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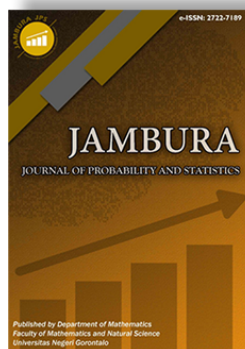
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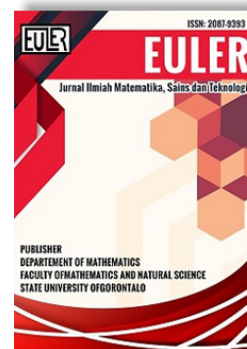
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# Stability and bifurcation of a two competing prey-one predator system with anti-predator behavior

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**ABSTRACT.** This article considers the impact of competitive response to interfering time and anti-predator behavior of a three species system in which one predator consumes both the competing prey species. Here one of the competing species shows anti-predator behavior. We have shown that its solutions are non-negative and bounded. Further, we analyze the existence and stability of all the feasible equilibria. Conditions for uniform persistence of the system are derived. Applying Bendixson's criterion for high-dimensional ordinary differential equations, we prove that the coexistence equilibrium point is globally stable under specific conditions. The system admits Hopf bifurcation when anti-predator behavior rate crosses a critical value. Analytical results are verified numerically.



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## 1. Introduction

In population biology, competition and predation are two fundamental interspecific interactions. Basic questions arise how predation affect competitive interactions. Usually predators attack weak competitor. In that case a superior competitor plays an important role in structuring the community. Actually, there are some situations where a superior competitor shows anti-predator behavior which in turn reduce the predation pressure. This fact occurs due to the evolutionary adaptation of prey and predator. The prey with the anti-predator behavior may promote coexistence of all the species. Several studies are carried out to focus anti-predator behavior [1–4]. Though there are evidence of the anti-predator behavior, mathematical model using this aspect is few [1, 2, 5]. Ives and Dobson [2] investigated anti-predator behavior in predator-prey model and found that anti-predator behavior increases the density of prey and reduces the ratio of predator-to-prey density and induces damps oscillation in the predator-prey system. Tang and Xiao [5] analyzed the dynamical behavior of predator-prey model with a non-monotonic functional response and anti-predator behavior. They showed that anti-predator behavior enhances the coexistence of the prey and predator and also damps the predator-prey oscillations.

Previous studies [2, 5] did not consider the density of prey in anti-predator behavior. Saitō [6] pointed out that the density of prey is a factor in anti-predator activity. Janssen et al. [7] remarked that prey population can kill the predator when the size of the prey attain a certain level. Sun et al. [8] investigated a piecewise dynamic system to address the impact of anti-predator behavior on predator-prey system. They remarked that an increasing amount of anti-predator behavior rate causes the prey population to persist though the coexistence of all species of the system depends upon the anti-predator behavior. Prasad and

Prasad [9] suggested additional food to a predator in predator-prey system with anti-predator behavior to control the loss in predator population. Tang and Qin [10] studied a predator-prey model with stage structure and anti-predator behavior. They obtained forward and backward bifurcation and mentioned the impact of anti-predator behavior upon the equilibrium level of prey equilibrium density. Morteja et al. [11] discussed the anti-predator behavior in stage structure predator-prey model with Holling type II and IV predator functional response. These studies are mainly confined into two interacting populations. But in real ecosystem, complicated dynamics emerges when there are more than two interacting species. Fujii [12] analyzed two prey-one predator model with competition between the prey species and observed globally stable limit cycle surrounding the unstable coexistence equilibrium point. Takeuchi and Adachi [13] discussed the stable behavior of two prey, one predator communities. They remarked that chaotic motion arises from periodic motion when one of two prey has greater competitive abilities than the other and predator mediated coexistence is possible depending on the preferences of a predator and competitive abilities of two prey. Deka et al. [14] studied the effect of predation on two competing prey species in the general Gauss type model.

In view of the above, we have interested to study predator-prey interaction with anti-predator behavior along with competition between the prey species where each species invest time in competing individual of the other species. The competition term suggested in [15] is considered here. Here we consider two competing prey species which share a common predator. Furthermore, one of the competing prey shows anti-predator behavior. For example, Uganda kobs and buffalo are two competing prey species. Lions predate both the prey species. The buffalo has adopted anti-predator behavior.

The rest of the paper is structured as follows. The model is

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presented in Section 2. Positivity and boundedness are checked in Section 3. The existence and stability of various steady states and persistence are discussed in Section 4. Global stability analysis is carried out in Section 5. Bifurcation result is prescribed in Section 6. Numerical computations are given in Section 7. A brief discussion follows in Section 8.

### 2. Model Formulation

Recently, Castillo-Alvino and Marva [15] revisited the classical two species Lotka-Volterra competition model considering the time spent in competition for the interacting species. They proposed the model by incorporating Holling type II competition response to interference time as follows:

$$\begin{aligned} \frac{dx_1}{dt} &= x_1 \left( r_1 - a_{11}x_1 - \frac{a_{12}x_2}{1 + a_1x_1} \right), \\ \frac{dx_2}{dt} &= x_2 \left( r_2 - a_{22}x_2 - \frac{a_{21}x_1}{1 + a_2x_2} \right), \end{aligned} \tag{1}$$

with  $x_i(0) > 0, i = 1, 2$ .

The variables  $x_1$  and  $x_2$  denote the densities of two competing species at time  $t$  respectively.  $r_1$  and  $r_2$  represent the intrinsic growth rate of both the species.  $a_{ii}$  denotes the intra-specific competition of species  $i$ .  $a_{ij}$  measures the action of species  $j$  upon the growth rate of species  $i$ .  $a_i$  denotes the searching rate. It is assumed that  $a_1 \neq a_2$ . In [15], the authors showed that the more time interfering with competition takes, the more likely coexistence and also obtained multi-stability scenarios. In the above study, the role of predator is not considered. Also anti-predator behavior of prey that exhibit complex dynamics has not been investigated yet. So we incorporate a predator  $y$  in system (1) with Holling type II predator functional response and the prey  $x_2$  exhibits anti-predator behavior in the form studied in [19] is considered here. Thus (1) transformed into:

$$\begin{aligned} \frac{dx_1}{dt} &= x_1 \left( r_1 - a_{11}x_1 - \frac{a_{12}x_2}{1 + a_1x_1} - \frac{p_1y}{1 + b_1x_1} \right), \\ \frac{dx_2}{dt} &= x_2 \left( r_2 - a_{22}x_2 - \frac{a_{21}x_1}{1 + a_2x_2} - \frac{p_2y}{1 + b_2x_2} \right), \\ \frac{dy}{dt} &= y \left( -d + \frac{c_1p_1x_1}{1 + b_1x_1} + \frac{c_2p_2x_2}{1 + b_2x_2} - \frac{\alpha x_2}{1 + \beta y} \right), \end{aligned} \tag{2}$$

with  $x_i(0) > 0, i = 1, 2, y(0) > 0$ .

Here  $p_1$  and  $p_2$  represent the per capita predator consumption rate,  $b_1$  and  $b_2$  denote the constant handling time for each prey captured.  $c_1$  and  $c_2$  are the conversion rate of prey biomass to predator biomass.  $1/\beta$  is the half saturation constant.  $\alpha/\beta$  is the maximal anti-predator efficiency of the prey  $x_2$ .  $\alpha$  denotes anti-predator rate. The anti-predator behavior of prey is assumed to resist predator aggression, though the growth of prey population is not increased still it can reduce the growth of the predator population.

### 3. Positivity and Boundedness of Solutions

In this section, we present positivity and boundedness of solutions of system (2) which ensure the biological validity of the model. We first check positivity.

**Lemma 1.** All solutions  $(x_1(t), x_2(t), y(t))$  of system (2) with initial values  $(x_{10}, x_{20}, y_0) \in \mathbb{R}_+^3$  remain positive for all  $t > 0$ .

