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Dynamics in two competing predators-one prey system with two types of Holling and fear effect

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Abstract

In this article, it is formulated a predator-prey model of two predators consuming a single limited prey resource. On the other hand, two predators have to compete with each other for survival. The predation function for two predators is assumed to be different where one predator uses Holling type I while the other uses Holling type II. It is also assumed that the fear effect is considered in this model as indirect influence evoked by both predator. Non-negativity and boundedness is written to show the biological justification of the model. Here, it is found that the model has five equilibrium points existed under certain condition. We also perform the local stability analysis on the equilibrium points with three equilibrium points are stable under certain condition and two equilibrium points are unstable. Hopf bifurcation is obtained by choosing the consumption rate of the second predator as the bifurcation parameter. In the last part, several numerical solutions are given to support the analysis results.

Keywords: Holling Functional Response; Hopf Bifurcation; Local Stability; Predator-Prey Model

1. Introduction

Since Lotka (1925) and Volterra (1931) proposed the predator-prey model known as the Lotka-Volterra model, the mathematical modeling of the ecological model has progressed rapidly. This model is the forerunner of other predator-prey model, such as Leslie-Gower model [1–5] and Rosenzweig-MacArthur model [6–9]. The dynamic behavior of predator-prey model can be influenced by many factors, such as mortality rates, competition between species, fear of prey, infectious disease, and environment conditions. In this paper, we analyze a predator-prey model to watch the dynamic behavior of predator-prey model with competition between two predators. According to [10], the competition factor represents a condition when the environment has limited resource so that both populations compete with each other for survival. Recently, several researchers have studied predator-prey models with competition between two predators. As in [11], Alebraheem and Yahya analyze a predator-prey model where two predators eat one prey. Prey and predators grow logistically where the carrying capacity of predator depends on the presence of prey. Holling type II is used to describe the predation rate of both predators and is assumed that two predators compete with other each for sharing food. A similar study is conducted by Sarwardi et al. [12] which investigates a predator-prey model where one prey provides protection and two predators consume the same prey as well as there is competition between two predators. This model also considers Holling type II as the predation rate of two predators on prey. Furthermore, the same predator-prey model is studied by Mukhopadhyay and Bhattacharyya [13] where two predators exploit the limited prey resource and they interfere each other because they want to capture food. However, the competition can harm its opponent. In this model, it is assumed that one of the predators is economically viable and is being harvested as well as the predation function for the two predators is different where the first predator uses Holling type I and the second predator uses Holling type II.

It is well known that predation and competition are important factors influencing the coexistence of species and are often considered in ecological model [14–16]. Gurevitch et al. [17] investigates that the predator can maintain the coexistence by reducing the power of competition. In the fact, the predators influence the prey by direct capturing. However, several researchers observe that the indirect influence of predators has a more profound impact than direct capturing [18–20]. Therefore, it is more realistic if the proposed model incorporates the fear effect where the model focuses on the character of both predators to maintain the coexistence of the prey. Furthermore, many studies have discussed the effect of fear in predator-prey models with various variations. As in [21], Zhang et al. observe the effect of anti-predator behavior because the fear evoked by predators in the predator-prey model where the predation rate follows Holling type II by including prey refuge. In this study,

the global stability is analyzed and Hopf bifurcation is found in the model. A mathematical model with two competing prey and one predator is presented in [22]. The model combines the effect of fear that influence the growth rate of both prey and the predation rate of predator and the intraspecific competition in predator. The analysis results show the global stability of equilibrium point and find Hopf bifurcation. In contrast to [22], the next study is conducted by Barman et al. [23] which investigates the infected predator-prey model with the predation rate uses Holling type II where the predators are built by two sub-classes, namely the susceptible and infected predator. In the model, it is assumed that the fear evoked by two predators on their prey has a different rate. As a result, local stability and Hopf bifurcation is found in the model.

In the present model, a mathematical model with one prey and two competing predators is presented to observe the dynamic behavior of the model. Here, it is assumed that the model includes the fear effect generated by both predators with the rate of fear is different and the predation function is given differently for the two predators where one of two predator follows Holling type I and the other uses Holling type II. The biological justification for the model is written on non-negativity and boundedness. In the last section, numerical solutions are performed to prove the analysis result.

2. Model Formulation

In this article, we consider a Lotka-Volterra model as in [13] where two predators exploit one prey. The density of prey at time T is denoted by $P(T)$ and the density of predator-1 and predator-2 at time T are denoted by $H_1(T)$ and $H_2(T)$, respectively. Both predators are assumed to have access to prey and they contend by eating the same prey and by harassing its rival. Here, we also consider the effect of fear as in [23] where the total population of predators is divided into two predators with the level of fear in prey population is different. Base on these considerations, the model can be written as follows.

