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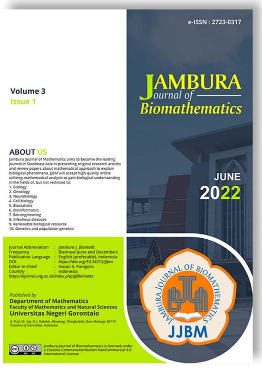

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Fear induced dynamics on Leslie-Gower predator-prey system with Holling-type IV functional response

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ABSTRACT. *This paper analyzes the effect of fear in a Leslie-Gower predator-prey system with Holling type IV functional response. Firstly, we show positivity and boundedness of the system. Then we discuss the structure of the positive equilibrium point, dynamical behavior of all the steady states and long term survival of all the populations in the system. It is shown that fear factor has an impact on the prey and predator equilibrium densities. We have shown the occurrence of transcritical bifurcation around the axial steady state. The presence of a Hopf bifurcation near the interior steady state has been developed by choosing the level of fear as a bifurcation parameter. Furthermore, we discuss the character of the limit cycle generated by Hopf bifurcation. A global stability criterion of the positive steady state point is derived. Numerically, we checked our analytical findings.*

1. Introduction

Predator-prey systems have been investigated extensively for over a hundred years. Most of the existing predator-prey models are based on Lotka-Volterra formalism. The functional response is a basic characteristic in any predator-prey interactions as it establishes the link between trophic levels. It describes the variation of biomass of the prey related to the predator biomass when prey biomass modify. In [1], the author introduced three types of functional response to various types of species to address the role of predation, which modified the well-known Lotka-Volterra system. There is an another type of functional response, Holling type IV suggested by Andrews [2] that is different from the above types on a group defense capability and does not follow monotonic response. This group defense activity demonstrates that when the biomass of prey attains an high degree, the prey increases its protection to limit the rising rate of predator. Such a situation is observed in aquatic ecosystem where aquatic snails (*Nucella lamellosa)* protect themselves by solidifying shells [3] due to predation by the crabs. As defense capability has a major influence on the reproduction of the prey, numerous studies on this issue on predator-prey interactions have been carried out.

In predator-prey model, Leslie-Gower [4] suggested another aspect which refers to the fact that the interrelated population develop following the logistic rule and that the carrying capacity for the predator depends on the number of prey. In case of scarcity of prey , predator can move for alternative food, but its development will be restricted as their primary prey is not available. To overcome this case, in [5], the authors suggested an improved Leslie-Gower model by incorporating a positive quantity that estimates the environmental defense for the predator. After then, several authors have investigated the Leslie-Gower model with different kinds of functional response [6–14].

In studying predator-prey interactions, it is usually observed that predator affects prey by direct killing only. But recent field studies [15] shows that predator can affect the behavior of prey by imposing fear on them. As a result, all creatures exhibit anti-predator defense by changing their dwelling, surveillance, hunting activity and biological conversion [6, 7, 16, 17]. To lessen the predation pressure, prey move from their habitat from a high risk area in a low risk area. Furthermore, the fearful prey may forage less which in turn cause hunger and less reproduction [18, 19]. For instance, mule deer limit foraging activities when lions predate them [6]. Elk modifies reproductive mechanism when wolves attack them [18]. Based on the experimental work [15], in [20], the authors studied the role of fear term in the prey's growth rate in a predator-prey system and showed the stabilizing effect of fear on the system. Subsequently, several models on prey-predator interaction considering the fear effect is formulated and analyzed [21–32].

Above investigations indicate that the fear can diminish prey reproduction as well as predator population, the density of predators can be regulated by extra food. So it is reasonable to study the predator-prey interaction, allowing fear factor and the alternative diet source for the predator. The above discussions motivated us to develop a system that incorporates both the aspects.

The main goal in this work is to address, the fear effect on a predator-prey system in the form of modified Leslie-Gower model with Holling type IV schemes.

