Meir Keeler’s Fixed-Point Theorem in Complex-Valued Modular Metric Space

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ABSTRACT. In this paper, we introduce the notion of Meir-Keeler contraction mapping, which is defined in complex-valued modular metric space. Some properties of sequences in this space, which are convergence, Cauchyness and completeness, are used to prove the fixed-point theorem under this mapping. Additionally, the Δ₂-type condition is also defined as the sufficient condition in order to have a unique fixed-point.

1. Introduction

A fixed-point is defined as a point within the domain of a function that is equal to the value of the function at that point. Stefan Banach introduced the concept of fixed-points and subsequently established a theorem concerning the existence and uniqueness of such fixed-points within certain metric spaces. Since then, this theory has evolved by defining new types of contractions through the generalization of the mapping itself [1–7], as well as the generalization of the space in which this mapping is defined [8–12].

One of the most interesting generalizations of contractions is Meir Keeler contraction in a complete metric space [13]. In 2013, Kiftiah [14] proposed the concept of fixed-points from several contraction mappings developed from metric spaces to modular spaces. One of these mappings is the Meir Keeler ρ contraction. Then, the existence and uniqueness of fixed-points under this mapping were proved. Following that, in 2018, Aksoya [15] additionally defined Meir Keeler type contraction mappings on modular metric space and successfully established its fixed-points theorem.

In 2013, Kiftiah [14] introduced the notion of Meir-Keeler contraction mapping, which is defined in complex-valued modular metric space. Our main goal is to investigate the existence and uniqueness of fixed-point of Meir Keeler type mapping in the context of complex-valued modular metric spaces.

2. Methods

The first step involves studying the concept of complex-valued modular metric spaces, as defined by Ozkan in [16], including the definitions, topology, convergent sequences, and fixed-points. Based on these concepts, the notion of Meir-Keeler ω-contraction mappings is constructed in complex-valued modular metric spaces as previously defined in metric spaces [13], modular spaces [14], and modular metric spaces [15]. Subsequently, the sufficient conditions that the Meir-Keeler ω-contraction mappings must satisfy to ensure the existence and uniqueness of their fixed-points are investigated. A fixed-point theorem is formulated from sufficient conditions for Meir-Keeler ω-contraction mappings in complex-valued modular metric spaces. Additionally, the proof of this theorem is presented.

3. Results and Discussion

Before investigating the main topic, let us first review some notations and definitions introduced by Azam [11], who studied the concepts of complex-valued metric spaces. These will serve as the foundation for our later discussion.

Definition 1. [11] Let C be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\preceq$ on C, satisfies: $z_1 \preceq z_2$ if and only if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$.

It implies that if $z_1 \preceq z_2$ then one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,
(ii) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(iii) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$,
(iv) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$.

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Homepage: http://ejurnal.ung.ac.id/index.php/euler/index / E-ISSN: 2776-3706 © 2024 by the Author(s).
If $z_1 \neq z_2$ and one of (i), (ii), or (iii) is satisfied, then we can write $z_1 \not\sim z_2$. Particularly, if only (iii) is satisfied, then we can write $z_1 \ll z_2$.

For every $z_1, z_2 \in \mathbb{C}$, the partial order on $\mathbb{C}$ has the following properties:

(i) $0 \not\preceq z_1 < z_2 \iff |z_1| < |z_2|$,
(ii) $z_2 \preceq z_2$ and $z_2 < z_3 \Rightarrow z_1 < z_3$,
(iii) $z \in \mathbb{C}, a, b \in \mathbb{R}, a \leq b \Rightarrow az \preceq bz$

Next, we recall some basic definitions and fundamental results on complex-valued modular metric space, which was proposed by Ozkan [16].

Let $X \neq \emptyset, \lambda > 0$ and a function $\omega : (0, \infty) \times X \times X \rightarrow \mathbb{C}$. In this article, for every $\lambda > 0$ and $x, y \in X$, then the function $\omega(\lambda, x, y)$ is denoted with $\omega_\lambda(x, y)$.