$$
\begin{aligned}
\frac{dP}{dT} &= \frac{rP}{1 + k_1 H_1 + k_2 H_2}\left(1 - \frac{P}{K}\right) - \xi_1 P H_1 - \frac{\xi_2 P H_2}{b + P}, \\
\frac{dH_1}{dT} &= -\delta_1 H_1 + c_1 \xi_1 P H_1 - \gamma_1 H_1 H_2, \\
\frac{dH_2}{dT} &= -\delta_2 H_2 + \frac{c_2 \xi_2 P H_2}{b + P} - \gamma_2 H_1 H_2,
\end{aligned}
\tag{1}
$$

with initial conditions $P(0) > 0, H_1(0) > 0, H_2(0) > 0$. The parameter r is the intrinsic growth rates and K is the environment carrying capacity. The parameters δ_i and c_i for $i = 1, 2$ denote predator's death rates and conversion rates of prey to predator for H_1 and H_2, respectively. Meanwhile, the parameters ξ_i and γ_i for $i = 1, 2$ denote consumption rates and interference rates for H_1 and H_2, respectively. The parameter b is half saturation constant for H_2. Next, the parameters k_i for $i = 1, 2$ are the level of fear generated by H_1 and H_2, respectively. The sense of biological and some conditions for the fear term can be found in [23].

The present model can be reduced into non-dimensional system. By choosing the scaling parameters $(x, y, z, t) \rightarrow \left(\frac{P}{K}, \frac{\xi_1 H_1}{r}, \frac{\xi_2 H_2}{rK}, rT\right)$, eq. (1) can be rewritten as follows.

$$
\begin{aligned}
\frac{dx}{dt} &= x\left[\frac{(1 - x)}{1 + \rho_1 y + \rho_2 z} - y - \frac{z}{\beta + x}\right] \equiv xF_1(x, y, z), \\
\frac{dy}{dt} &= y\left[-\mu + \nu x - \eta z\right] \equiv yF_2(x, y, z), \\
\frac{dz}{dt} &= z\left[-\sigma + \frac{\kappa x}{\beta + x} - \psi y\right] \equiv zF_3(x, y, z),
\end{aligned}
\tag{2}
$$

where

$$
\rho_1 = \frac{k_1 r}{\xi_1}, \rho_2 = \frac{K k_2 r}{\xi_2}, \beta = \frac{b}{K}, \mu = \frac{\delta_1}{r}, \nu = \frac{c_1 \xi_1 K}{r}, \eta = \frac{\gamma_1 K}{\xi_2}, \sigma = \frac{\delta_2}{r}, \kappa = \frac{c_2 \xi_2}{r}, \psi = \frac{\gamma_2}{\xi_1},
$$

with initial conditions $x(0) = x_0 > 0, y(0) = y_0 > 0$, and $z(0) = z_0 > 0$. The functions F_i for $i = 1, 2, 3$ are smooth continuous function in $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0\}$. All parameters of eq. (2) are positive values.

3. Non-negativity and Boundedness of Solutions

In this section, we present some theorems below to show the non-negative and boundedness of solution for eq. (2), These theorems are very important because it involves the biological validity of model.

3.1. Non-negativity

Theorem 1. *All solutions of eq. (2) with $x_0 > 0, y_0 > 0, z_0 > 0$ are non-negative.*

proof. The non-negative of solution can be proved by the equation as follows.

$$x(t) = x_0 \exp\left\{\int_0^t F_1(x, y, z)d\theta\right\} \geq 0,$$

$$y(t) = y_0 \exp\left\{\int_0^t F_2(x, y, z)d\theta\right\} \geq 0,$$

$$z(t) = z_0 \exp\left\{\int_0^t F_3(x, y, z)d\theta\right\} \geq 0.$$

It is known the exponential functions have non-negative values for any number real and the initial conditions $x_0 > 0, y_0 > 0, z_0 > 0$. Thus, all solutions of eq. (2) are always non-negative at any time t. ∎

3.2. Boundedness

Theorem 2. *All solutions of eq. (2) are uniformly bounded.*

proof. We start by defining the function as follows.

$$Q(t) = x(t) + \frac{y(t)}{\nu} + \frac{z(t)}{\kappa}. \tag{3}$$

We take the time derivative of eq. (3) along the solution and choose $\phi \in \mathbb{R}_+$ with $\phi = \min(\mu, \sigma)$. Thus, we obtain

$$\frac{dQ}{dt} + \phi Q \leq \frac{(1+\phi)^2}{4} \equiv M (say).$$

Next, by integrating the differential inequality between t_0 and t, we get

$$0 \leq Q(t) \leq Q(t_0)e^{-\phi t} + \frac{M}{\phi}\left(1 - e^{-\phi t}\right).$$

For $t \to \infty$, we have $0 \leq Q(t) \leq \frac{M}{\phi}$. Hence, all solution of eq. (2) are confined in the region

$$\Omega = \left\{(x, y, z) \in \mathbb{R}_+^3 : 0 \leq Q(t) \leq \frac{M}{\phi} + \epsilon, \text{for any } \epsilon > 0\right\}.$$

∎

4. Equilibrium Point, Local Stability Analysis, and Hopf Bifurcation

4.1. Existence of Equilibrium Points

By setting the right side equal to zero, it is obtained that eq. (2) has five non-negative equilibrium points as follows.

