059609365x + 0.899e^{-6}.$$

For $\alpha = 0.5$, we obtained an approximate solution

$$y_4(x) = -0.1914391829e^{-5}x^4 + 0.4556244360e^{-5}x^3 + 1.000022333x^2 + 0.1177e^{-5}x - 1.30410^{-7}.$$

For $\alpha = 1.5$, we obtained an approximate solution and

$$y_4(x) = -0.6135873344e^{-2}x^4 + 0.3402463791e^{-1}x^3 + 0.8752745178x^2 - 0.55563752e^{-2}x + 0.9337354e^{-3}.$$

The results are shown in Table 3 and Figure 3.

3.3. Discussion of Results

Three problems were solved in this seminar using the proposed methods; Least Squares Method with the shifted Chebychev polynomials as basis functions. It was observed generally that, using the approximate solutions, the results were closed to the exact solutions at $\alpha = 0.5$ for problems 1 and 2 which is the mid point of the interval of consideration of the problems. It is observed also that as α is taken closer to the lower or upper limits

Table 3. Error of results for Example 3

x	Exact	$\alpha = 0.25$	Error	$\alpha = 0.5$	Error	$\alpha = 1.5$	Error
0.0	0.0000000	0.000000142	1.4210e-07	0.00000008	8.0000e-09	0.00000089	8.9900e-07
0.1	0.1100000	0.120038724	1.0039e-02	0.11296763	2.9676e-03	0.10998977	1.0222e-05
0.2	0.2400000	0.252438085	1.2438e-02	0.28007738	4.0077e-02	0.23997869	2.1334e-05
0.3	0.3900000	0.419282585	2.9283e-02	0.48011612	9.0116e-02	0.38996753	3.2467e-05
0.4	0.5600000	0.720155044	1.6016e-01	0.61435307	5.4353e-02	0.55995635	4.3650e-05
0.5	0.7500000	0.838482284	8.8482e-02	1.00019419	2.5019e-01	0.74994508	5.4912e-05
0.6	0.9600000	1.092483720	1.3248e-01	1.32023350	3.6023e-01	0.95993369	6.6283e-05
0.7	1.1900000	1.377151690	1.8715e-01	1.68027340	4.9027e-01	1.18992220	7.7792e-05
0.8	1.4400000	1.693261268	2.5326e-01	2.08031358	6.4031e-01	1.43991053	8.9468e-05
0.9	1.710000	2.041568298	3.3157e-01	2.52035420	8.1035e-01	1.70989865	1.0134e-04
1.0	2.0000000	2.422809420	4.2281e-01	3.00039530	1.0004e+00	1.99988650	1.1344e-04

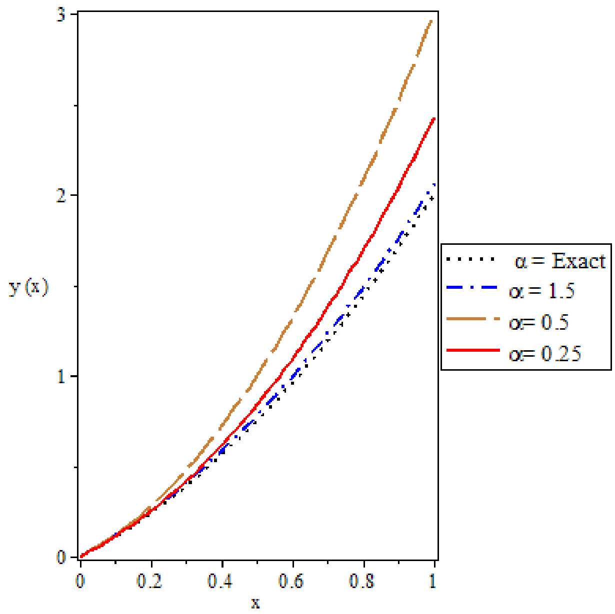

Figure 3. Graphical representation of Example 3

of the interval, the result refuse to converge rapidly to the exact solution. This shows that the actual value α used at the original point of modeling of the problem may be $\alpha = 0.5$. However, for problem 3, that $\alpha = 1.5$ the obtained approximate solutions is better than other α values confirming our earlier comment since the α value of the problem was clearly stated as $\alpha = 1.5$. In all, it is easy to say that the proposed method is suitable solving the fractional differential equations for which they were applied. The fact that Chebyshev polynomials were used as basis polynomial shows that the method is compatible with the Chebyshev polynomials or vice versa. The advantage of these method is the fact that it is capable of giving accurate approximate results comparable to the exact solution at appropriate α value.