The article is organized as follows. In Section 2, we propose our model. Positivity, boundedness, the existence of equilibria, stability, bifurcation, uniform persistence and the influ-

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ence of fear factor on the prey and predator equilibrium densities are discussed in the same section. A criterion for global stability is derived in Section 3. We verify our results numerically in Section 4. Finally, we give precise discussion in Section 5.

2. The Model

In this part, we develop predator-prey model incorporating the following issues: (i) modified Leslie-Gower system with Holling type IV schemes [33], (ii) fear effect [20]. Recent field experiments showed that fear factor can reduce the production of prey populations. This motivated Wang et al. [20] to introduce the fear factor $\frac{1}{1+ky}$ in predator-prey model, after then it is used in [23, 25–27]. For more details, see Wang et al. [20]. Based on the above facts, we propose the following model:

$$\begin{aligned} \frac{dx}{dt} &= \frac{rx}{1+ky} - \alpha x^2 - \frac{pxy}{a+bx+x^2} := f_1(x,\ y), \\ \frac{dy}{dt} &= y\left(h - \frac{qy}{x+c}\right) := \ f_2(x,\ y), \end{aligned} \tag{1}$$

with $x(0) = x_0 > 0$ and $y(0) = y_0 > 0$. Here $x(t)$ and $y(t)$ stands for the density of prey and predator species at time t, respectively. $\frac{1}{1+ky}$ stands for the fear effect on the natural growth rate of prey population and k represents the level of fear. α is the intraspecific competition coefficient among the prey species. The non-monotonic functional response [34] $g(x) = \frac{px}{a+bx+x^2}$, where p and a are positive constants, and $b > -\sqrt{2}a$ (so that $a + bx + x^2 > 0$ for all $x \geq 0$ and hence $g(x) > 0$ for all $x > 0$) which describes the antipredator behavior (APB) phenomenon called defense group formation [35, 36], or else, the phenomenon of aggregation [37], or inhibitory effects [38].

The alternative food for predators is represented by the addition of a parameter $c > 0$, in the variable environmental carrying capacity of predators. Implicitly, this implies the predators are generalist [39]. r and h represent the intrinsic growth rate of the prey and the predator species respectively. p is the consumption rate. a is a positive constant. q is an estimate of the food standard that the prey supplies for transformation into the predator's birth. System parameters r, k, h, α, and q are considered to be positive.

2.1. *Positivity and boundedness of solutions*

In this part, we first examine positivity and boundedness of (1). These are crucial as they relate to the biological validation. We first show the positivity.

Lemma 1. *All solutions $(x(t), y(t))$ of system (1) with initial values $(x_0, y_0) \in \mathbb{R}^2_+$ will be positive for all $t > 0$.*

Proof. The positivity of $x(t)$ and $y(t)$ can be checked by the following equations:

$$\begin{aligned} x(t) = x_0 \exp\left\{\int_0^t \left[\frac{r}{1+ky(s)} - \alpha x(s) - \frac{py(s)}{a} + bx(s)\right.\right. \\ \left.\left. + x^2(s)\right] ds\right\} \end{aligned}$$

$$y(t) = y_0 \exp \int_0^t \left[h - \frac{qy(s)}{x(s)+c}\right] ds$$

with $x_0, y_0 > 0$. As $x_0 > 0$ then $x(t) > 0$ for all $t > 0$. Similarly, we can show that $y(t) > 0$. □

Lemma 2. *All solutions (1) that initiate in $\mathbb{R}^2_+$ will enter the set*

$$B = \left\{(x,\ y) \in \mathbb{R}^2_+ : x \leq \frac{r}{\alpha},\ y \leq \frac{h(r+c\alpha)}{q\alpha}\right\}.$$

Proof. From the first equation of (1), we have

$$\frac{dx}{dt} \leq x(r - \alpha x) \tag{2}$$

which implies that

$$\lim_{t\to\infty} \sup x(t) \leq \frac{r}{\alpha}.$$

Using the estimate of $x(t)$ on the second equation of (1), we have

$$\frac{dy}{dt} \leq y\left(h - \frac{q\alpha y}{r+c\alpha}\right)$$

which implies that

$$\lim_{t\to\infty} \sup y(t) \leq \frac{h(r+c\alpha)}{q\alpha} = \mu(\text{say}).$$

Hence system (1) is bounded. □

2.2. *Valid steady states and their behavior*

Clearly, there are three non-negative steady states for system (1) namely $E_0 = (0, 0)$, $E_1 = \left(\frac{r}{\alpha}, 0\right)$ and $E_2 = \left(0, \frac{ch}{q}\right)$.

