

Fear effect in discrete prey-predator model incorporating square root functional response

P. K. Santra

To cite this article:

P. K. Santra, "Fear effect in discrete prey-predator model incorporating square root functional response", *Jambura J. Biomath*, vol. 2, no. 2, pp. 51–57, 2021

DOI: <https://doi.org/10.34312/jjbm.v2i2.10444>

© 2021 Author(s).

Articles You may be interested in

Bifurkasi Hopf pada model prey-predator-super predator dengan fungsi respon yang berbeda

D. Savitri and H. S. Panigoro

<http://dx.doi.org/10.34312/jjbm.v1i2.8399>

Analisis kestabilan model predator-prey dengan infeksi penyakit pada prey dan pemanenan proporsional pada predator

S. Maisaroh, Resmawan, and E. Rahmi

<http://dx.doi.org/10.34312/jjbm.v1i1.5948>

Bifurkasi Hopf pada model Lotka-Volterra orde-fraksional dengan Efek Allee aditif pada predator

H. S. Panigoro and D. Savitri

<http://dx.doi.org/10.34312/jjbm.v1i1.6908>

Discrete-time prey-predator model with θ -logistic growth for prey incorporating square root functional response

P. K. Santra

<http://dx.doi.org/10.34312/jjbm.v1i2.7660>

A stage-structure Rosenzweig-MacArthur model with effect of prey refuge

L. K. Beay and M. Saija

<http://dx.doi.org/10.34312/jjbm.v1i1.6891>

Fear effect in discrete prey-predator model incorporating square root functional response

P.K. Santra

Abada Nsup School, Abada, Howrah-711313, India

Corresponding author. Email: prasunsantra5@gmail.com

Abstract

In this work, an interaction between prey and its predator involving the effect of fear in presence of the predator and the square root functional response is investigated. Fixed points and their stability condition are calculated. The conditions for the occurrence of some phenomena namely Neimark-Sacker, Flip, and Fold bifurcations are given. Base on some hypothetical data, the numerical simulations consist of phase portraits and bifurcation diagrams are demonstrated to picturise the dynamical behavior. It is also shown numerically that rich dynamics are obtained by the discrete model as the effect of fear.

Keywords: Discrete prey-predator; Stability; Neimark-Sacker bifurcation; Flip bifurcation; Fold bifurcation; Fear effect

1. Introduction

The prey-predator model, still now an exciting topic in mathematical biology. Most of this ecological problem which studies the interaction between a prey and its predator is modeled by deterministic approach using first-order differential equation [1–5], fractional-order differential equation [6–8], or with discrete-time equation [9–13]. Particularly, discrete models are an essential tool for mathematical biology problems. The discrete-time population models are based on a phenomenon in which time is not considered a continuous function. These models focus on such biological situations in which it is natural to view an event at discrete time intervals. The discrete-time population model is applicable for non-overlapping generation models. Such models appear to be more realistic than continuous ones when the population size is small.

Din [9] discussed chaos control in a discrete-time prey-predator system. Zhao and Du [10] investigated a discrete-time prey-predator model with an Allee effect. Santra and Mahapatra [11] studied the dynamics of a discrete-time prey-predator model under imprecise biological parameters. Santra et al. [12] investigated bifurcation and chaos of a discrete predator-prey model with Crowley-Martin functional response. For some more dynamical investigations related to different versions of prey-predator models, we refer to Singh and Deolia [14], Khan and Khalique [15], and references therein.

If we look further in nature, the predation process depends on which organisms interacted. For example, a bilinear predation process that appears in most marine ecosystems, a saturated predation process in the forest ecosystem, and a ratio-dependent predation process that assumes both prey and predator densities affect the predator's ability in predation. This mechanism is called the predation functional response which corresponds to the prey and predator natural behaviors. One popular predation mechanism is square root functional response which states the prey has herd behavior so that the predator is difficult to hunt when the population density of prey is high [3, 16, 17]. Although the prey has herd behavior, naturally they have fear of the presence of predator. The effect of fear has a direct impact on prey reproduction [18–21]. Based on those descriptions, we study the dynamical behaviors of a discrete-time prey-predator system involving both fear effect and square root functional response.