4. Conclusion

In this study, three problems were solved using the suggested Least Squares Method (LSM) using shifted Chebyshev polynomials as basis functions. The results showed that approximation answers were generally near to exact solutions for situations in the middle of the studied interval. However, when α approached the interval's boundaries, the convergence of approximate solutions to the precise solution increased slower. Optimizing α is crucial for proper problem modeling. In a situation with clearly specified α, the resulting approximate solutions outperformed alternative values, demonstrating the method's robustness in such cases. In conclusion, the proposed method is effective for solving fractional differential equations. Using Chebyshev polynomials improves the method's compatibility and efficiency. Overall, it provides a dependable method for generating precise approximate solutions, making it an appropriate instrument for solving fractional differential equations.

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References

[1] S. Ahmed and T. Elzaki, "The solution of nonlinear Volterra integro-differential equations of second kind by combine Sumudu transforms and Adomian decomposition method," *International Journal of Advanced and Innovative Research*, vol. 2, no. 12, pp. 90–93, 2014.

[2] N. M. Dabwan and Y. Q. Hasan, "Solution of Fractional Differential Equations by Modified Adomian Decomposition Method," *Advances in Mathematics: Scientific Journal*, vol. 9. No. 7, pp. 4761–4767, 2020. doi: 10.37418/amsj.9.7.42.

[3] A. Khalouta and A. Kadem, "A New Method to Solve Fractional Differential Equations: Inverse Shehu Transform Method," *International Journal: Applications and Applied Mathematics*, vol. 14, no. 2, pp. 926–941, 2019.

[4] H. Bulut, H. M. Baskonus, and F. B. M. Belgacem, "The Analytical Solution of Some Fractional Ordinary Differential Equations by the Sumudu Transform Method," *Abstract and Applied Analysis*, vol. 2013, Article ID 203875, pp. 1-6, 2013. doi: 10.1155/2013/203875.

[5] B. Ibis, M. Bayram, and A. G. Agargun, "Applications of Frac-

tional Differential Transform Method to Fractional Differential-Algebraic Equations," *European Journal of Pure and Applied Mathematics*, vol. 4, no. 2, pp. 129–141, May 2011. [Online]. Available at: https://www.ejpam.com/index.php/ejpam/article/view/1172.

[6] A. Kehaili, A. Hakem, and A. Benali, "Homotopy Perturbation Transform Method for Solving the Partial and the Time-Fractional Differential Equations with Variable Coefficients," *Global Journal of Pure and Applied Sciences*, vol. 26, no. 1, pp. 35–55, Jun. 2020. DOI: 10.4314/gjpas.v26i1.6.

[7] M. S. Mohamed, M. R. Alharthi, and R. A. Alotabi, "Solving Fractional Integro Differential Equations by Homotopy Analysis Transform Method," *International Journal of Pure and Applied Mathematics*, vol. 106, no. 4, pp. 1037–1055, 2016. DOI: 10.12732/ijpam.v106i4.6.

[8] C. Bota, B. Caruntu, D. Tucu, M. Lapadat, and M. S. Paşca, "A Least Squares Differential Quadrature Method for a Class of Nonlinear Partial Differential Equations of Fractional Order," *Mathematics*, vol. 8, no. 8, p. 1336, 2020. DOI: 10.3390/math8081336.

[9] T. Oyedepo, O. B. Akinduko, O. E. Faniyi, and E. P. Okperhie, "Bernstein Least-Squares Technique for Solving Fractional Integro-Differential Equations," *Science World Journal*, vol. 14, no. 3, pp. 56–60, Sep. 2019.

[10] O. A. Taiwo and M. O. Fesojaye, "Perturbation Least-Squares Chebyshev Method for Solving Fractional Order Integro-Differential Equations," *Theoretical Mathematics and Applications*, vol. 5, no. 4, pp. 37–47, 2015.

[11] O. A. Uwaheren, A. F. Adebisi, and O. A. Taiwo, "Perturbed Collocation Method For Solving Singular Multi-order Fractional Differential Equations of Lane-Emden Type," *Journal of the Nigerian Society of Physical Sciences*, vol. 3, pp. 141–148, 2020. DOI: 10.46481/jnsps.2020.69.