1. $E_0(0, 0, 0)$ shows that the extinction of all populations.
2. $E_1(1, 0, 0)$ shows that prey population exist but two predators are extinct.
3. The predator-1 free equilibrium point $E_2(\hat{x}, 0, \hat{z})$ with $\hat{x} = \frac{\sigma\beta}{\kappa - \sigma}$ and $\hat{z}$ is the roots of quadratic equation $\omega_1(\hat{z})^2 + \omega_2\hat{z} + \omega_3 = 0$ where $\omega_1 = \rho_2(\kappa - \sigma)^2$, $\omega_2 = (\kappa - \sigma)^2$, and $\omega_3 = -[\kappa - \sigma(1 + \beta)]\beta\kappa$. The point E_2 exits when $\kappa > \sigma(1 + \beta)$.
4. The predator-2 free equilibrium point $E_3(\bar{x}, \bar{y}, 0)$ with $\bar{x} = \frac{\mu}{\nu}$ and $\bar{y}$ is the roots of quadratic equation $\nu\rho_1(\bar{y})^2 + \nu\bar{y} + (\mu - \nu) = 0$. The point E_3 exists if $\nu > \mu$.

5. The existence point for all populations $E_*(x_*, y_*, z_*)$ with $y_* = \frac{\kappa x_* - \sigma(\beta + x_*)}{\psi(\beta + x_*)}$, $z_* = \frac{\nu x_* - \mu}{\eta}$, and x_* is the positive root of cubic equation

$$A_1(x_*)^3 + A_2(x_*)^2 + A_3 x_* + A_4 = 0, \tag{4}$$

where

$$
\begin{aligned}
A_1 &= \psi \left[\eta \nu \rho_2(\kappa - \sigma) + \nu^2 \psi \rho_2 + \eta^2 \psi \right], \\
A_2 &= (2\beta\eta^2 - 2\mu\nu\rho_2 - \eta^2 + \eta\nu)\psi^2 + \eta^2 \rho_1(\kappa - \sigma)^2 + \beta\nu\rho_2\psi(\nu\psi - \eta\sigma) \\
&\quad + \eta\psi(\kappa - \sigma)(\beta\nu\rho_2 - \mu\rho_2 + \nu\rho_1 + \eta), \\
A_3 &= (\beta^2\eta^2 - 2\beta\mu\nu\rho_2 - 2\beta\eta^2 + \beta\eta\nu + \mu^2\rho_2 - \eta\mu)\psi^2 \\
&\quad - \eta\psi \left[\beta\rho_2\sigma(\beta\nu - \mu) + \beta\sigma(\nu\rho_1 + \eta) + (\kappa - \sigma)(\beta\mu\rho_2 - \beta\eta + \mu\rho_1) \right] \\
&\quad - 2\beta\eta^2\rho_1\sigma(\kappa - \sigma), \\
A_4 &= (\mu^2\rho_2 - \eta\mu - \beta\eta^2)\beta\psi^2 + \beta\eta\sigma\psi(\beta\mu\rho_2 - \beta\eta + \mu\rho_1) + \beta^2\eta^2\rho_1\sigma^2.
\end{aligned}
$$

The explicit form and existence of positive root at eq. (4) are be solved by using Cardan's method as in [1].

4.2. Local Stability of Equilibrium Points

To investigate the local stability of each equilibrium point, we have to calculate the Jacobian matrix and find the eigenvalue of each equilibrium point. Here, the Jacobian matrix of eq. (2) at $E(x, y, z)$ is $J = (a_{ij}) \in \mathbb{R}^{3\times3}$ with

$$
J = (a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \tag{5}
$$

where

$$
\begin{aligned}
a_{11} &= \frac{1 - 2x}{1 + \rho_1 y + \rho_2 z} - \frac{\beta z}{(\beta + x)^2} - y, \\
a_{12} &= -\frac{x(1 - x)\rho_1}{(1 + \rho_1 y + \rho_2 z)^2} - x, \\
a_{13} &= -\frac{x(1 - x)\rho_2}{(1 + \rho_1 y + \rho_2 z)^2} - \frac{x}{\beta + x}, \\
a_{21} &= \nu y, \\
a_{22} &= -\eta z + \nu x - \mu, \\
a_{23} &= -\eta y, \\
a_{31} &= \frac{\beta \kappa z}{(\beta + x)^2}, \\
a_{32} &= -\psi z, \\
a_{33} &= -\sigma + \frac{\kappa x}{\beta + x} - \psi y.
\end{aligned}
$$

In this article, the Jacobian matrix of each equilibrium point is denoted as $J_s = J(E_s) = \left(a_{ij}^{[s]} \right)$ at E_s for $s = 0, ..., 3$ and $J_* = J(E_*) = \left(a_{ij}^{[*]} \right)$ at E_*. The following theorems are given to to show the local stability of each equilibrium point of eq. (2).

Theorem 3. *The point E_0 is unstable while the point E_1 is locally asymptotically stable when $\nu < \mu$ and $\kappa < \sigma(1 + \beta)$.*

proof. The eigenvalues of J_0 are $\lambda_1 = 1, \lambda_2 = -\mu, \lambda_3 = -\sigma$. It is confirmed that $\lambda_1 > 0$. Thus, E_0 is unstable. Furthermore, the eigenvalues of J_1 are $\lambda_1 = -1, \lambda_2 = -\mu + \nu, \lambda_3 = -\sigma + \frac{\kappa}{1+\beta}$. It is noted that E_1 is locally

asymptotically stable when $\lambda_2 < 0$ and $\lambda_3 < 0$. Thus, the condition for the stability of E_1 is $\nu < \mu$ and $\kappa < \sigma(1+\beta)$. ∎

Base on theorem 3, we remark that if the equilibrium point E_1 is locally asymptotically stable, then the points E_2 and E_3 do not exist.