We arrange this paper as follows. In Section 2, the discrete-time model is formulated using Euler's scheme. The existence of fixed points and their local stability are given in Section 3. In Section 4, the existence condition of Neimark-Sacker, Flip, and Fold bifurcations are proposed. To support the theoretical findings, we present some numerical simulations in Section 5. Finally, this paper ends with a conclusion in Section 6.

2. Model formulation

These works study the impact of fear to the dynamics of a modified Lotka-Volterra model with square root functional response [19]. The following system of the equation governs the prey-predator dynamics.

$$\begin{aligned}\frac{dx}{dt} &= rx \left(1 - \frac{x}{k}\right) \frac{1}{1 + \phi y} - by\sqrt{x}, \\ \frac{dy}{dt} &= cy\sqrt{x} - dy,\end{aligned}\tag{1}$$

where $x(t)$ and $y(t)$ represent the density of prey and predator populations respectively, with initial condition $x(0) \geq 0$, and $y(0) \geq 0$. The parameters r , k , ϕ , b , c , and d respectively denote the intrinsic per capita growth rate of prey, the environmental carrying capacity of prey, the fear effect due to predation, the maximal per capita consumption rate of predators, the efficiency with which predators convert consumed prey into new predators, and the per capita death rate of predators. Now, by applying the forward Euler's scheme to (1), we achieve the discrete model as follows.

$$\begin{aligned}x &\mapsto x + h \left[rx \left(1 - \frac{x}{k}\right) \frac{1}{1 + \phi y} - by\sqrt{x} \right] \\ y &\mapsto y + h [cy\sqrt{x} - dy]\end{aligned}\tag{2}$$

where h is the step size and $\{r, k, \phi, b, c, d\} \in \mathbb{R}_+$. The dynamics are investigated in the region $\Omega = \{(x, y) : x \geq 0, y \geq 0\}$ for the biological reason.

3. General stability analysis

3.1. Fixed points

The following equations are solved to determine fixed points of the system (2).

$$\begin{aligned}x &= x + h \left[rx \left(1 - \frac{x}{k}\right) \frac{1}{1 + \phi y} - by\sqrt{x} \right] \\ y &= y + h [cy\sqrt{x} - dy]\end{aligned}$$

Therefore, three types biological fixed points are obtained namely the origin $P_0 = (0, 0)$, the extinction of predator point $P_1 = (k, 0)$, and the co-existence point $P_2 = (x_2, y_2)$, where $x_2 = \left(\frac{d}{c}\right)^2$ and y_2 is a positive solution of $y^2 + \frac{y}{\phi} - \frac{rd}{\phi bc} \left(1 - \frac{d^2}{kc^2}\right) = 0$.

3.2. Local stability analysis

In this section, we discuss the local stability of fixed points. By linearization around (x, y) , the Jacobian matrix J for the system (2) is given by

$$J = \begin{bmatrix} 1 + h \left[r \left(1 - \frac{2x}{k}\right) \frac{1}{1 + \phi y} - \frac{by}{2\sqrt{x}} \right] & h \left[-rx \left(1 - \frac{x}{k}\right) \frac{\phi}{(1 + \phi y)^2} - b\sqrt{x} \right] \\ \frac{hcy}{2\sqrt{x}} & 1 + h [c\sqrt{x} - d] \end{bmatrix}.\tag{3}$$

Therefore, from matrix J , we acquire the characteristic equation $\lambda^2 - T\lambda + D = 0$, where

$$\begin{aligned}T &= 2 + h \left[r \left(1 - \frac{2x}{k}\right) \frac{1}{1 + \phi y} - \frac{by}{2\sqrt{x}} + c\sqrt{x} - d \right] \\ D &= \left[1 + h \left[r \left(1 - \frac{2x}{k}\right) \frac{1}{1 + \phi y} - \frac{by}{2\sqrt{x}} \right] \right] [1 + h [c\sqrt{x} - d]] \\ &\quad - \left[h \left[-rx \left(1 - \frac{x}{k}\right) \frac{\phi}{(1 + \phi y)^2} - b\sqrt{x} \right] \right] \left[\frac{hcy}{2\sqrt{x}} \right] \\ &= 1 + h [c\sqrt{x} - d] + h \left[r \left(1 - \frac{2x}{k}\right) \frac{1}{1 + \phi y} - \frac{by}{2\sqrt{x}} \right] [1 + h [c\sqrt{x} - d]]\end{aligned}$$

$$- \left[h \left[-rx \left(1 - \frac{x}{k} \right) \frac{\phi}{(1 + \phi y)^2} - b\sqrt{x} \right] \right] \left[\frac{hcy}{2\sqrt{x}} \right]$$

Hence, the following statements hold.