*Proof.* The positivity of  $x_1(t), x_2(t), y(t)$  can be shown by the equations

$$\begin{aligned} x_1(t) &= x_{10} \exp \left( \int_0^t \left\{ r_1 - a_{11}x_1(s) - \frac{a_{12}x_2(s)}{1 + a_1x_1(s)} - \frac{p_1y(s)}{1 + b_1x_1(s)} \right\} ds \right), \\ x_2(t) &= x_{20} \exp \left( \int_0^t \left\{ r_2 - a_{22}x_2(s) - \frac{a_{21}x_1(s)}{1 + a_2x_2(s)} - \frac{p_2y(s)}{1 + b_2x_2(s)} \right\} ds \right), \\ y(t) &= y_0 \exp \left( \int_0^t \left\{ -d + \frac{c_1p_1x_1(s)}{1 + b_1x_1(s)} + \frac{c_2p_2x_2(s)}{1 + b_2x_2(s)} - \frac{\alpha x_2(s)}{1 + \beta y(s)} \right\} ds \right), \end{aligned}$$

with  $x_{10}, x_{20}, y_0 > 0$ . As  $x_{10} > 0$  then  $x_1(t) > 0$  for all  $t > 0$ . Similarly we can show that  $x_2(t) > 0$  and  $y(t) > 0$  for all  $t > 0$ . Hence the interior of  $\mathbb{R}_+^3$  is an invariant set of system (2).  $\square$

**Lemma 2.** All solutions  $(x_1(t), x_2(t), y(t))$  of system (2) with initial values  $(x_{10}, x_{20}, y_0) \in \mathbb{R}_+^3$  are bounded; furthermore, they enter the region

$$B = \left\{ (x_1, x_2, y) \in \mathbb{R}_+^3 : 0 \leq c_1x_1 + c_2x_2 + y \leq \frac{(r_1 + \tau)^2}{4} a_{11}\tau + \frac{(r_2 + \tau)^2}{4a_{22}\tau} \right\}, \text{ where } \tau < d.$$

*Proof.* Define the function

$$W(t) = c_1x_1 + c_2x_2 + y.$$

The time derivative along a solution of (2) is

$$\begin{aligned} \frac{dW(t)}{dt} &= c_1x_1 \left( r_1 - a_{11}x_1 - \frac{a_{12}x_2}{1 + a_1x_1} \right) \\ &\quad + c_2x_2 \left( r_2 - a_{22}x_2 - \frac{a_{21}x_1}{1 + a_2x_2} \right) \\ &\quad - y \left( d + \frac{\alpha x_2}{1 + \beta y} \right). \end{aligned}$$

For each  $\tau > 0$ , the following inequality is fulfilled.

$$\begin{aligned} \frac{dW}{dt} + \tau W &\leq c_1x_1(r_1 - a_{11}x_1 + \tau) \\ &\quad + c_2x_2(r_2 - a_{22}x_2 + \tau) + y(\tau - d) \\ &\leq -c_1a_{11} \left\{ \left( x_1 - \frac{r_1 + \tau}{2a_{11}} \right)^2 - \left( \frac{r_1 + \tau}{2a_{11}} \right)^2 \right\} \\ &\quad - c_2a_{22} \left\{ \left( x_2 - \frac{r_2 + \tau}{2a_{22}} \right)^2 - \left( \frac{r_2 + \tau}{2a_{22}} \right)^2 \right\} \end{aligned}$$

$$+ y(\tau - d). \tag{3}$$

Now choose  $\tau < d$ . Therefore (3) leads to

$$\frac{dW}{dt} + \tau W \leq \frac{(r_1 + \tau)^2}{4a_{11}} + \frac{(r_2 + \tau)^2}{4a_{22}}.$$

By applying comparison theorem [16], we have

$$0 \leq W(x_1(t), x_2(t), y(t)) \leq \frac{(r_1 + \tau)^2}{4a_{11}\tau} + \frac{(r_2 + \tau)^2}{4a_{22}\tau} + \frac{W(x_1(0), x_2(0), y(0))}{e^{\tau t}}.$$

Passing to the limit as  $t \rightarrow \infty$ , we get

$$0 < W(t) \leq \frac{(r_1 + \tau)^2}{4a_{11}\tau} + \frac{(r_2 + \tau)^2}{4a_{22}\tau} = N \text{ (say).}$$

Hence system (2) is bounded. From above lemma we can conclude that  $x_1(t) \leq \frac{N}{c_1}$ ,  $x_2(t) \leq \frac{N}{c_2}$ , and  $y(t) \leq N$ .  $\square$

#### 4. Existence of Equilibria and Stability Analysis

##### 4.1. Existence of equilibria

Evidently, system (2) has six non-negative equilibrium points. The population free equilibrium point  $E_0 = (0, 0, 0)$ . The first prey only equilibrium point  $E_1 = (\frac{r_1}{a_{11}}, 0, 0)$ . The second prey only equilibrium point  $E_2 = (0, \frac{r_2}{a_{22}}, 0)$ . Here  $E_0$ ,  $E_1$ , and  $E_2$  always exist. The predator free equilibrium point  $E_{12} = (\bar{x}_1, \bar{x}_2, 0)$ . The equilibrium point  $E_{12}$  can be found in  $x_1 - x_2$  plane provided it satisfies the following equations:

$$r_1 - a_{11}x_1 - \frac{a_{12}x_2}{1 + a_1x_1} = 0, \tag{4}$$

$$r_2 - a_{22}x_2 - \frac{a_{21}x_1}{1 + a_2x_2} = 0. \tag{5}$$

From eq. (4), we find the value of  $x_2$  as

$$x_2 = \frac{(r_1 - a_{11}x_1)(1 + a_1x_1)}{a_{12}}. \tag{6}$$

Now using the value of  $x_2$  in eq. (5), we get the following equation in  $x_1$ ,

$$f(x_1) = \alpha_0x_1^4 + \alpha_1x_1^3 + \alpha_2x_1^2 + \alpha_3x_1 + \alpha_4 = 0 \tag{7}$$

where

$$\alpha_0 = a_2a_{22}a_{11}^2a_1^2,$$

$$\alpha_1 = 2a_1a_2a_{11}a_{22}(a_{11} - r_1a_1),$$

$$\alpha_2 = a_2a_{22} \left\{ (r_1a_{11} - a_{11})^2 - 2a_{11}a_1r_1 \right\} + a_{11}a_1a_{12}(r_2a_2 - a_{22}a_{12}),$$

$$\alpha_3 = a_2(r_1a_1 - a_{11})(2a_{22}r_1 - r_2a_{12}) + a_{12}^2a_{21},$$

$$\alpha_4 = (a_{22}r_1 - r_2a_2)(r_2a_2 + a_{12}).$$

Now  $f(0) = (a_{22}r_1 - r_2a_2)(r_2a_2 + a_{12})$  and  $f(\frac{r_1}{a_{11}}) = \frac{r_1a_{21} - r_2a_{11}}{a_{11}}$ . Clearly, eq. (7) has a positive root between 0 and  $\frac{r_1}{a_{11}}$  if  $f(0)$  and  $f(\frac{r_1}{a_{11}})$  are of opposite sign. We note that if

$$\frac{a_{12}}{a_{22}} > \frac{r_1}{r_2} > \frac{a_{11}}{a_{21}} \text{ or } \frac{a_{12}}{a_{22}} < \frac{r_1}{r_2} < \frac{a_{11}}{a_{21}} \tag{8}$$

then eq. (7) has a positive root. Thus  $E_{12}$  is feasible if (8) and  $r_1 > a_{11}\bar{x}_1$  hold.

The first prey and predator only equilibrium point  $E_{13} = (\hat{x}_1, 0, \hat{y})$  where  $\hat{x}_1 = \frac{d}{c_1p_1 - db_1}$  and  $\hat{y} = \frac{(r_1 - a_{11}\hat{x}_1)(1 + b_1\hat{x}_1)}{p_1}$ . It can be shown that  $E_{13}$  is feasible if  $db_1 < c_1p_1$  and  $r_1 > a_{11}\hat{x}_1$ .

The second prey and predator only equilibrium point  $E_{23} = (0, \tilde{x}_2, \tilde{y})$ . The equilibrium point  $E_{23}$  can be found in  $x_2 - y$  plane provided it satisfies the following equations:

$$r_2 - a_{22}x_2 - \frac{p_2y}{1 + b_2x_2} = 0, \tag{9}$$

$$-d + \frac{c_2p_2x_2}{1 + b_2x_2} - \frac{\alpha x_2}{1 + \beta y} = 0. \tag{10}$$

From eq. (9), we find the value of  $y$  as

$$y = \frac{(r_2 - a_{22}x_2)(1 + b_2x_2)}{p_2}.$$

Now using the value of  $y$  in eq. (10), we get the following equation in  $x_2$ ,

$$g(x_2) = \beta_0x_2^3 + \beta_1x_2^2 + \beta_2x_2 + \beta_3 = 0 \tag{11}$$

where

$$\beta_0 = \beta a_{22}(c_2p_2b_2 - d),$$

$$\beta_1 = d\beta(r_2 - 2a_{22}b_2) + p_2(\alpha b_2 + \beta c_2a_{22} - \beta c_2r_2b_2),$$

$$\beta_2 = p_2(db_2 + \alpha - c_2p_2 - c_2\beta r_2) + d\beta(2r_2b_2 - a_{22}),$$

$$\beta_3 = d(p_2 + \beta r_2).$$

We note that

$$g(0) = d(p_2 + \beta r_2) > 0 \text{ and}$$

$$g\left(\frac{r_2}{a_{22}}\right) = \frac{p_2}{a_{22}^2} \{ (da_{22} + \alpha r_2)(a_{22} + b_2r_2) - c_2p_2r_2a_{22} \}$$

Clearly,  $g(\frac{r_2}{a_{22}}) < 0$  if

$$(da_{22} + \alpha r_2)(a_{22} + b_2r_2) < c_2p_2r_2a_{22}. \tag{12}$$

If the inequality (12) hold then eq. (11) has a positive root  $\tilde{x}_2$  between 0 and  $\frac{r_2}{a_{22}}$ . Thus  $E_{23}$  is feasible if (12) and  $r_2 > a_{22}\tilde{x}_2$  hold.