Theorem 4. *Let*

$$\mathbf{B_1} : \mu > \frac{\nu\sigma\beta}{(\kappa - \sigma)} - \eta\hat{z},$$

$$\mathbf{B_2} : \rho_2 > \frac{(\kappa\beta)^2 \left[\kappa - \sigma(1+2\beta)\right]}{\beta(\hat{z})^2(\kappa - \sigma)^3} - \frac{1}{\hat{z}}.$$

When $\mathbf{B_1}$ and $\mathbf{B_2}$ hold, then the point E_2 is locally asymptotically stable.

proof. The characteristic equation of J_2 is given as follows.

$$\left(a_{22}^{[2]} - \lambda\right)\left(\lambda^2 - a_{11}^{[2]}\lambda + a_{13}^{[2]}a_{31}^{[2]}\right) = 0. \tag{6}$$

Thus, the eigenvalue of eq. (6) is

$$\lambda_1 = a_{22}^{[2]},$$

$$\lambda_{2,3} = \frac{a_{11}^{[2]}}{2} \pm \sqrt{\left(\frac{a_{11}^{[2]}}{2}\right)^2 - a_{13}^{[2]}a_{31}^{[2]}}.$$

It is confirmed that $\lambda_1 < 0$ when $\mathbf{B_1}$ holds. It is noted that the condition of existence for E_2 is $\kappa > \sigma(1+\beta)$. Thus, we get $a_{13}^{[2]} > 0$ and $a_{31}^{[2]} > 0$. In the other hand, if $\mathbf{B_2}$ holds, then $a_{11}^{[2]} < 0$. Therefore, we obtain $Re\left(\lambda_{2,3}\right) < 0$. So, E_2 is locally asymptotically stable. ∎

Theorem 5. *The point E_3 is locally asymptotically stable if $\sigma > \frac{\kappa\mu}{(\beta\nu+\mu)} - \psi\bar{y}$ and $\mu > 0.5\nu\left[1 - \bar{y}\left(1 + \rho_1\bar{y}\right)\right]$.*

proof. The characteristic equation of J_3 is expressed with

$$\left(a_{33}^{[3]} - \lambda\right)\left(\lambda^2 - a_{11}^{[3]}\lambda + a_{12}^{[3]}a_{21}^{[3]}\right) = 0. \tag{7}$$

Thus, the eigenvalues of eq. (7) is

$$\lambda_1 = a_{33}^{[3]},$$

$$\lambda_{2,3} = \frac{a_{11}^{[3]}}{2} \pm \sqrt{\left(\frac{a_{11}^{[3]}}{2}\right)^2 - a_{12}^{[3]}a_{21}^{[3]}}.$$

It is easily confirmed that $\lambda_1 < 0$ when $\sigma > \frac{\kappa\mu}{(\beta\nu+\mu)} - \psi\bar{y}$. Meanwhile, it is known that the condition of existence for E_3 is $\nu > \mu$. Thus, we have $a_{12}^{[3]} > 0$ and $a_{21}^{[3]} > 0$. Moreover, we get that $a_{11}^{[3]} < 0$ when $\mu > 0.5\nu\left[1 - \bar{y}\left(1 + \rho_1\bar{y}\right)\right]$. Therefore, we obtain that $Re\left(\lambda_{2,3}\right) < 0$. So, E_3 is locally asymptotically stable. ∎

Theorem 6. *The point E_* is unstable.*

proof. By using the Jacobian matrix J_*, we get the characteristic equation as follows.

$$\lambda^3 + \zeta_1 \lambda^2 + \zeta_2 \lambda + \zeta_3 = 0, \tag{8}$$

where

$$\zeta_1 = -a_{11}^{[*]},$$
$$\zeta_2 = -\left(a_{12}^{[*]} a_{21}^{[*]} + a_{23}^{[*]} a_{32}^{[*]} + a_{13}^{[*]} a_{31}^{[*]} \right),$$
$$\zeta_3 = \left(a_{11}^{[*]} a_{23}^{[*]} a_{32}^{[*]} - a_{12}^{[*]} a_{23}^{[*]} a_{31}^{[*]} - a_{13}^{[*]} a_{21}^{[*]} a_{32}^{[*]} \right).$$

By using Routh–Hurwitz criterion, eq. (8) have the negative real roots if the necessary and sufficient conditions are

$$\zeta_1 > 0, \zeta_2 > 0, \zeta_3 > 0, \zeta_1 \zeta_2 > \zeta_3.$$

It is noted that if $a_{11}^{[*]} > 0$, then $\zeta_1 < 0$. Meanwhile, if we choose $a_{11}^{[*]} < 0$, then $\zeta_3 < 0$. Therefore, the Routh–Hurwitz criterion is not satisfied and E_* is unstable. ∎

4.3. Existence of Hopf Bifurcation

Now, we want to check the Hopf bifurcation of eq. (2). This subject guarantees a change in the stability of model by taking into account several parameters. A minor change to parameter can make a transform of dynamical model which shows that dynamic populations may be fluctuating.