- (i) If $|D| < 1$ then system (2) is a dissipative dynamical system.
- (ii) $|D| = 1$ if and only if system (2) is a conservative dynamical system.
- (iii) system (2) is an undissipated dynamical system otherwise.

Now, the following theorems are given to describe the dynamical behavior around each fixed points.

Theorem 1. The fixed point $P_1 = (k, 0)$ is

- (i) Sink if $|1 - rh| < 1$, and $\left| 1 + h [c\sqrt{k} - d] \right| < 1$,
- (ii) Source if $|1 - rh| > 1$, and $\left| 1 + h [c\sqrt{k} - d] \right| > 1$,
- (iii) Saddle if $|1 - rh| > 1$, and $\left| 1 + h [c\sqrt{k} - d] \right| < 1$; or $|1 - rh| < 1$, and $\left| 1 + h [c\sqrt{k} - d] \right| > 1$,
- (iv) Non-hyperbolic if $|1 - rh| = 1$ or $\left| 1 + h [c\sqrt{k} - d] \right| = 1$.

proof. By substituting $P_1 = (k, 0)$ to (3), we obtain

$$J = \begin{bmatrix} 1 - rh & -bh\sqrt{k} \\ 0 & 1 + h [c\sqrt{k} - d] \end{bmatrix},$$

which gives a pair of eigenvalues $\lambda_1 = 1 - rh$ and $\lambda_2 = 1 + h [c\sqrt{k} - d]$. Obeying Lemma 1 in [13], all statements are proven. ■

Theorem 2. If $1 - T + D > 0$, then interior fixed point $P_2 (x_2, y_2)$ is: (i) Sink if $1 + T + D > 0$ and $D < 1$, (ii) Source if $1 + T + D > 0$ and $D > 1$, (iii) Saddle if $1 + T + D < 0$, (iv) Non-hyperbolic if $1 + T + D = 0$ and $T \neq 0, 2$, or $T^2 - 4D < 0$ and $D = 1$.

proof. From the Jacobian matrix at the interior fixed point $P_2 (x_2, y_2)$, we get

$$\begin{aligned} 1 - T + D &= -1 - h \left[r \left(1 - \frac{2x_2}{k} \right) \frac{1}{1 + \phi y_2} - \frac{by_2}{2\sqrt{x_2}} + c\sqrt{x_2} - d \right] \\ &+ \left[1 + h \left[r \left(1 - \frac{2x_2}{k} \right) \frac{1}{1 + \phi y_2} - \frac{by_2}{2\sqrt{x_2}} \right] \right] [1 + h [c\sqrt{x_2} - d]] \\ &- \left[h \left[-rx_2 \left(1 - \frac{x_2}{k} \right) \frac{\phi}{(1 + \phi y_2)^2} - b\sqrt{x_2} \right] \right] \left[\frac{hcy_2}{2\sqrt{x_2}} \right] \end{aligned}$$

$$\begin{aligned} 1 + T + D &= 3 + h \left[r \left(1 - \frac{2x_2}{k} \right) \frac{1}{1 + \phi y_2} - \frac{by_2}{2\sqrt{x_2}} + c\sqrt{x_2} - d \right] \\ &+ \left[1 + h \left[r \left(1 - \frac{2x_2}{k} \right) \frac{1}{1 + \phi y_2} - \frac{by_2}{2\sqrt{x_2}} \right] \right] [1 + h [c\sqrt{x_2} - d]] \\ &- \left[h \left[-rx_2 \left(1 - \frac{x_2}{k} \right) \frac{\phi}{(1 + \phi y_2)^2} - b\sqrt{x_2} \right] \right] \left[\frac{hcy_2}{2\sqrt{x_2}} \right] \end{aligned}$$