To locate the coexistence equilibrium point  $E^* = (x_1^*, x_2^*, y^*)$  of system (2), we use isocline method.  $x_1^*$ ,  $x_2^*$  and  $y^*$  are the positive solutions of the following system of equations:

$$r_1 - a_{11}x_1 - \frac{a_{12}x_2}{1 + a_1x_1} - \frac{p_1y}{1 + b_1x_1} = 0, \tag{13}$$

$$r_2 - a_{22}x_2 - \frac{a_{21}x_1}{1 + a_2x_2} - \frac{p_2y}{1 + b_2x_2} = 0, \quad (14)$$

$$-d + \frac{c_1p_1x_1}{1 + b_1x_1} + \frac{c_2p_2x_2}{1 + b_2x_2} - \frac{\alpha x_2}{1 + \beta y} = 0. \quad (15)$$

From, equation (13), we get

$$y = \frac{(1 + b_1x_1)}{p_1} \left\{ r_1 - a_{11}x_1 - \frac{a_{12}x_2}{1 + a_1x_1} \right\} = y_e. \text{ (say)}$$

For positivity of  $y$ ,  $r_1 > a_{11}x_1 + \frac{a_{12}x_2}{1 + a_1x_1}$ . Now, we substitute the value of  $y$  in (14) and (15) and obtain

$$f_1(x_1, x_2) = r_2 - a_{22}x_2 - \frac{a_{21}x_1}{1 + a_2x_2} - \frac{p_2y_e}{1 + b_2x_2} = 0, \quad (16)$$

$$f_2(x_1, x_2) = -d + \frac{c_1p_1x_1}{1 + b_1x_1} + \frac{c_2p_2x_2}{1 + b_2x_2} - \frac{\alpha x_2}{1 + \beta y_e} = 0. \quad (17)$$

In eq. (16), when  $x_2 \rightarrow 0$ , we have  $r_2 - a_{21}x_1 - p_2y_e = 0$ . Now, we have

$$\frac{dx_1}{dx_2} = -\frac{\partial f_1}{\partial x_2} / \frac{\partial f_1}{\partial x_1} = -\frac{M_1}{N_1}$$

where

$$M_1 = -a_{22} + \frac{a_{21}a_2x_1}{(1 + a_2x_2)^2} + p_2 \left\{ \frac{a_{12}(1 + b_1x_1)}{p_1(1 + a_1x_1)(1 + b_2x_2)} + \frac{b_2y_e}{(1 + b_2x_2)^2} \right\},$$

$$N_1 = -\frac{a_{21}}{1 + a_2x_2} - \frac{p_2}{1 + b_2x_2} \left\{ \frac{b_1}{p_1} \left( r_1 - a_{11}x_1 - \frac{a_{12}x_2}{1 + a_1x_1} \right) + \frac{(1 + b_1x_1)}{p_1} \left( -a_{11} + \frac{a_{12}a_1x_1}{(1 + a_1x_1)^2} \right) \right\}.$$

It is clear that  $\frac{dx_1}{dx_2} > 0$  if either (i)  $M_1 > 0$  and  $N_1 < 0$  or (ii)  $M_1 < 0$  and  $N_1 > 0$  hold. Also we get,

$$\frac{dx_1}{dx_2} = -\frac{\partial f_2}{\partial x_2} / \frac{\partial f_2}{\partial x_1} = -\frac{M_2}{N_2}$$

where

$$M_2 = \frac{c_2p_2}{(1 + b_2x_2)^2} - \alpha \left\{ \frac{1}{1 + \beta y_e} + \frac{\beta a_{12}x_2(1 + b_1x_1)}{(1 + a_1x_1)(1 + \beta y_e)^2} \right\},$$

$$N_2 = \frac{c_1p_1}{(1 + b_1x_1)^2} + \alpha \beta x_2 \left\{ \frac{b_1}{p_1} \left( r_1 - a_{11}x_1 - \frac{a_{12}x_2}{1 + a_1x_1} \right) + \frac{(1 + b_1x_1)}{p_1} \left( -a_{11} + \frac{a_{12}a_1x_2}{(1 + a_1x_1)^2} \right) \right\}.$$

We note that  $\frac{dx_1}{dx_2} < 0$  if either (i)  $M_2 > 0$  and  $N_2 > 0$  or (ii)  $M_2 < 0$  and  $N_2 < 0$ .

From the above analysis, we conclude that the two isoclines (16) and (17) intersect at the point  $(x_1^*, x_2^*)$  under certain conditions. Throughout this paper we assume that  $E^*$  exists.

## 4.2. Stability analysis

The local stability properties of the equilibrium points can be determined through the Jacobian matrix around each equilibrium point. Clearly,  $E_0$  is always unstable. Otherwise, we have

- $E_1$  is locally stable if  $r_2 a_{11} < a_{21} r_1$  and  $d(a_{11} + b_1 r_1) > c_1 p_1 r_1$ .
- $E_2$  is locally stable if  $r_1 a_{22} < a_{12} r_2$  and  $(da_{22} + \alpha r_2)(a_{22} + b_2 r_2) > a_{22} c_2 p_2 r_2$ .
- $E_{12}$  is locally stable if  $d + \alpha \bar{x}_2 > \frac{c_1 p_1 \bar{x}_1}{1 + b_1 \bar{x}_1} + \frac{c_2 p_2 \bar{x}_2}{1 + b_2 \bar{x}_2}$  and  $a_{11} \bar{x}_1 + a_{22} \bar{x}_2 > \bar{x}_1 \bar{x}_2 \left\{ \frac{a_{12} a_1}{(1 + a_1 \bar{x}_1)^2} + \frac{a_{21} a_2}{(1 + a_2 \bar{x}_2)^2} \right\}$ .
- $E_{13}$  is locally stable if  $r_2 < a_{21} \hat{x}_1 + p_2 \hat{y}$  and  $a_{11} \hat{x}_1 > \frac{b_1 p_1 \hat{x}_1 \hat{y}}{(1 + b_1 \hat{x}_1)^2}$ .
- $E_{23}$  is locally stable if  $r_1 < a_{12} \tilde{x}_2 + p_1 \tilde{y}$ ,  $a_{22} > \frac{\tilde{y} \left\{ \frac{p_2 b_2}{(1 + b_2 \tilde{x}_2)^2} + \frac{\alpha \beta}{(1 + \beta \tilde{y})^2} \right\}}{1 + \beta \tilde{y}}$  and  $\frac{c_2 p_2}{(1 + b_2 \tilde{x}_2)^2} > \frac{\alpha}{1 + \beta \tilde{y}}$ .

To determine the stability of interior equilibrium point  $E^*$ , we find out the characteristic equation around  $E^*$  which is given by

$$\lambda^3 + \gamma_1 \lambda^2 + \gamma_2 \lambda + \gamma_3 = 0 \quad (18)$$

where

$$\gamma_1 = a_{11}x_1^* - \frac{a_{12}a_1x_1^*x_2^*}{(1 + a_1x_1^*)^2} - \frac{b_1p_1x_1^*y^*}{(1 + b_1x_1^*)^2} + a_{22}x_2^* - \frac{a_{21}a_2x_1^*x_2^*}{(1 + a_2x_2^*)^2} - \frac{b_2p_2x_2^*y^*}{(1 + b_2x_2^*)^2} - \frac{\alpha\beta x_2^*y^*}{(1 + \beta y^*)^2},$$

$$\gamma_2 = \left\{ a_{11}x_1^* - \frac{a_{12}a_1x_1^*x_2^*}{(1 + a_1x_1^*)^2} - \frac{b_1p_1x_1^*y^*}{(1 + b_1x_1^*)^2} \right\} \left\{ a_{22}x_2^* - \frac{a_{21}a_2x_1^*x_2^*}{(1 + a_2x_2^*)^2} - \frac{b_2p_2x_2^*y^*}{(1 + b_2x_2^*)^2} \right\} - \frac{\alpha\beta x_2^*y^*}{(1 + \beta y^*)^2} \left\{ a_{11}x_1^* - \frac{a_{12}a_1x_1^*x_2^*}{(1 + a_1x_1^*)^2} - \frac{b_1p_1x_1^*y^*}{(1 + b_1x_1^*)^2} \right\} + a_{22}x_2^* - \frac{a_{21}a_2x_1^*x_2^*}{(1 + a_2x_2^*)^2} - \frac{b_2p_2x_2^*y^*}{(1 + b_2x_2^*)^2} - \frac{p_2x_2^*y^*}{1 + b_2x_2^*} \left\{ \frac{c_2p_2}{(1 + b_2x_2^*)^2} - \frac{\alpha}{1 + \beta y^*} \right\} - \frac{a_{12}a_{21}x_1^*x_2^*}{(1 + a_1x_1^*)(1 + a_2x_2^*)} + \frac{c_1p_1^2x_1^*y^*}{(1 + b_1x_1^*)^3},$$