To identify Hopf bifurcation around the point E_2, we need to show that the coefficient λ in eq. (6) is zero and the constant term is positive value. Therefore, the linear terms in eq. (6) can be formed into $a_{11}^{[2]} = 0$. Eventually, we get that the critical value for Hopf bifurcation is $\kappa = \kappa_c$ (see Theorem 4).

5. Numerical Solutions

In this section, we provide several numerical solutions by using the fourth-order predictor-corrector method expressed through phase portrait in Python. It is known that our model doesn't refer to an actual case that occurs in the interaction of three species. Therefore, the parameter selections are customized to the stability conditions obtained in the previous results. The parameter values are given in the following table.

Table 1. The parameter values for numerical simulations

Simulations	Parameters								
	ρ_1	ρ_2	β	μ	ν	η	σ	κ	ϕ
Simulation 1	0.1	0.2	0.2	0.9	0.155	0.01	0.2	0.24	0.011
Simulation 2	0.1	0.2	0.2	0.08	0.155	0.01	0.2	0.29	0.011
Simulation 3	0.1	0.2	0.2	0.08	0.155	0.01	0.2	0.26	0.011
Simulation 4	0.1	0.2	0.2	0.08	0.155	0.01	0.2	0.31	0.011
Simulation 5	0.1	0.01	1.2	0.08	0.155	0.01	0.5	1.7	0.1

5.1. The Local Stability for E_1

By using the parameter values in simulation 1 and different initial conditions, we obtain the points E_2, E_3, E_* don't exist while the points E_0, E_1 exist. It is noted that the point E_1 is locally asymptotically stable because all solutions tend to the point E_1 (see Figure 1). Thus, it proves that Theorem 3 is satisfied and the prey populations will exist while the both of predators are extinct.
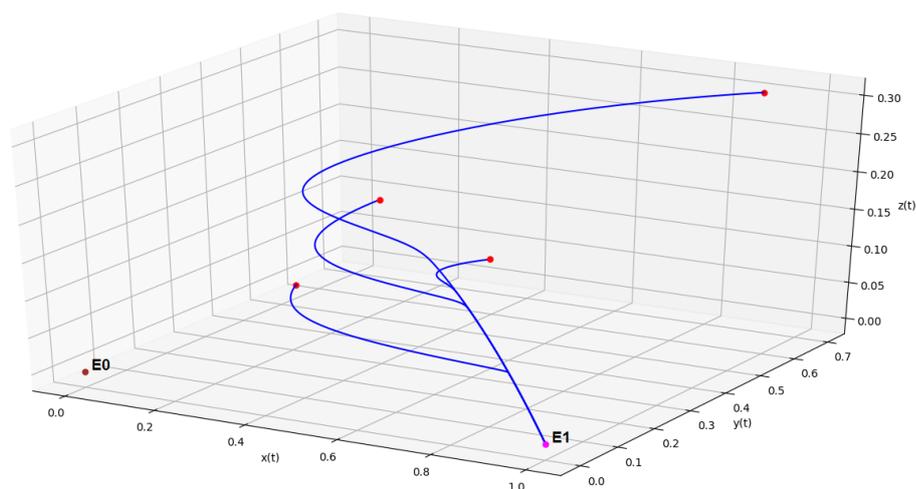
Figure 1. The phase portrait for simulation 1

5.2. The Local Stability for E_2

By using the parameter values in simulation 2, we observe that the points E_0, E_1, E_2, E_3 exist but the point E_* doesn't exist. In this simulation, the Theorem 4 is satisfied so that all solutions tend to the point E_2 (see Figure 2). Thus, the point E_2 is locally asymptotically stable. Therefore, the predator-1 is extinct while the prey populations and predator-2 will survive.
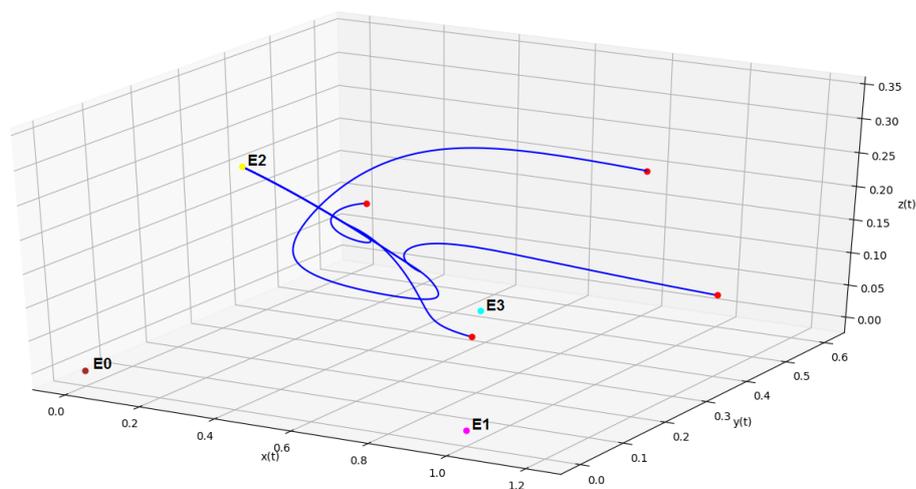

Figure 2. The phase portrait for simulation 2

5.3. The Local Stability for E_3

When we use the parameter value in simulation 3, the Theorem 5 is satisfied. Eventually, all solutions tend to the point E_3 with several initial conditions and the points E_0, E_1, E_2, E_3 will exist while the point E_* doesn't exist (see Figure 3). Thus, the point E_3 is locally asymptotically stable which means that the prey populations and predator-1 can survive while the predator-2 is extinct.