By using Lemma 1 and 2 in [13] and obeying Juri condition [22], the dynamics given by Theorem 2 are completely proven. ■

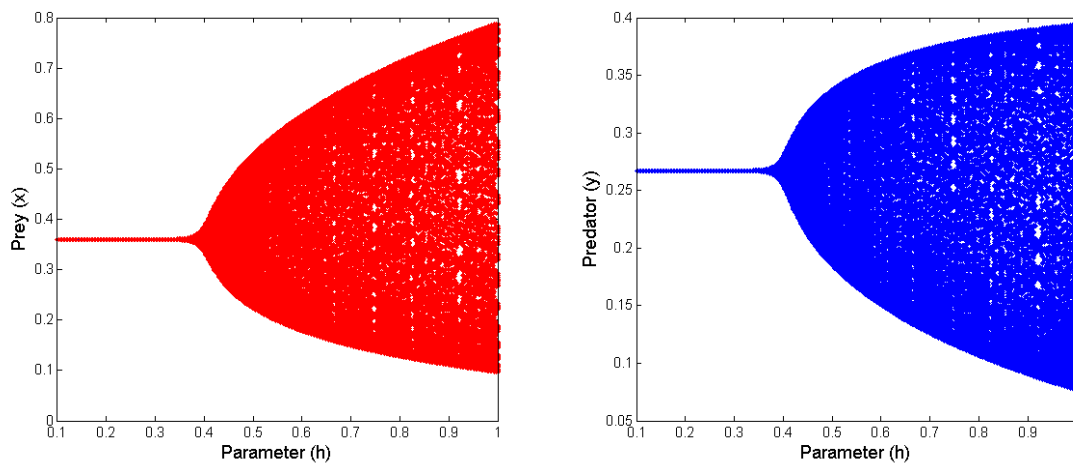


Figure 1. The bifurcation diagram with respect to the step size h

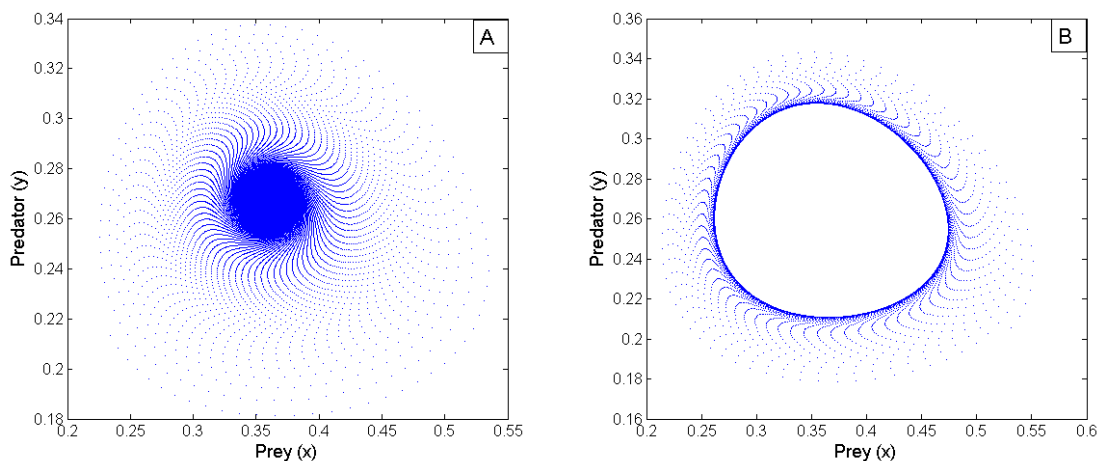


Figure 2. Phase portraits of the system for different values of step size h

4. Bifurcation Analysis

Bifurcation is a non-linear phenomenon that exhibits the change of dynamical behavior when one or more parameters are varied. In this section, we propose some one-parameter bifurcations namely Neimark-Sacker bifurcation, flip, and fold bifurcations. Neimark-Sacker bifurcation indicates the occurrence of closed invariant curves that isolates a fixed point after its stability change sign. Another bifurcation, flip bifurcation (also known as period-doubling bifurcation), occurs when the system switches to a new limit-cycle twice the period of the existing one. Fold bifurcation, in which two fixed points collide and disappear into the system. The sufficient conditions for the occurrence of those bifurcations are given as follows.