$$\gamma_3 = \frac{\alpha\beta x_2^*y^*}{(1 + \beta y^*)^2} - \frac{a_{12}a_{21}x_1^*x_2^*}{(1 + a_1x_1^*)(1 + a_2x_2^*)} - \left( a_{11}x_1^* - \frac{a_{12}a_1x_1^*x_2^*}{(1 + a_1x_1^*)^2} - \frac{b_1p_1x_1^*y^*}{(1 + b_1x_1^*)^2} \right) \left( a_{22}x_2^* - \frac{a_{21}a_2x_1^*x_2^*}{(1 + a_2x_2^*)^2} - \frac{b_2p_2x_2^*y^*}{(1 + b_2x_2^*)^2} \right) - \left( \frac{c_2p_2}{(1 + b_2x_2^*)^2} - \frac{\alpha}{1 + \beta y^*} \right)$$



$$\begin{aligned} & \left( \frac{p_2 x_2^* y^*}{1 + b_2 x_2^*} \left( a_{11} x_1^* - \frac{a_{12} a_1 x_1^* x_2^*}{(1 + a_1 x_1^*)^2} - \frac{b_1 p_1 x_1^* y^*}{(1 + b_1 x_1^*)^2} \right) \right. \\ & \left. + \frac{p_1 a_{21} x_1^* x_2^* y^*}{(1 + b_1 x_1^*)(1 + a_2 x_2^*)} \right) - \frac{1}{(1 + b_1 x_1^*)^3} \\ & \left( c_1 p_1^2 x_1^* \left( a_{22} x_2^* - \frac{a_{21} a_2 x_1^* x_2^*}{(1 + a_2 x_2^*)^2} - \frac{b_2 p_2 x_2^* y^*}{(1 + b_2 x_2^*)^2} \right) \right. \\ & \left. + \frac{a_{12} c_2 p_1 p_2 x_1^* x_2^* y^*}{1 + a_1 x_1^*} \right) \end{aligned}$$

From the Routh-Hurwitz criterion, we can say that  $E^*$  is locally asymptotically stable if the following conditions are satisfied.

$$\gamma_1 > 0, \gamma_3 > 0 \text{ and } \gamma_1 \gamma_2 > \gamma_3. \tag{19}$$

### 4.3. Persistence

If all the solutions of system (2) enter the compact region  $M \subset G = \{(x_1, x_2, y) : x_1 > 0, x_2 > 0, y > 0\}$  then the system is said to be persistent.

We now present persistence criterion.

**Proposition 1.** Suppose  $E_{12}, E_{13}$  and  $E_{23}$  exist. Further suppose that there are no limit cycles in  $x_1 - x_2, x_1 - y$  and  $x_2 - y$  plane. If  $r_2 a_{11} > a_{21} r_1, r_1 a_{22} > a_{12} r_2, d + \alpha \bar{x}_2 < \frac{c_1 p_1 \bar{x}_1}{1 + b_1 \bar{x}_1} + \frac{c_2 p_2 \bar{x}_2}{1 + b_2 \bar{x}_2}, r_2 > a_{21} \hat{x}_1 + p_2 \hat{y}$  and  $r_1 > a_{12} \tilde{x}_2 + p_1 \tilde{y}$  then system (2) is uniformly persistent.

*Proof.* Proceeding along the lines in [17], we can prove the theorem and is deleted here.  $\square$

**Remark 1.** System (2) is uniformly persistent when the conditions in Proposition 1 are satisfied. Thus we infer that there exist a time  $T$  such that  $x_1(t), x_2(t), y(t) > K, 0 < K < \frac{r_1}{a_{11}}$  for  $t > T$ .

**Remark 2.** If there are finite number of limit cycles, then persistence conditions in Proposition 1 becomes

$$\begin{aligned} & \int_0^\sigma \left( -d + \frac{c_1 p_1 \bar{\phi}(t)}{1 + b_1 \bar{\phi}(t)} + \frac{c_2 p_2 \bar{\psi}(t)}{1 + b_2 \bar{\psi}(t)} - \alpha \bar{\psi}(t) \right) dt > 0, \\ & \int_0^\sigma \left( r_2 - a_{21} \hat{\phi}(t) - p_2 \hat{\psi}(t) \right) dt > 0, \\ & \int_0^\sigma \left( r_1 - a_{12} \tilde{\phi}(t) - p_1 \tilde{\psi}(t) \right) dt > 0. \end{aligned}$$

For each limit cycle,  $(\bar{\phi}(t), \bar{\psi}(t))$  in the  $x_1 - x_2$  plane,  $(\hat{\phi}(t), \hat{\psi}(t))$  in the  $x_1 - y$  plane and  $(\tilde{\phi}(t), \tilde{\psi}(t))$  in the  $x_2 - y$  respectively where  $\sigma$  is the appropriate period.

## 5. Global Stability Analysis

We have already observed that the coexistence equilibrium point  $E^*$  will be locally stable when the inequalities (19) are sat-

isfied. So it will be of interest to know whether this equilibrium point is globally stable or not. Usually Lyapunov function is used to examine the global stability. But it is not always possible to find a suitable Lyapunov function to prove global stability. In that case, an alternative approach developed in [18] is used. Now we apply a high-dimensional Bendixson’s criterion of Li and Muldowney [18], which is demonstrated below.

Let  $D \subset \mathbb{R}^n$  be an open set and  $F \in C^1$ . Consider a system of differential equations

$$\frac{dX}{dt} = F(X). \tag{20}$$

According to the theory described in [18], it is sufficient to show that the second compound equation

$$\frac{dU}{dt} = \frac{\partial F^{[2]}}{\partial X}(X(t, X_0)) U(t) \tag{21}$$

with respect to a solution  $X(t, X_0)$  of system (20) is equi-uniformly asymptotically stable, namely, for each  $X_0 \in D$ , system (21) is uniformly asymptotically stable and the exponential decay rate is uniform for  $X_0$  in each compact subset of  $D$ , where  $D \subset \mathbb{R}^n$  is an open connected set. Here  $\partial F / \partial X^{[2]}$  is the second additive compound matrix of the Jacobian matrix  $\partial F^{[2]} / \partial X$ . It is an  $\binom{n}{2} \times \binom{n}{2}$  matrix and thus (21) is a linear system of dimension  $\binom{n}{2}$  (see Fiedler [19] and Muldowney [20]). For a general  $3 \times 3$  matrix

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix},$$

Its compound matrix  $M^{[2]}$  is

$$M^{[2]} = \begin{pmatrix} m_{11} + m_{22} & m_{23} & -m_{13} \\ m_{32} & m_{11} + m_{33} & m_{12} \\ -m_{31} & m_{21} & m_{22} + m_{33} \end{pmatrix}. \tag{22}$$

The equi-uniform asymptotic stability of (21) implies the exponential decay of the surface area of any compact two-dimensional surface  $D$ . If  $D$  is simply connected, then it will not allow any invariant simple closed rectifiable curve in  $D$ , including periodic orbits. The following result is proved in [18].

**Proposition 2.** Let  $D \subset \mathbb{R}^n$  be simply connected region. Assume that the family of linear systems (21) is equi-uniformly asymptotically stable. Then

- (i)  $D$  contains no simple closed invariant curves including periodic orbits, homoclinic orbits, heteroclinic cycles;
- (ii) each semi-orbit in  $D$  converges to a single equilibrium.

In particular, if  $D$  is positively invariant and contains an unique equilibrium  $\bar{X}$ , then  $\bar{X}$  is globally asymptotically stable in  $D$ .

One can prove uniform asymptotic stability of system (21) by constructing a Lyapunov function. For example, (21) is equi-uniformly asymptotically stable if there exists a positive definite function  $V(U)$ , such that  $dV(U)/dt|_{(21)}$  is negative definite and  $V$  and  $dV(U)/dt|_{(21)}$  are independent of  $X_0$ .

We need the following assumptions to show the global stability of the coexistence equilibrium point  $E^*$  of system (2).

(A<sub>1</sub>) There exist positive numbers  $\rho$  and  $\eta$  such that

$$\max \left\{ c_{11} + \frac{c_{12}\rho}{\eta} + c_{13}\rho, \frac{c_{21}\eta}{\rho} + c_{22} + c_{23}\eta, \frac{c_{31}}{\rho} + \frac{c_{32}}{\eta} + c_{33} \right\} < 0$$

(A<sub>2</sub>) All the assumptions of Proposition 1 hold.