[12] A. M. S. Mahdy and R. T. Shwayye, "Numerical Solution of Fractional Integro-Differential Equations by Least Squares Method and Shifted Laguerre Polynomials Pseudo-Spectral Method," *International Journal of Scientific & Engineering Research*, vol. 7, no. 4, pp. 1589–1596, Apr. 2016.

[13] S. N. Shehab, H. A. Ali, and H. M. Yaseen, "Least Squares Method for Solving Integral Equations with Multiple Time Lags," *Engineering and Technology Journal*, vol. 28, no. 10, pp. 1893–1899, 2010.

[14] O. A. Uwaheren and O. A. Taiwo, "A Collocation Technique Based on an Orthogonal Polynomial for Solving Multi-Fractional Order Integro Differential Equations," *Journal of the Mathematical Society of Nigeria (ABACUS)*, vol. 43, pp. 215–220, 2016.

[15] A. Verma and M. Kumar, "Numerical Solution of Lane–Emden Type Equations Using Multilayer Perceptron Neural Network Method," *International Journal of Applied and Computational Mathematics*, vol. 5, no. 5, p. 141, Oct. 2019, doi: 10.1007/s40819-019-0728-6.

[16] Y. Öztürk, "An Efficient Numerical Algorithm for Solving System of Lane–Emden Type Equations," *Nonlinear Engineering*, vol. 8, no. 1, pp. 429–437, 2019, doi: 10.1515/nleng-2018-0062.

[17] M. Weilbeer, "Efficient Numerical Methods for Fractional Differential Equations and Their Analytical Background," Ph.D. dissertation, Carl-Friedrich-Gauss Faculty of Mathematics and Computer Science, Technische Universität Braunschweig, 2005.

[18] P. Guo, "The Adomian Decomposition Method for a Type of Fractional Differential Equations," *Journal of Applied Mathematics and Physics*, vol. 7, no. 10, pp. 2459–2466, Oct. 2019, doi: 10.4236/jamp.2019.710166.

[19] W. Al-Hayani, L. Alzubaidy, and A. Entesar, "Solutions of Singular IVP's of Lane-Emden Type by Homotopy Analysis Method with Genetic Algorithm," *Applied Mathematics and Information Sciences*, vol. 11, no. 2, pp. 407–416, 2017, doi: 10.18576/amis/110208.

[20] M. M. Khader and K. M. Saad, "A numerical study using Chebyshev collocation method for a problem of biological invasion: Fractional Fisher equation," *International Journal of Biomathematics*, vol. 11, no. 8, p. 1850099, 2018, doi: 10.1142/S1793524518500997.

[21] A. Owolanke, O. D. Ogwumu, T. Y. Kyagya, C. O. Okorie, and G. I. Amakoromo, "A hybrid two-step method for direct solutions of general second order initial value problem," *Engineering Mathematics Letters*, vol. 2021, Article ID 1, Nov. 2020. DOI: 10.28919/eml/4877.

[22] A. O. Owolanke, O. Uwaheren, and F. O. Obarhua, "An eight order two-step Taylor series algorithm for the numerical solutions of initial value problems of second order ordinary differential equations," *Open Access Library Journal*, vol. 4, p. e3486, 2017. DOI: 10.4236/oalib.1103486.

[23] O. K. Ogunbamike and A. O. Owolanke, "Convergence of analytical solution of the initial-boundary value moving mass problem of beams resting on Winkler foundation," *Electronic Journal of Mathematical Analysis and Applications*, vol. 10, no. 1, pp. 129–136, 2022, doi: 10.21608/ejmaa.2022.311006.

[24] O. K. Ogunbamike, I. T. Awolere, and O. A. Owolanke, "Dynamic response of uniform cantilever beams on elastic foundation," *African Journal of Mathematics and Statistics Studies*, vol. 4, no. 1, pp. 47–62, 2021.

[25] T. Oyedepo, A. K. Bello, J. U. Abubakar, A. M. Ayinde, and T. F. Mohammed, "Numerical approximation of multi-order fractional differential equations by Galerkin method with Chebyshev polynomial basis," *Journal of Fractional Calculus and Applications*, vol. 15, no. 2, pp. 1–20, 2024, doi: 10.21608/jfca.2024.348330.

[26] S. Sadiq, "Spectral Methods for Fractional Differential Equations," Ph.D. thesis, *School of Natural Sciences, National University of Sciences and Technology*, 2024.