- (i) Condition for the occurrence of Neimark Sacker bifurcation [23] at an interior fixed point $P_2(x_2, y_2)$ is $D = 1$. i.e.

$$\begin{aligned}
 h [c\sqrt{x_2} - d] + h \left[r \left(1 - \frac{2x_2}{k} \right) \frac{1}{1 + \phi y_2} - \frac{b y_2}{2\sqrt{x_2}} \right] [1 + h [c\sqrt{x_2} - d]] \\
 = \left[h \left[-r x_2 \left(1 - \frac{x_2}{k} \right) \frac{\phi}{(1 + \phi y_2)^2} - b\sqrt{x_2} \right] \right] \left[\frac{h c y_2}{2\sqrt{x_2}} \right]
 \end{aligned}$$

- (ii) Condition for the occurrence of Flip bifurcation [23] at an interior fixed point $P_2(x_2, y_2)$ is $1 + T + D = 0$. i.e.

$$3 + h \left[r \left(1 - \frac{2x_2}{k} \right) \frac{1}{1 + \phi y_2} - \frac{b y_2}{2\sqrt{x_2}} + c\sqrt{x_2} - d \right]$$

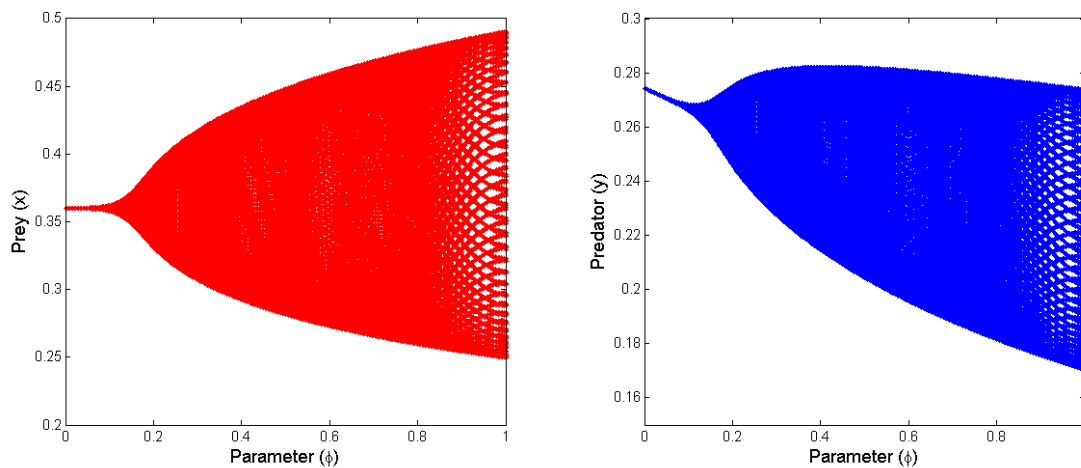


Figure 3. The bifurcation diagram with respect to the fear effect ϕ

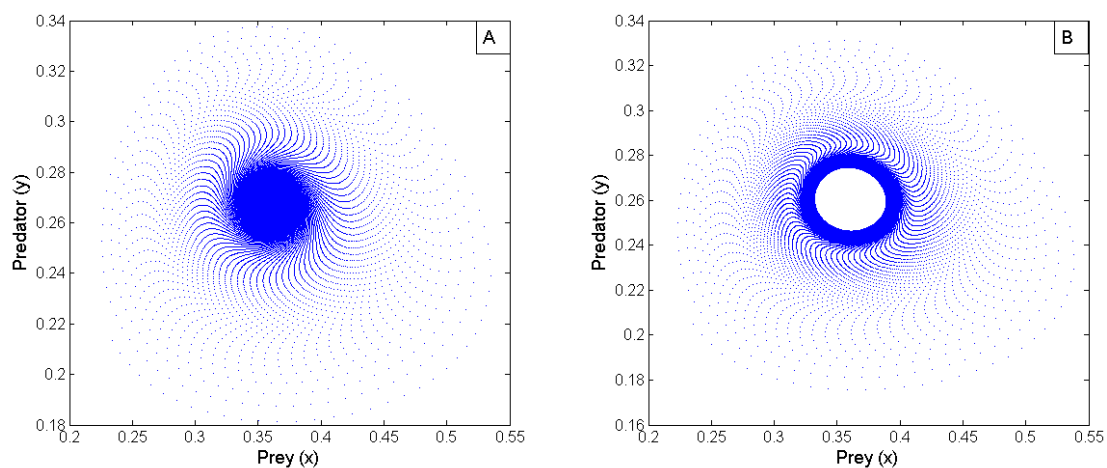


Figure 4. Phase portraits of the system for different values of fear effect ϕ

$$\begin{aligned}
 &+ \left[1 + h \left[r \left(1 - \frac{2x_2}{k} \right) \frac{1}{1 + \phi y_2} - \frac{by_2}{2\sqrt{x_2}} \right] \right] [1 + h [c\sqrt{x_2} - d]] \\
 &= \left[h \left[-rx_2 \left(1 - \frac{x_2}{k} \right) \frac{\phi}{(1 + \phi y_2)^2} - b\sqrt{x_2} \right] \right] \left[\frac{hcy_2}{2\sqrt{x_2}} \right]
 \end{aligned}$$

(iii) Condition for the occurrence of Fold bifurcation [23] at an interior fixed point $P_2(x_2, y_2)$ is $1 - T + D = 0$. i.e.

$$\begin{aligned}
 &-1 - h \left[r \left(1 - \frac{2x_2}{k} \right) \frac{1}{1 + \phi y_2} - \frac{by_2}{2\sqrt{x_2}} + c\sqrt{x_2} - d \right] \\
 &+ \left[1 + h \left[r \left(1 - \frac{2x_2}{k} \right) \frac{1}{1 + \phi y_2} - \frac{by_2}{2\sqrt{x_2}} \right] \right] [1 + h [c\sqrt{x_2} - d]] \\
 &= \left[h \left[-rx_2 \left(1 - \frac{x_2}{k} \right) \frac{\phi}{(1 + \phi y_2)^2} - b\sqrt{x_2} \right] \right] \left[\frac{hcy_2}{2\sqrt{x_2}} \right]
 \end{aligned}$$