We again denote  $X = (x_1, x_2, y)^T$  and

$$F(X) = \begin{pmatrix} x_1 \left( r_1 - a_{11}x_1 - \frac{a_{12}x_2}{1 + a_1x_1} - \frac{p_1y}{1 + b_1x_1} \right), \\ x_2 \left( r_2 - a_{22}x_2 - \frac{a_{21}x_1}{1 + a_2x_2} - \frac{p_2y}{1 + b_2x_2} \right), \\ y \left( -d + \frac{c_1p_1x_1}{1 + b_1x_1} + \frac{c_2p_2x_2}{1 + b_2x_2} - \frac{\alpha x_2}{1 + \beta y} \right) \end{pmatrix}^T$$

and by (22)

$$\frac{\partial F^{[2]}}{\partial X} = \begin{pmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{pmatrix}$$

where

$$\begin{aligned} n_{11} &= r_1 + r_2 - 2(a_{11}x_1 + a_{22}x_2) - \frac{a_{12}x_2}{(1 + a_1x_1)^2} \\ &\quad - \frac{a_{21}x_1}{(1 + a_2x_2)^2} - \frac{p_1y}{(1 + b_1x_1)^2} - \frac{p_2y}{(1 + b_2x_2)^2}, \\ n_{12} &= -\frac{p_2x_2}{1 + b_2x_2}, n_{13} = \frac{p_1x_1}{1 + b_1x_1}, \\ n_{21} &= y \left\{ \frac{c_2p_2}{(1 + b_2x_2)^2} - \frac{\alpha}{1 + \beta y} \right\}, \\ n_{22} &= r_1 - 2a_{11}x_1 - \frac{a_{12}x_2}{(1 + a_1x_1)^2} - \frac{p_1y}{(1 + b_1x_1)^2} - d \\ &\quad + \frac{c_1p_1x_1}{1 + b_1x_1} + \frac{c_2p_2x_2}{1 + b_2x_2} - \frac{\alpha x_2}{(1 + \beta y)^2}, \\ n_{23} &= -\frac{a_{12}x_1}{1 + a_1x_1}, \\ n_{31} &= -\frac{c_1p_1y}{(1 + b_1x_1)^2}, \\ n_{32} &= -\frac{a_{21}x_2}{1 + a_2x_2}, \\ n_{33} &= r_2 - 2a_{22}x_2 - \frac{a_{21}x_1}{(1 + a_2x_2)^2} - \frac{p_2y}{(1 + b_2x_2)^2} - d \\ &\quad + \frac{c_1p_1x_1}{1 + b_1x_1} + \frac{c_2p_2x_2}{1 + b_2x_2} - \frac{\alpha x_2}{(1 + \beta y)^2}. \end{aligned}$$

The second compound system is

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{pmatrix} = \frac{\partial F^{[2]}}{\partial X} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

is

$$\dot{u}_1 = \left\{ r_1 + r_2 - 2(a_{11}x_1 + a_{22}x_2) - \frac{a_{12}x_2}{(1 + a_1x_1)^2} \right.$$

$$\begin{aligned} &\left. - \frac{a_{21}x_1}{(1 + a_2x_2)^2} - \frac{p_1y}{(1 + b_1x_1)^2} - \frac{p_2y}{(1 + b_2x_2)^2} \right\} u_1 \\ &- \frac{p_2x_2}{1 + b_2x_2} u_2 + \frac{p_1x_1}{1 + b_1x_1} u_3, \\ \dot{u}_2 &= y \left\{ \frac{c_2p_2}{(1 + b_2x_2)^2} - \frac{\alpha}{1 + \beta y} \right\} u_1 \\ &+ \left\{ r_1 - 2a_{11}x_1 - \frac{a_{12}x_2}{(1 + a_1x_1)^2} - \frac{p_1y}{(1 + b_1x_1)^2} - d \right. \\ &\left. + \frac{c_1p_1x_1}{1 + b_1x_1} + \frac{c_2p_2x_2}{1 + b_2x_2} - \frac{\alpha x_2}{(1 + \beta y)^2} \right\} u_2 \\ &- \frac{a_{12}x_1}{1 + a_1x_1} u_3, \\ \dot{u}_3 &= -\frac{c_1p_1y}{(1 + b_1x_1)^2} u_1 - \frac{a_{21}x_2}{1 + a_2x_2} u_2 \\ &+ \left\{ r_2 - 2a_{22}x_2 - \frac{a_{21}x_1}{(1 + a_2x_2)^2} - \frac{p_2y}{(1 + b_2x_2)^2} \right. \\ &\left. - d + \frac{c_1p_1x_1}{1 + b_1x_1} + \frac{c_2p_2x_2}{1 + b_2x_2} - \frac{\alpha x_2}{(1 + \beta y)^2} \right\} u_3. \end{aligned} \tag{23}$$

where  $X(t) = (x_1(t), x_2(t), y(t))^T$  is arbitrary solution of system (2) with  $X_0(t) = (x_{10}(t), x_{20}(t), y_0(t))^T \in \mathbb{R}_+^3$ . Set  $W(U) = \max\{\rho|u_1|, \eta|u_2|, |u_3|\}$  where  $\rho, \eta > 0$  are constants. The direct calculations yield the following inequalities:

$$\begin{aligned} \frac{d^+}{dt} \rho|u_1| &\leq c_{11}\rho|u_1| + \frac{c_{12}\rho}{\eta}|u_2| + c_{13}\rho|u_3|, \\ \frac{d^+}{dt} \eta|u_2| &\leq \frac{c_{21}\eta\rho}{\rho}|u_1| + c_{22}\eta|u_2| + c_{23}|u_3|\eta, \\ \frac{d^+}{dt} |u_3| &\leq \frac{c_{31}\rho}{\rho}|u_1| + \frac{c_{32}\eta}{\eta}|u_2| + c_{33}|u_3|, \end{aligned}$$

where  $d^+/dt$  represents the right hand derivative and

$$\begin{aligned} c_{11} &= r_1 + r_2 - 2K(a_{11} + a_{22}) - \frac{Ka_{12}c_1^2}{(c_1 + a_1N)^2} \\ &\quad - \frac{Ka_{21}c_2^2}{(c_2 + a_2N)^2} - \frac{Kp_1c_1^2}{(c_1 + b_1N)^2} - \frac{Kp_2c_2^2}{(c_2 + b_2N)^2}, \\ c_{12} &= -\frac{Kp_2c_2}{c_2 + b_2N}, \\ c_{13} &= \frac{NP_1}{c_1(1 + b_1K)}, \\ c_{21} &= \frac{Nc_2p_2}{(1 + b_2K)^2} - \frac{\alpha K}{1 + \beta N}, \\ c_{22} &= r_1 - 2a_{11}K - \frac{Ka_{12}c_1^2}{(c_1 + a_1N)^2} - \frac{Kp_1c_1^2}{(c_1 + b_1N)^2} \\ &\quad - d + \frac{p_1N}{1 + b_1K} + \frac{p_2N}{1 + b_2N} - \frac{\alpha K}{(1 + \beta N)^2}, \\ c_{23} &= -\frac{Ka_{12}c_1}{c_1 + a_1N}, \\ c_{31} &= -\frac{Kp_1c_1^2}{(c_1 + b_1N)^2}, \end{aligned}$$

$$c_{32} = -\frac{K a_{21} c_2}{c_2 + a_2 N},$$

$$c_{33} = r_2 - 2a_{22}K - \frac{K a_{21} c_2^2}{(c_2 + a_2 N)^2} - \frac{K p_2 c_2^2}{(c_2 + b_2 N)^2} - d + \frac{p_1 N}{1 + b_1 K} + \frac{p_2 N}{1 + b_2 N} - \frac{\alpha K}{(1 + \beta N)^2}.$$

Therefore,

$$\frac{d^+}{dt} W(U(t)) \leq LW(U(t))$$

with

$$L = \max \left\{ c_{11} + \frac{c_{12} \rho}{\eta} + c_{13} \rho, \frac{c_{21} \eta}{\rho} + c_{22} + c_{23} \eta, \frac{c_{31}}{\rho} + \frac{c_{32}}{\eta} + c_{33} \right\}.$$

Thus under assumptions  $(A_1)$  and  $(A_2)$ , we find a positive constant  $\delta$  such that  $L \leq -\delta < 0$  and thus

$$W(U(t)) \leq W(U(s)) \exp(-\delta(t-s)), t \geq s > 0.$$

This ensures the equi-uniform asymptotic stability of the second compound system (23) and hence the coexistence equilibrium point  $E^*$  is globally stable following Proposition 2. From above analysis, we now state our global stability result.

**Theorem 1.** *If the assumptions  $(A_1)$  and  $(A_2)$  hold then system (2) has no non-trivial periodic solutions. Furthermore, the coexistence equilibrium point  $E^*$  is globally stable in  $\mathbb{R}_+^3$ .*

### 6. Bifurcation Study

$$\text{Set } h(\alpha) = \gamma_1(\alpha)\gamma_2(\alpha) - \gamma_3(\alpha).$$

**Theorem 2.** *If there exists  $\alpha = \alpha^*$  such that (i)  $\gamma_i(\alpha^*) > 0, i = 1, 2, 3$ , (ii)  $h(\alpha^*) = 0$ , (iii)  $h'(\alpha^*) > 0$  then the positive equilibrium point  $E^*$  is unstable if  $\alpha < \alpha^*$  but is stable for  $\alpha > \alpha^*$  and a Hopf bifurcation of periodic solution appears at  $\alpha = \alpha^*$ .*

*Proof.* Proceeding along the lines in [21], we can prove the theorem and is deleted here.  $\square$

### 7. Numerical Simulations

In this section, we will discuss some examples to validate our results found in this paper. Numerical simulations are carried out with the help of a Matlab software package for a hypothetical set of data.