**Definition 2.** [16] Let $X \neq \emptyset$. A function $\omega : (0, \infty) \times X \times X \rightarrow \mathbb{C}$ is said to be complex valued modular metric space on $X$, if it satisfies:

- (M1) $\omega(\lambda, z_1, z_2) > 0$ and $\omega(\lambda, z_1, z_2) = 0 \iff z_1 = z_2$.
- (M2) $\omega(\lambda, z_1, z_2) = \omega(\lambda, z_2, z_1)$.
- (M3) $\omega(\lambda + \mu, z_1, z_2) \leq \omega(\lambda, z_1, z_3) + \omega(\mu, z_3, z_2)$, for all $\lambda, \mu > 0$ and $z_1, z_2, z_3 \in X$.

**Definition 3.** [16] Let $X \neq \emptyset$ and $\omega : (0, \infty) \times X \times X \rightarrow \mathbb{C}$ be a complex modular metric on $X$. For all $x_0 \in X$, the set

$$X_\omega = \left\{ x \in X \mid \lim_{\lambda \rightarrow \infty} \omega_\lambda(x, x_0) = 0 \right\}$$

is said to be modular metric space (around $x_0$).

**Definition 4.** Let $X_\omega$ be a complex valued modular metric space and a sequence $x_n$ in $X_\omega$.

(i) A sequence $x_n \subseteq X_\omega$ is said to be $\omega$-complex convergent to $x \in X_\omega$ if for every $\varepsilon \in \mathbb{C}$ with $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and some $\lambda > 0$, we have $\omega_\lambda(x_n, x) < \varepsilon$. Further, $x$ is called a $\omega$-limit of $x_n$, and we write $\lim_{n \rightarrow \infty} \omega_\lambda(x_n, x) = 0$.

(ii) A sequence $x_n \subseteq X_\omega$ is said to be $\omega$-complex Cauchy sequence, if for every $\varepsilon \in \mathbb{C}$ with $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for every $m, n \geq n_0$ and some $\lambda > 0$, we have $\omega_\lambda(x_n, x_m) < \varepsilon$.

(iii) Complex modular metric space $X_\omega$ is said to be $\omega$-complex complete if every $\omega$-complex Cauchy sequence in $X_\omega$ is $\omega$-complex convergent.

Furthermore, we give some basic properties of $\omega$-complex convergent.

**Lemma 1.** [17] Let $X_\omega$ be a complex valued modular metric space and a sequence $x_n$ in $X_\omega$. A sequence $x_n \subseteq X_\omega$ is $\omega$-complex convergent to $x \in X_\omega$ if and only if $\lim_{n \rightarrow \infty} |\omega_\lambda(x_n, x)| = 0$.

**Lemma 2.** [17] Let $X_\omega$ be a complex valued modular metric space and a sequence $x_n$ in $X_\omega$. A sequence $x_n \subseteq X_\omega$ is $\omega$-complex Cauchy sequence in $X_\omega$ if and only if $\lim_{m, n \rightarrow \infty} |\omega_\lambda(x_n, x_m)| = 0$.

**Lemma 3.** [16] Let $\omega, z \in \mathbb{C}$. If $\omega > 0, |z| < 1$ and $\omega \ll z\omega$ then $\omega = 0 \in \mathbb{C}$.

The following is the definition of $\Delta_2$-type condition in a complex-valued modular metric space by adopting the description of the $\Delta_2$-type condition in modular metric space case in Abdou [18].

**Definition 5.** Let $X_\omega$ be a complex valued modular metric space and a sequence $x_n$ in $X_\omega$.

(i) A function $\omega$ satisfies $\Delta_2$-condition if $\lim_{m \rightarrow \infty} \omega_\lambda(x_n, x) = 0$, for some $\lambda > 0$ implies $\lim_{m \rightarrow \infty} \omega_\lambda(x_n, x) = 0$, for all $\lambda > 0$.