5. Numerical simulations

In order to support our analytical results, we perform some numerical simulations consist of bifurcation diagrams and their appropriate phase portraits. We set the hypothetical parameter values as follows

$$r = 0.5, k = 1.0, \phi = 0.1, b = 0.7, c = 0.5, d = 0.3.$$

By varying the step size h in interval $[0.1, 1]$, we portray the bifurcation diagram in Figure 1. For $0.1 \leq h < 0.45$, the interior point is a sink. This behavior is changed when h crosses 0.415 and nearby solutions converge to a stable limit-cycle till $h = 1$. This phenomenon confirms the occurrence of Neimark Sacker bifurcation driven by step size h given by the previous analytical study. We choose $h = 0.4$ and 0.45 to ensure the dynamical behaviors for each condition using phase portraits, see Figure 2.

Now, the influence of the fear effect is studied numerically. By keeping the parameter same as before, using step size $h = 0.4$, and varying ϕ in interval $(0, 1]$, we obtain the bifurcation diagram as in Figure 3. The interior point which is stable for $0 < \phi < 0.155$ loses its stability via Neimark-Sacker bifurcation when ϕ passes through 0.155. Again, we give two phase portraits to show the dynamics for each case i.e. when $\phi = 0.1$ which gives a stable interior point as in Figure 4A and when $\phi = 0.2$ where the stability of interior point is gone and the solution converge to the limit-cycle as in Figure 4B.

From numerical results, we conclude that the step size h and the fear effect due to predator ϕ are the parameters for Neimark-Sacker bifurcations. Those parameters play crucial roles in controlling the dynamical behaviors of the system. The biological meanings of these numerical phenomena show us that there exists a condition when the interior point loses its stability, the existence of both populations still maintained by changing their densities periodically.