5.4. Hopf Bifurcation

By considering the parameter values in simulation 2 and simulation 4, we identify Hopf bifurcation around the point E_2 where $\kappa = \kappa_c$ is chosen as the bifurcation parameter. When κ crosses a critical value $\kappa_c = 0.305$, then the point E_2 loses the stability via Hopf bifurcation. To describe The dynamical behavior for E_2, we choose $\kappa = 0.29 < \kappa_c = 0.305$ so that we obtain the phase portrait as Figure 4 in point (a). Here, the point E_2 is locally
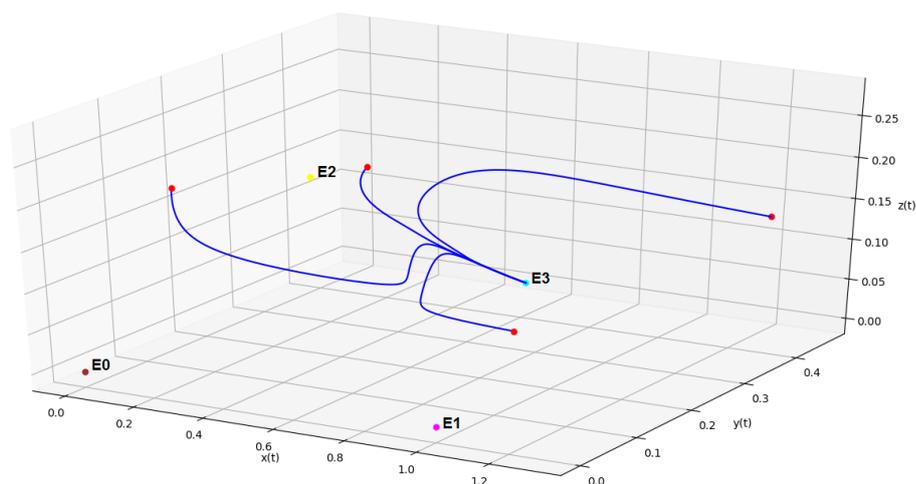
Figure 3. The phase portrait for simulation 3

asymptotically stable because all simulations convergent to the point E_2. It means that the prey population and predator-2 will survive but predator-1 is extinct. But, when we choose $\kappa > \kappa_c = 0.305$, then the point E_2 loses stability via Hopf bifurcation. By considering $\kappa = 0.31$, we obtain the phase portrait as Figure 4 in point (b). All solutions undergo to limit cycle so that the point E_2 is unstable.
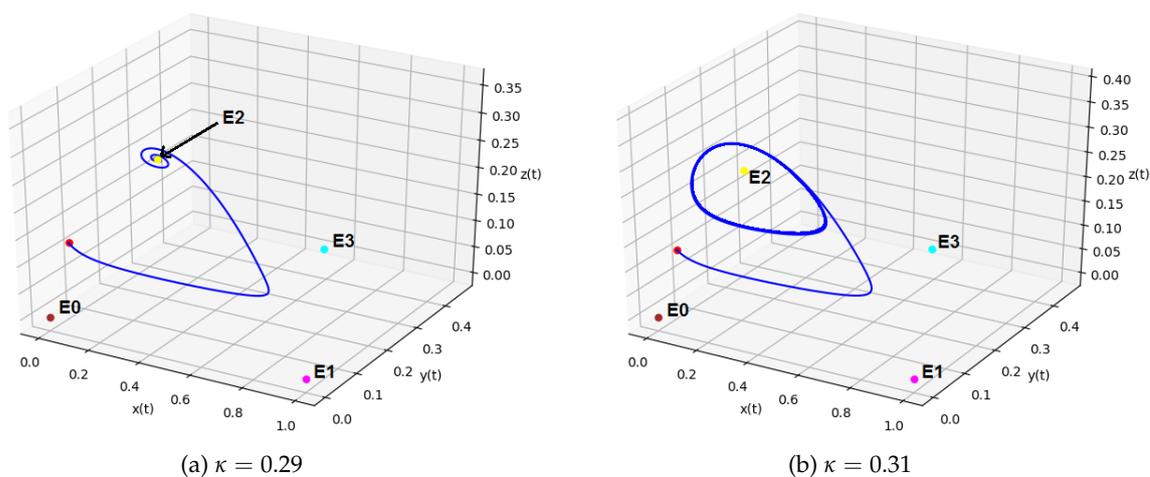

(a) $\kappa = 0.29$ (b) $\kappa = 0.31$

Figure 4. The phase portrait around the equilibrium point E_2

5.5. Bi-stability

By using the parameter values in simulation 5, all equilibrium points E_0, E_1, E_2, E_3, and E_* exist. From Theorem 4 and Theorem 5, the points E_2 and E_3 are locally asymptotically stable. Hence, for simulation 5, the model shows a phenomenon of bi-stability because there are two points of stability. This phenomenon is shown in the Figure 5.