(ii) A function $\omega$ satisfies $\Delta_2$-type condition if for any $\alpha > 0$ there exist $C > 0$ such that

$$\omega_{\alpha}(z_1, z_2) \leq C \cdot \omega_{\alpha}(z_1, z_2)$$

for all $\lambda > 0, z_1, z_2 \in X_\omega, z_1 \neq z_2$.

It is clear that if $\omega$ satisfies the $\Delta_2$-type condition then $\omega$ satisfies the $\Delta_2$-condition.

Inspired from the definitions of Meir-Keeler contractions in modular metric space, we define the following complex-valued modular space versions of such type of mapping.

**Definition 6.** Let $X_\omega$ be a complete complex valued modular metric space and $T : X_\omega \rightarrow X_\omega$ is a mapping. A mapping $T$ is said Meir Keeler $\omega$-complex contraction if and only if for every $\varepsilon \in \mathbb{C}$ with $\varepsilon > 0$, there exists $\delta \in \mathbb{C}$ with $\delta > 0$, such that for any

$$x, y \in X_\omega \land \lambda > 0 \land \varepsilon \leq \omega_\lambda(x, y) \land \varepsilon + \delta, \quad \text{we have } \omega_\lambda(Tx, Ty) < \varepsilon.$$ 

Since $T$ is a Meir Keeler $\omega$-complex contraction mapping, we can derive the following equivalence based on Definition 6.

**Definition 7.** Let $X_\omega$ be a complete complex valued modular metric space and $T : X_\omega \rightarrow X_\omega$ is a mapping. A mapping $T$ is said Meir Keeler $\omega$-complex contraction if and only if for every $\varepsilon \in \mathbb{C}$ with $\varepsilon > 0$ there exists $\delta \in \mathbb{C}$ with $\delta > 0$ such that

$$\omega_\lambda(Tx, Ty) < \omega_\lambda(x, y),$$

for any $x, y \in X_\omega$ and $\lambda > 0 \land \varepsilon \leq \omega_\lambda(x, y) \land \varepsilon + \delta$.

Before stating and proving our fixed-point result for the
Hence, sequence $R\varepsilon$ converges to its infimum, that is, there exists $\varepsilon_R \geq 0$ with $\varepsilon_R = \inf \{ |\omega_\lambda(T^n x_0, T^{n+1} x_0)| : n \in \{0, 1, 2, \cdots \} \}$ such that $\lim_{n \to \infty} |\omega_\lambda(T^n x_0, T^{n+1} x_0)| = \varepsilon_R$.

We will prove $\varepsilon_R = 0$. Since $T$ is a Meir Keeler $\omega$-complex contraction mapping, using Lemma 4, we obtain that there exists $\delta_R > 0$ such that $\delta_R < \varepsilon_R$. Furthermore, since $\lim_{n \to \infty} |\omega_\lambda(T^n x_0, T^{n+1} x_0)| = \varepsilon_R$, there then exists $N \in \mathbb{N}$ such that $\delta_R < \varepsilon_R$. This implies that for $\varepsilon_R = \inf \{ |\omega_\lambda(T^n x_0, T^{n+1} x_0)| : n \in \{0, 1, 2, \cdots \} \}$, then $\varepsilon_R = 0$. So, we conclude that $\lim_{n \to \infty} |\omega_\lambda(T^n x_0, T^{n+1} x_0)| = 0$. 

In the following, we present the Meir Keeler’s fixed-point theorem based on the mapping given in Definition 6.

### Theorem 1
Let $X_\omega$ be a complete complex-valued modular metric space. Assume $\omega$ satisfies the $\Delta_\delta$-type condition. If $T : X_\omega \to X_\omega$ is a Meir Keeler $\omega$-complex contraction mapping, then $T$ has a unique fixed-point on $X_\omega$.