6. Conclusion

The impact of the step size and the effect of fear on the dynamical behaviors of the prey-predator interaction have been investigated both analytically and numerically. The biological conditions of the local dynamics for each fixed point have been given. The sufficient condition for the occurrence of Neimark-Sacker, flip, and fold bifurcation also have been identified analytically. Some numerical simulations exhibit that the step size and the effect of fear have an impact on the dynamics of the system especially in the interior of the system. The impact of the Neimark-Sacker bifurcation shows that the density of both populations changes periodically when the interior point loses its stability.

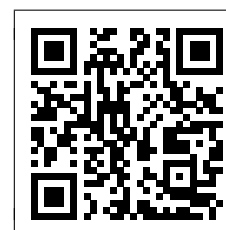
References

- [1] L. K. Beay and M. Saija, "Dynamics of a stage-structure Rosenzweig-MacArthur model with linear harvesting in prey and cannibalism in predator," *Jambura Journal of Biomathematics (JJBM)*, vol. 2, no. 1, pp. 42–50, 2021.
- [2] P. Majumdar, S. Debnath, S. Sarkar, and U. Ghosh, "The Complex Dynamical Behavior of a Prey-Predator Model with Holling Type-III Functional Response and Non-Linear Predator Harvesting," *International Journal of Modelling and Simulation*, vol. 00, no. 00, pp. 1–18, 2021.
- [3] M. G. Mortuja, M. K. Chaube, and S. Kumar, "Dynamic analysis of a predator-prey system with nonlinear prey harvesting and square root functional response," *Chaos, Solitons and Fractals*, vol. 148, p. 111071, 2021.
- [4] L. K. Beay and M. Saija, "A Stage-Structure Rosenzweig-MacArthur Model with Effect of Prey Refuge," *Jambura Journal of Biomathematics (JJBM)*, vol. 1, no. 1, pp. 1–7, 2020.
- [5] H. S. Panigoro, E. Rahmi, N. Achmad, and S. L. Mahmud, "The Influence of Additive Allee Effect and Periodic Harvesting to the Dynamics of Leslie-Gower Predator-Prey Model," *Jambura Journal of Mathematics*, vol. 2, no. 2, pp. 87–96, 2020.
- [6] H. S. Panigoro, A. Suryanto, W. M. Kusumawinahyu, and I. Darti, "A Rosenzweig-MacArthur model with continuous threshold harvesting in predator involving fractional derivatives with power law and mittag-leffler kernel," *Axioms*, vol. 9, no. 4, p. 122, 2020.
- [7] L. Dai, J. Wang, Y. Ni, and B. Xu, "Dynamical analysis of a new fractional-order predator-prey system with Holling type-III functional," *Advances in Difference Equations*, vol. 2021, no. 1, 2021.
- [8] J. Alidousti and E. Ghafari, "Dynamic behavior of a fractional order prey-predator model with group defense," *Chaos, Solitons and Fractals*, vol. 134, p. 109688, 2020.
- [9] Q. Din, "Complexity and chaos control in a discrete-time prey-predator model," *Communications in Nonlinear Science and Numerical Simulation*, vol. 49, pp. 113–134, 2017.
- [10] M. Zhao and Y. Du, "Stability of a discrete-time predator-prey system with Allee effect," *Nonlinear Analysis and Differential Equations*, vol. 4, no. 5, pp. 225–233, 2016.
- [11] P. K. Santra and G. S. Mahapatra, "Dynamical study of discrete-time prey-predator model with constant prey refuge under imprecise biological parameters," *Journal of Biological Systems*, vol. 28, no. 03, pp. 681–699, 2020.
- [12] P. K. Santra, G. S. Mahapatra, and G. R. Phajoo, "Bifurcation and chaos of a discrete predator-prey model with Crowley-Martin functional response incorporating proportional prey refuge," *Mathematical Problems in Engineering*, vol. 2020, pp. 1–18, 2020.
- [13] H. S. Panigoro and E. Rahmi, "The Dynamics of a Discrete Fractional-Order Logistic Growth Model with Infectious Disease," *Contemporary Mathematics and Applications*, vol. 3, no. 1, pp. 1–18, 2021.