We know that both the predator-1 free equilibrium point E_2 and the predator-2 free equilibrium point E_3 are locally asymptotically stable. It is noted that the initial conditions in this simulation are very sensitive to influence the dynamics of model. For relatively large initial conditions, all solutions tend to the predator-1 free equilibrium point E_2 (see green-trajectories in Figure 5). Therefore, the prey populations and the predator-2 populations exist but the predator-1 populations are extinct. Meanwhile, for relatively small initial conditions, all solutions convergent to the predator-2 free equilibrium point E_3 (see blue-trajectories in Figure 5). Hence, the predator-2 populations are extinct while the prey populations and the predator-1 populations exist.
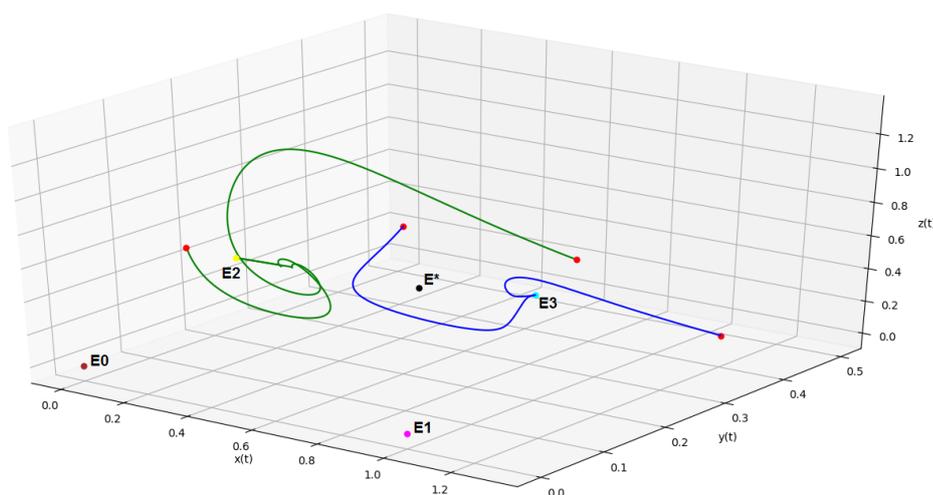
Figure 5. The phase portrait for simulation 5

6. Conclusion

In this article, we have observed the local stability in the predator-prey model with competition between predators including two response functions. It is noted that the model has five non-negative equilibrium points existed under certain condition. It is found that the points E_1, E_2, E_3 are stable under certain conditions while the points E_0, E_* are unstable. Hopf bifurcation is found by taking $\kappa = \kappa_c$ as the bifurcation parameter. When we take $\kappa = \kappa_c = 0.305$, then Hopf bifurcation appears. Thus, if we choose the parameter $\kappa < \kappa_c = 0.305$, then the point E_2 is stable. Meanwhile, if we take the parameter $\kappa > \kappa_c = 0.305$, then the point E_2 loses stability and undergoes to limit cycle. Therefore, the phenomenon illustrated by the phase portrait has supported the results obtained above. For the next research, we can analyze the global stability of the model.

References

[1] Y. Cai, C. Zhao, W. Wang, and J. Wang, "Dynamics of a Leslie-Gower predator-prey model with additive Allee effect," *Applied Mathematical Modelling*, vol. 39, no. 7, pp. 2092–2106, 2015.

[2] J. Wang, Y. Cai, S. Fu, and W. Wang, "The effect of the fear factor on the dynamics of a predator-prey model incorporating the prey refuge," *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 29, no. 8, 2019.

[3] U. Salamah, A. Suryanto, and W. M. Kusumawinahyu, "Leslie-Gower predator-prey model with stage-structure, Beddington-DeAngelis functional response, and anti-predator behavior," *AIP Conference Proceedings*, vol. 2084, 2019.

[4] H. S. Panigoro, E. Rahmi, N. Achmad, and S. L. Mahmud, "The influence of additive Allee effect and periodic harvesting to the dynamics of Leslie-Gower predator-prey model," *Jambura Journal of Mathematics*, vol. 2, no. 2, pp. 87–96, 2020.

[5] A. L. Firdiansyah, "Effect of prey refuge and harvesting on dynamics of eco-epidemiological model with Holling type III," *Jambura Journal of Mathematics*, vol. 3, no. 1, pp. 16–25, 2021.

[6] M. L. Rosenzweig and R. H. MacArthur, "Graphical representation and stability conditions of predator-prey interactions," *The American Naturalist*, vol. 97, no. 895, pp. 209–223, 1963.

[7] L. K. Beay, A. Suryanto, I. Darti, and Trisilowati, "Stability of a stage-structure Rosenzweig-MacArthur model incorporating Holling type-II functional response," *IOP Conference Series: Materials Science and Engineering*, vol. 546, no. 5, pp. 0–7, 2019.