### Proof.
Let $x_0 \in X_\omega$ be arbitrary. For any $n \in \{0, 1, 2, \cdots \}$, we define $T^0 x_0 : x_0; T^{n+1} x_0 = T(T^n x_0)$, and $x_n = T^n x_0$.

If $\omega_\lambda(x_n, x_{n+1}) = \omega_\lambda(x_n, T x_n) = 0$, for every $\lambda > 0$ and some $n \in \{0, 1, 2, \cdots \}$, then $T$ has a unique fixed-point, that is $x_n \in X_\omega$. If $\omega_\lambda(x_n, x_{n+1}) > 0$, we will prove that $T$ has a unique fixed-point for any $n \in \{0, 1, 2, \cdots \}$. Let $n \in \{0, 1, 2, \cdots \}$ be arbitrary. Using Definition 7, we have $\omega_\lambda(x_n, x_{n+1}) = \omega_\lambda(T^n x_0, T^{n+1} x_0)$.

Taking modulus on both sides, we obtain

$$0 < |\omega_\lambda(x_n, x_{n+1})| < |\omega_\lambda(x_{n-1}, x_n)| .$$

Thus, the sequence $\{ |\omega_\lambda(x_n, x_{n+1})| \}$ is decreasing on $\mathbb{R}$ and bounded by 0. From Lemma 5, we derive $\lim_{n \to \infty} |\omega_\lambda(x_n, x_{n+1})| = 0$. Using the property of a partial order, we obtain

$$\lim_{n \to \infty} \omega_\lambda(x_n, x_{n+1}) = 0 .$$

Further, we will prove that $x_n$ is $\omega$-complex Cauchy sequence. Let $\varepsilon \in \mathbb{C}(\varepsilon > 0)$, then there is $\delta \in \mathbb{C}(\delta > 0)$ such that for $x, y \in X_\omega$ with $\varepsilon < \omega_\lambda(x, y) < \varepsilon + \delta$ implies $\omega_\lambda(T x, T y) < \varepsilon$.

Since $\omega_\lambda(T x, T y) < \omega_\lambda(x, y)$, for any $\varepsilon \in \mathbb{C}$ with $\varepsilon > 0$ implies the above Definition 6 is still satisfied if we choose $\delta < \varepsilon$ such that when $\omega_\lambda(x, y) < \varepsilon$ implies $\omega_\lambda(T x, T y) < \varepsilon$. Since $\lim_{n \to \infty} \omega_\lambda(x_n, x_{n+1}) = 0$, then there exists $K \in \mathbb{N}$ such that $\omega_\lambda(x_{n-1}, x_n) < \delta$ for any $n > K$.

Let $m, n \in \mathbb{N}$ such that $m, n > K$. Without loss of generality, we assume $m > n$, then $m = n + p$, for some $p \in \mathbb{N}$. In what...
follows, we prove that \( \{x_n\} \) is \( \omega \)-complex Cauchy sequence. That is,
\[
\omega_\lambda (x_n, x_{n+p}) < \varepsilon.
\]

We will use mathematical induction. For \( p = 1 \). By Definition 7, we have
\[
\omega_\lambda (x_n, x_{n+1}) < \omega_\lambda (x_{n-1}, x_n) < \delta < \varepsilon.
\]
We assume that the statement holds for some fixed \( p \in \mathbb{N} \),
\[
\omega_\lambda (x_n, x_{n+p}) < \varepsilon.
\]

Now, we show that the statement also holds for \( p + 1 \). Since \( \omega \) satisfies the \( \Delta_2 \)-type condition, there exist \( C = 1 \) such that
1. \( \omega_\lambda (x_{n-1}, x_n) < \omega_\lambda (x_{n-1}, x_{n+1}) \), which implies
2. \( \omega_\lambda (x_{n-1}, x_n) < \delta < \varepsilon \).
Furthermore, we have
\[
\omega_\lambda (x_n, x_{n+p}) \leq \omega_\lambda (x_n, x_{n+p+1}) < \varepsilon.
\]

Now, we consider two cases.