**Example 1.** Suppose  $r_1 = 2, a_{11} = 0.6, a_{12} = 1, a_1 = 0, p_1 = 1, b_1 = 0.1, r_2 = 4, a_{22} = 3, a_{21} = 2, a_2 = 0, p_2 = 0.1, b_2 = 0.1, d = 0.5, c_1 = 1, c_2 = 0.1, \alpha = 0$  and  $\beta = 0$ . In absence of interfering time and anti-predator behavior, an oscillation is observed in the system around the equilibrium point  $E^*(0.0211, 1.3038, 0.4755)$  (see Figure 1).

**Example 2.** Suppose  $r_1 = 2, a_{11} = 0.6, a_{12} = 1, a_1 = 3, p_1 = 1, b_1 = 0.1, r_2 = 4, a_{22} = 3, a_{21} = 2, a_2 = 1, p_2 = 0.1, b_2 = 0.1, d = 0.5, c_1 = 1, c_2 = 0.1, \alpha = 0.1$  and  $\beta = 0.1$ . In presence of interfering time and anti-predator behavior, an oscillation persist in the system around the equilibrium point  $E^*(1.8581, 0.7916, 0.3890)$ (see Figure 2).

**Example 3.** Taking  $\alpha = 0.2$ , keeping all other parameters in Example 2, unchanged, we observe multiple limit cycles surrounding the equilibrium point  $E^*(0.4992, 1.1764, 0.7116)$  (see Figure 3).

**Example 4.** Taking  $\alpha = 0.8$ , keeping all other parameters in Example 2, unchanged, a stable behavior is observed and the solutions converge to the equilibrium point  $E^*(1.2363, 0.8515, 1.2080)$  (see Figure 4). Bifurcation diagram with respect to the parameter  $\alpha$  is depicted in Figure 5.

**Example 5.** Suppose  $r_1 = 2, a_{11} = 1.5, a_{12} = 1, a_1 = 3, p_1 = 1, b_1 = 0.1, r_2 = 4, a_{22} = 3, a_{21} = 2, a_2 = 1, p_2 = 0.1, b_2 = 0.1, d = 0.5, c_1 = 1, c_2 = 0.1, \alpha = 0.1$  and  $\beta = 0.1$ . Then, system (2) has an equilibrium point  $E^*(0.6314, 1.1128, 0.7106)$ . Conditions of Proposition 1 are satisfied, hence system (2) is uniformly persistent. We now choose  $K = 1$ . With the above choice of parameters, we obtain  $c_{11} = -3.8781, c_{12} = -0.028, c_{13} = 2.336, c_{21} = -0.0583, c_{22} = 0.3134, c_{23} = -0.1148, c_{31} = -0.6328, c_{32} = -0.0749, c_{33} = -0.0334$ . the positive numbers  $\rho = 1, \eta = 4$  such that  $\max\{-1.5491, -0.3790, -0.6849\} < 0$ . Therefore  $E^*$  is globally stable (see Figure 6).

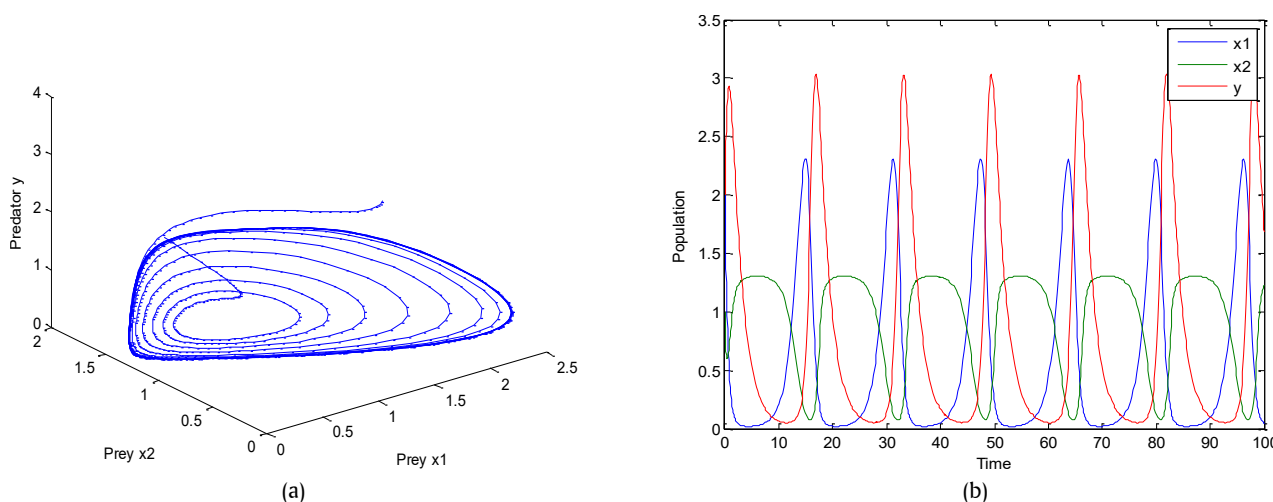
**Example 6.** Suppose  $r_1 = 2, a_{11} = 0.6, a_{12} = 1, p_1 = 1, b_1 = 0.1, r_2 = 4, a_{22} = 3, a_{21} = 2, a_2 = 1, p_2 = 0.1, b_2 = 0.1, d = 0.5, c_1 = 1, c_2 = 0.1, \alpha = 0.1$  and  $\beta = 0.1$ . Bifurcation diagram with respect to the parameter  $a_1$  is depicted in Figure 7.

### 8. Discussion

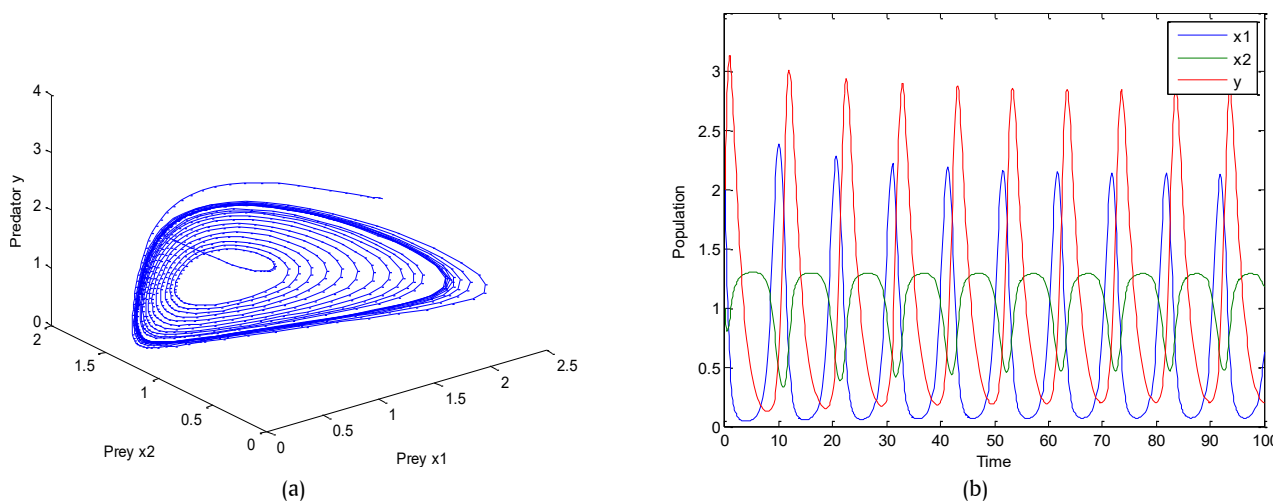
In this paper, we have proposed and analyzed the dynamical behavior of two competing prey-one predator model where competition process obeys Holling type II competitive response to interfering time and anti-predator behavior. Here we have assumed that the prey (superior competitor) can counter attack their predators. There is an upper threshold value of the anti-predator efficiency of the prey when predator density increases. Predation process follows Holling type II response function.