[8] L. K. Beay and M. Saija, "A stage-structure Rosenzweig-MacArthur model with effect of prey refuge," *Jambura Journal of Biomathematics*, vol. 1, no. 1, pp. 1–7, 2020.

[9] I. Djakaria, M. B. Gaib, and Resmawan, "Analysis of the Rosenzweig-MacArthur predator-prey model with anti-predator behavior," *CAUCHY: Jurnal Matematika Murni dan Aplikasi*, vol. 6, no. 4, pp. 260–269, 2021.

[10] D. Savitri, A. Suryanto, W. M. Kusumawinahyu, and Abadi, "A dynamics behaviour of two predators and one prey interaction with competition between predators," *IOP Conference Series: Materials Science and Engineering*, vol. 546, no. 5, 2019.

[11] J. Alebraheem and Y. Abu-Hasan, "Persistence of predators in a two predators-one prey model with non-periodic solution," *Applied Mathematical Sciences*, vol. 6, no. 17-20, pp. 943–956, 2012.

[12] S. Sarwardi, P. K. Mandal, and S. Ray, "Dynamical behaviour of a two-predator model with prey refuge," *Journal of Biological Physics*, vol. 39, no. 4, pp. 701–722, 2013.

[13] B. Mukhopadhyay and R. Bhattacharyya, "Effects of harvesting and predator interference in a model of two-predators competing for a single prey," *Applied Mathematical Modelling*, vol. 40, no. 4, pp. 3264–3274, 2016.

[14] R. D. Holt, "Predation, apparent competition, and the structure of prey communities," *Theoretical Population Biology*, vol. 12, no. 2, pp. 197–229, 1977.

[15] R. R. Vance, "Predation and resource partitioning in one predator – two prey model communities," *The American Naturalist*, vol. 112, no. 987, pp. 797–813, 1978.

[16] B. P. Kotler and R. D. Holt, "Predation and competition: The interaction of two types of species interactions," *Oikos*, vol. 54, no. 2, pp. 256–260, 1989.

[17] J. Gurevitch, J. A. Morrison, and L. V. Hedges, "The interaction between competition and predation: A meta-analysis of field experiments," *The American Naturalist*, vol. 155, no. 4, pp. 435–453, 2000.

[18] S. L. Lima, "Nonlethal effects in the ecology of predator-prey interactions," *BioScience*, vol. 48, no. 1, pp. 25–34, 1998.

[19] S. Creel, D. Christianson, S. Liley, and J. A. Winnie, "Predation risk affects reproductive physiology and demography of elk," *Science*, vol. 315, no. 5814, p. 960, 2007.

[20] S. Creel and D. Christianson, "Relationships between direct predation and risk effects," *Trends in Ecology & Evolution*, vol. 23, no. 4, pp. 194–201, 2008.

[21] H. Zhang, Y. Cai, S. Fu, and W. Wang, "Impact of the fear effect in a prey-predator model incorporating a prey refuge," *Applied Mathematics and Computation*, vol. 356, pp. 328–337, 2019.

[22] D. Mukherjee, "Impact of predator fear on two competing prey species," *Jambura Journal of Biomathematics*, vol. 2, no. 1, pp. 1–12, 2021.

[23] D. Barman, J. Roy, and S. Alam, "Dynamical Behaviour of an Infected Predator-Prey Model with Fear Effect," *Iranian Journal of Science and Technology, Transaction A*, vol. 45, no. 1, pp. 309–325, 2021.
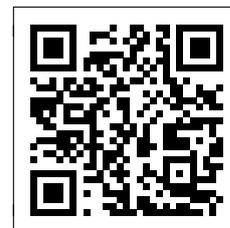
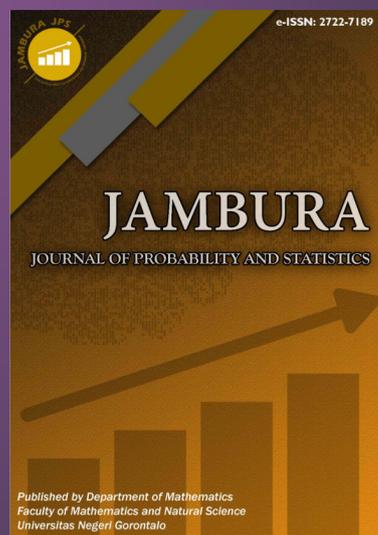Submit your **manuscript** at
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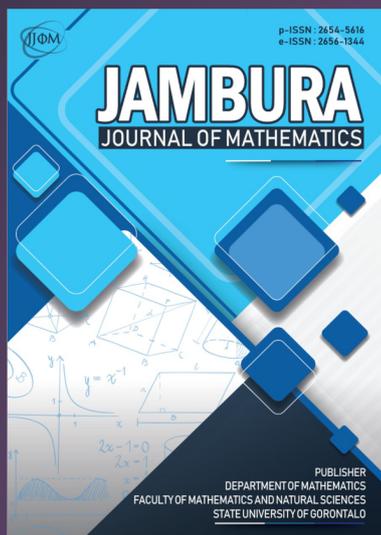
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