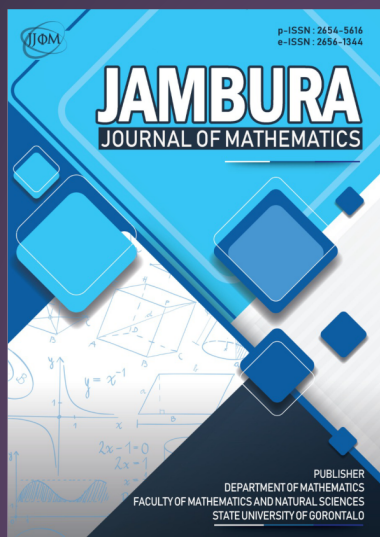
- [14] A. Singh and P. Deolia, "Dynamical analysis and chaos control in discrete-time prey-predator model," *Communications in Nonlinear Science and Numerical Simulation*, vol. 90, p. 105313, 2020.
- [15] A. Q. Khan and T. Khalique, "Bifurcations and chaos control in a discrete-time biological model," *International Journal of Biomathematics*, vol. 13, no. 04, p. 2050022, 2020.
- [16] P. Chakraborty, U. Ghosh, and S. Sarkar, "Stability and bifurcation analysis of a discrete prey-predator model with square-root functional response and optimal harvesting," *Journal of Biological Systems*, vol. 28, no. 01, pp. 91–110, 2020.
- [17] P. Santra, "Discrete-time prey-predator model with θ -logistic growth for prey incorporating square root functional response," *Jambura Journal of Biomathematics*, vol. 1, no. 2, pp. 41–48, 2020.
- [18] R. Ma, Y. Bai, and F. Wang, "Dynamical behavior analysis of a two-dimensional discrete predator-prey model with prey refuge and fear factor," *Journal of Applied Analysis and Computation*, vol. 10, no. 4, pp. 1683–1697, 2020.
- [19] X. Wang, L. Zanette, and X. Zou, "Modelling the fear effect in predator-prey interactions," *Journal of Mathematical Biology*, vol. 73, no. 5, pp. 1179–1204, 2016.
- [20] Z. Zhu, R. Wu, L. Lai, and X. Yu, "The influence of fear effect to the Lotka–Volterra predator–prey system with predator has other food resource," *Advances in Difference Equations*, vol. 2020, no. 1, pp. 1–13, 2020.
- [21] D. Barman, J. Roy, H. Alrabaiah, P. Panja, S. P. Mondal, and S. Alam, "Impact of predator incited fear and prey refuge in a fractional order prey predator model," *Chaos, Solitons and Fractals*, vol. 142, p. 110420, 2021.
- [22] E. I. Jury, *Inners and stability of dynamic systems*. Wiley, 1976.
- [23] S. N. Elaydi, *Discrete chaos: with applications in science and engineering*, 2nd ed. Chapman & Hall/CRC, 2007.



©2021 by the Author(s). This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution-NonCommercial 4.0 International License. **Editorial of JJBM:** Department of Mathematics, State University of Gorontalo, Jln. Prof. Dr. Ing. B. J. Habibie, Bone Bolango 96119, Indonesia.



Submit your manuscript at
<http://ejournal.ung.ac.id/>



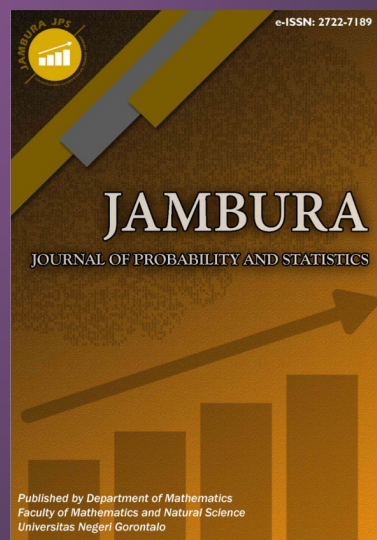
**Jambura Journal of
Mathematics**



**Jambura Journal of
Biomathematics**



**Jambura Journal of
Mathematics Education**



**Jambura Journal of
Probability and Statistics**

Published by
Department of Mathematics
Faculty of Mathematics and Natural Sciences
State University of Gorontalo