For biological reasons, we have shown positivity and boundedness of solutions. The existence of all possible steady

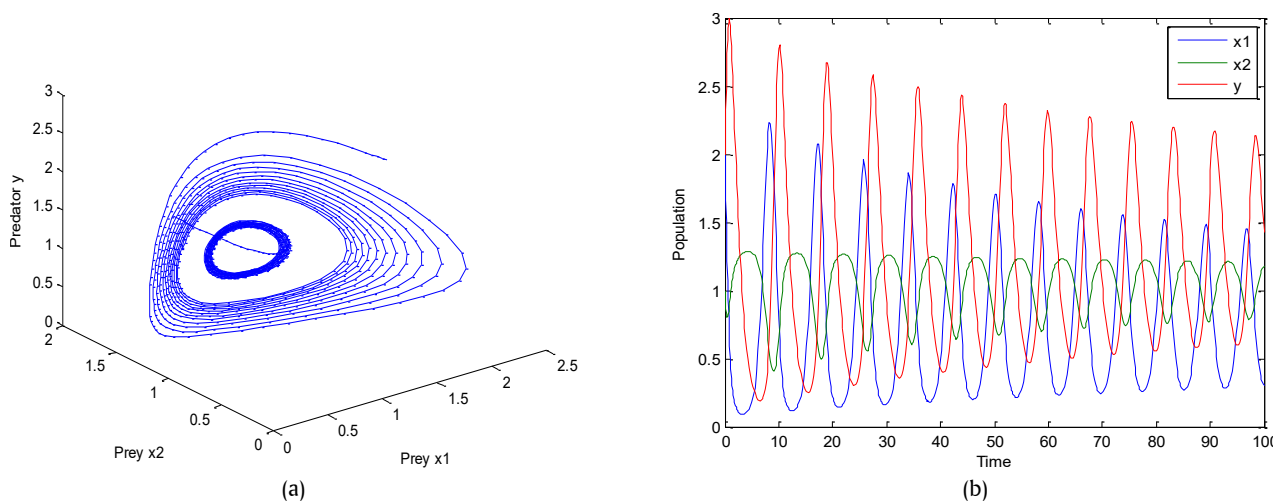




**Figure 1.** Phase portrait along with time series plot of system (2) for parameter values  $r_1 = 2, a_{11} = 0.6, a_{12} = 1, a_1 = 0, p_1 = 1, b_1 = 0.1, r_2 = 4, a_{22} = 3, a_{21} = 2, a_2 = 0, p_2 = 0.1, b_2 = 0.1, d = 0.5, c_1 = 1, c_2 = 0.1, \alpha = 0$  and  $\beta = 0$



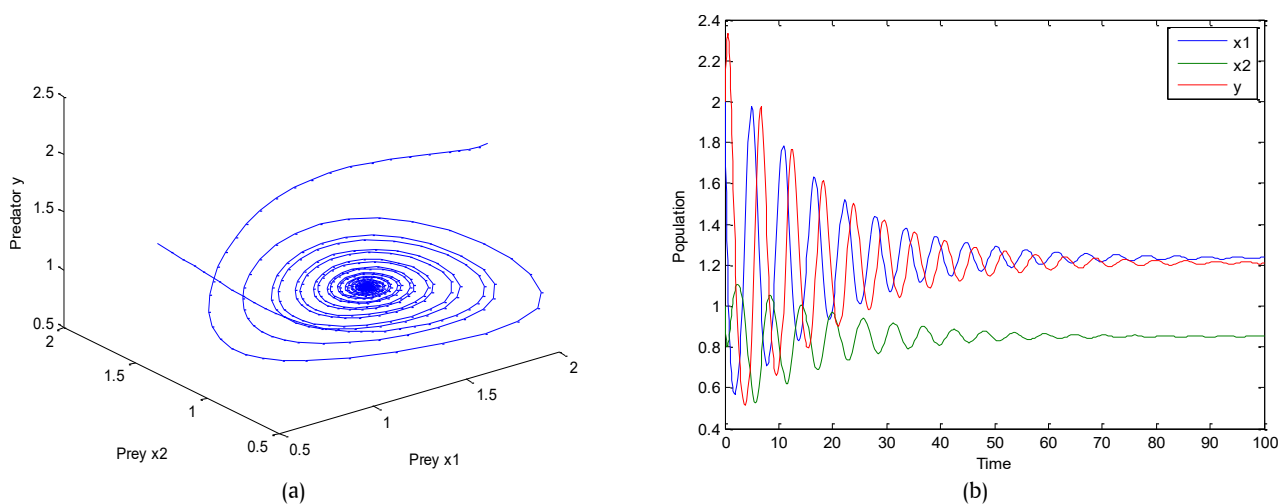
**Figure 2.** Phase portrait along with time series plot of system (2) for parameter values  $r_1 = 2, a_{11} = 0.6, a_{12} = 1, a_1 = 3, p_1 = 1, b_1 = 0.1, r_2 = 4, a_{22} = 3, a_{21} = 2, a_2 = 1, p_2 = 0.1, b_2 = 0.1, d = 0.5, c_1 = 1, c_2 = 0.1, \alpha = 0.1$  and  $\beta = 0.1$



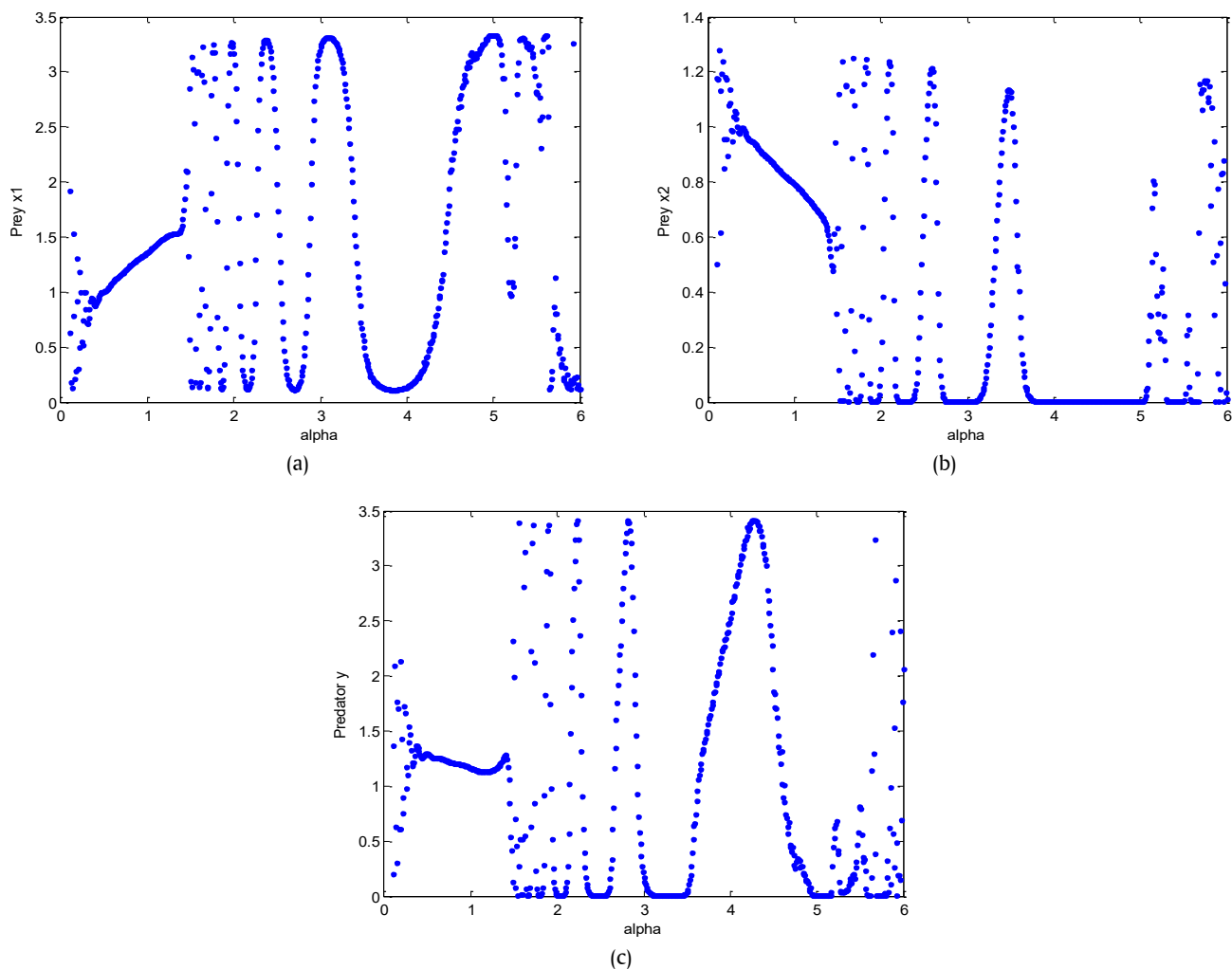
**Figure 3.** Phase portrait along with time series plot of system (2) for parameter values  $r_1 = 2, a_{11} = 0.6, a_{12} = 1, a_1 = 3, p_1 = 1, b_1 = 0.1, r_2 = 4, a_{22} = 3, a_{21} = 2, a_2 = 1, p_2 = 0.1, b_2 = 0.1, d = 0.5, c_1 = 1, c_2 = 0.1, \alpha = 0.2$  and  $\beta = 0.1$

states is described. We have pointed out the existence criteria for positive equilibrium point by isoclines method. Though, the

uniform persistence criterion can also ensure the existence of the positive equilibrium point. Still, it is very difficult to find the co-