1. If \( \omega_\lambda (x_{n-1}, x_{n+p}) \not\leq \varepsilon \), then using Definition 6 we get
2. If \( \omega_\lambda (x_{n-1}, x_{n+p}) \not\leq \varepsilon \), then using Definition 7 we get
\[
\omega_\lambda (x_n, x_{n+p}) < \varepsilon.
\]

Hence, we conclude that \( \omega_\lambda (x_n, x_{n+p+1}) \not\leq \varepsilon \). So, \( x_n \) is \( \omega \)-complex Cauchy sequence on \( X_\omega \). By completeness of \( X_\omega \), there exist \( u \in X_\omega \) such that sequence \( x_n \) converges to \( u \). Hence, \( \lim_{n \to \infty} \omega_\lambda (x_n, u) = 0 \).

Next, we show that \( u \) is a fixed-point of \( T \). Since \( T \) is a Meir-Keeler \( \omega \)-complex contraction mapping, we obtain
\[
\omega_\lambda (u, Tu) \leq \omega_{\lambda/2} (u, T^{n+1}u) + \omega_{\lambda/2} (T^{n+1}u, Tu) + \varepsilon.
\]

Taking modulus on both sides, we get
\[
|\omega_\lambda (u, Tu)| \leq |\omega_{\lambda/2} (u, T^{n+1}u)| + |\omega_{\lambda/2} (T^{n+1}u, u)|.
\]

Since \( \lim_{n \to \infty} \omega_\lambda (x_n, u) = 0 \), Lemma 1 implies \( \lim_{n \to \infty} \omega_\lambda (x_n, u) = 0 \). Since \( \omega \) satisfies the \( \Delta_2 \)-type condition, we have
\[
0 \leq \lim_{n \to \infty} \omega_\lambda (u, Tu) \leq \lim_{n \to \infty} \omega_{\lambda/2} (u, T^{n+1}u) + \lim_{n \to \infty} \omega_{\lambda/2} (T^{n+1}u, u).
\]

Hence, \( \lim_{n \to \infty} \omega_\lambda (u, Tu) = 0 \), which implies \( \lim_{n \to \infty} \omega_\lambda (u, Tu) = 0 \). Therefore, \( Tu = u \). So, \( u \) is a fixed-point of \( T \).

Finally, we show the uniqueness of the fixed-point \( u \) of the mapping \( T \). We assume there exists \( u, v \in X_\omega \) such that \( Tu = u \) and \( Tv = v \). We deduce
\[
\omega_\lambda (u, v) = \omega_\lambda (Tu, Tv) < \omega_\lambda (u, v),
\]
as \( \omega_\lambda (u, v) \in \mathbb{C} \), this leads to a contradiction. Then, \( u \) is a unique fixed-point of \( T \). This completes the proof.

4. Conclusion

Based on the discussion, we conclude that the fixed-point theorem for Meir-Keeler contraction mapping can be extended to complex-valued modular metric spaces by adding sufficient conditions for such a contraction mapping to have a unique fixed-point. To ensure the existence of a fixed-point for a Meir-Keeler contraction mapping in this space, then \( \omega \) must satisfy the \( \Delta_2 \)-type condition.

Author Contributions. Mariatul Kiftiah: conceptualization, investigation, methodology, and writing - original draft. Yudhi: investigation and validation. All authors discussed the results and contributed to the final manuscript.

Acknowledgement. The authors express their gratitude to the editor and reviewers for their meticulous reading, insightful critiques, and practical recommendations, all of which have greatly enhanced the quality of this work.

Funding. This research received no external funding.

Conflict of interest. The authors declare that there are no conflicts of interest related to this article.

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