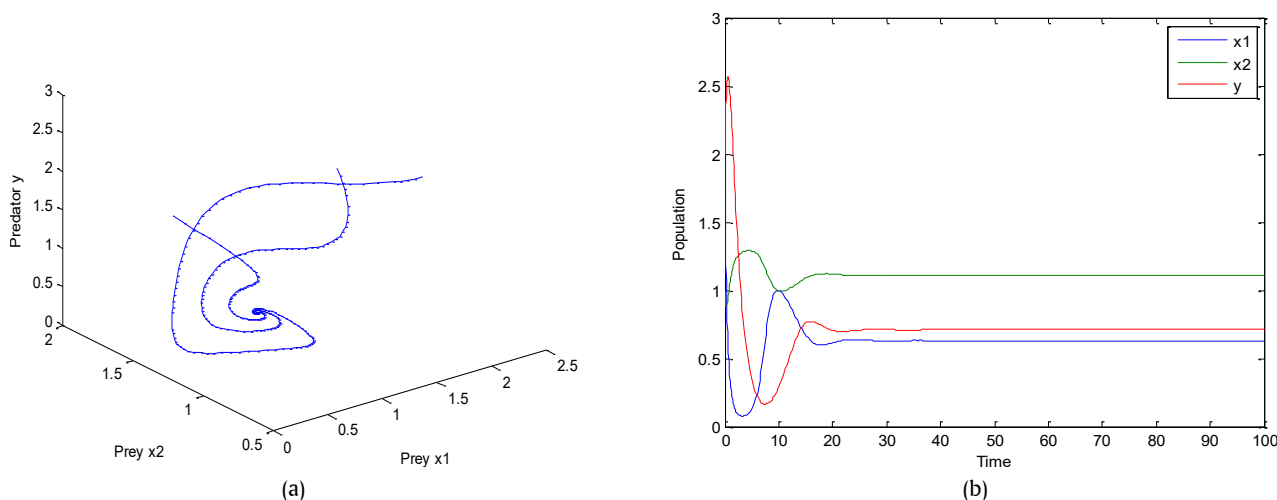
**Figure 4.** Phase portrait along with time series plot of system (2) for parameter values  $r_1 = 2, a_{11} = 0.6, a_{12} = 1, a_1 = 3, p_1 = 1, b_1 = 0.1, r_2 = 4, a_{22} = 3, a_{21} = 2, a_2 = 1, p_2 = 0.1, b_2 = 0.1, d = 0.5, c_1 = 1, c_2 = 0.1, \alpha = 0.8$  and  $\beta = 0.1$ .



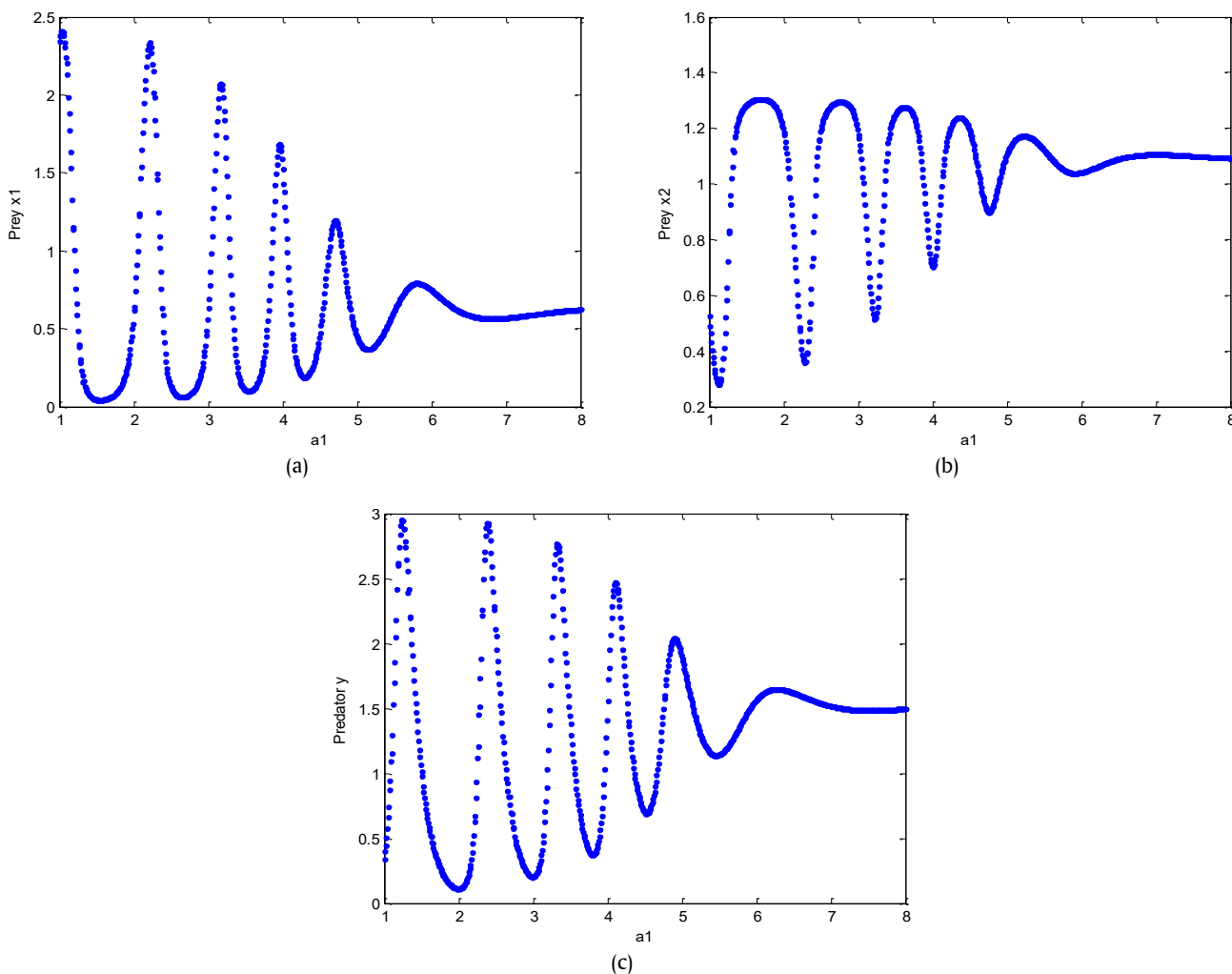
**Figure 5.** Bifurcation diagram with respect to the parameter  $\alpha$  where other parameter values are  $r_1 = 2, a_{11} = 0.6, a_{12} = 1, a_1 = 3, p_1 = 1, b_1 = 0.1, r_2 = 4, a_{22} = 3, a_{21} = 2, a_2 = 1, p_2 = 0.1, b_2 = 0.1, d = 0.5, c_1 = 1, c_2 = 0.1$  and  $\beta = 0.1$

ordinates of the positive equilibrium point in a specific form in system parameters. As it is known to us that if the positive equilibrium point is globally stable it must be unique. To examine the uniqueness of the positive equilibrium point, we have devel-

oped the global stability criterion by the use of high-dimensional Bendixson’s criterion due to Li and Muldowney [18]. By choosing ant-predator behavior rate  $\alpha$  as bifurcation parameter, we have shown the existence of limit cycles emerging through Hopf bi-



**Figure 6.** Phase portrait along with time series plot of system (2) for parameter values  $r_1 = 2, a_{11} = 1.5, a_{12} = 1, a_1 = 3, p_1 = 1, b_1 = 0.1, r_2 = 4, a_{22} = 3, a_{21} = 2, a_2 = 1, p_2 = 0.1, b_2 = 0.1, d = 0.5, c_1 = 1, c_2 = 0.1, \alpha = 0.1$  and  $\beta = 0.1$



**Figure 7.** Bifurcation diagram with respect to the parameter  $a_1$  where other parameter values are  $r_1 = 2, a_{11} = 0.6, a_{12} = 1, p_1 = 1, b_1 = 0.1, r_2 = 4, a_{22} = 3, a_{21} = 2, a_2 = 1, p_2 = 0.1, b_2 = 0.1, d = 0.5, c_1 = 1, c_2 = 0.1, \alpha = 0.1$  and  $\beta = 0.1$

furcation. Bifurcation diagram with respect to the parameter is shown in Figure 5. From numerical simulation, we observe similar phenomena with respect to the parameter  $a_1$  and is depicted in Figure 7.

Deka et al. [14], Fujii [12] and Takeuchi and Adachi [13] addressed an ecological system with the same type of species, but no interfering time to competitive response and anti-predator behavior for obtaining coexistence results. Finally, we note that if

competing takes time to both competing species, then competition pressure becomes low, which enhances the coexistence when there is no predator. But in the presence of predators along with the anti-predator behavior of prey, whether the coexistence is possible or not is chiefly depend on the preference of the predator. It is noted that due to anti-predator behavior, the growth of prey (inferior competitor) species increases while the growth of prey (superior competitor) species and predator species decrease. If the prey can further increase their anti-predator behavior, the predator population can persist with stable, positive equilibrium as there is a choice of the other prey in the system.

The main novelty in our work is the inclusion of competition term other than the classical competition law and anti-predator behavior of prey, which are not considered in [12–14].

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