Theorem 1. *(i) E_0 and E_1 are always unstable.*

(ii) E_2 is locally asymptotically stable if $r < \frac{pch(q+kch)}{aq^2}$.

Proof. Proof can be shown by the linearization technique at the steady states. □

Now we find out the restriction for existence of interior steady state $E^* = (x^*, y^*)$. Here x^* and y^* must be positive and satisfy equations below:

$$\frac{r}{1+ky} - \alpha x - \frac{py}{a+bx+x^2} = 0, \tag{3}$$

$$h - \frac{qy}{x+c} = 0. \tag{4}$$

From eq. (3), we find the value of y as $y = \frac{h(x+c)}{q}$. Putting the value of y in eq. (3), we have:

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0 \tag{5}$$

where

$$\begin{aligned}
a_0 &= \alpha qkh, \\
a_1 &= \frac{\alpha q}{4}\left(q + bkh + khc\right), \\
a_2 &= \frac{1}{6}\left\{b\alpha q\left(q + khc\right) + akh\alpha q + kph^2 - rq^2\right\}, \\
a_3 &= \frac{1}{4}\left\{\left(\alpha qa + ph\right)\left(q + khc\right) + kpch^2 - rbq^2\right\}, \\
a_4 &= phc\left(q + khc\right) - raq^2.
\end{aligned}$$

Before investigating the existence of roots of eq. (5), we require $I = a_0a_4 - 4a_1a_3 + 3a_2^2$, $J = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix}$, $I^3 - 27J^2 = D$, where D is the discriminant of (5). The criteria of the positive steady states is stated below.

Theorem 2. *Suppose that $phc\left(q + khc\right) < raq^2$. Equation (5) admits*
(i) only one positive root x^* when $D < 0$.
(ii) multiple when $D > 0$.

Proof. Since $a_4 < 0$, the eq. (5) possesses at least one positive and one negative roots. If $D < 0$, eq. (5) admit two real and two imaginary roots. Consequently, in case (i), eq. (5) has exactly one positive root. When $D > 0$, all roots of eq. (5) are either real or imaginary. As it is already shown that eq. (5) admits at least one positive root, so in case (ii), all the roots are real. This ensures the presence of multiple roots. Hence the theorem. □

If one choose $r = \frac{76}{35}$, $\alpha = \frac{2}{35}$, $p = 1$, $q = 1$, $a = 1$. $b = 1$, $c = 1$, $h = 1$ then system (1) has three positive equilibrium points (1, 2), (2, 3) and (0.24037, 1.24037). As we are not investigating multiple equilibria, so condition for stability of unique equilibrium point will be derived.

Theorem 3. *Consider the condition (i) of Theorem 2 be fulfilled. Further assume that $\alpha > \frac{(b+2x^*)py^*}{\left(a+bx^*+x^{*2}\right)^2}$. Then E^* becomes stable.*

Proof. The variational matrix of system (1) at E^* is

$$J\left(E^*\right) = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix},$$

where

$$\begin{aligned}
b_1 &= -x^*\{\alpha - \frac{(b+2x^*)py^*}{\left(a+bx^*+x^{*2}\right)^2}\}, \\
b_2 &= -x^*\{\frac{rk}{\left(1+ky^*\right)^2} + \frac{p}{a+bx^*+x^{*2}}\}, \\
b_3 &= \frac{qy^{*2}}{\left(x^*+c\right)^2}, \\
b_4 &= -\frac{qy^*}{x^*+c}.
\end{aligned}$$

The characteristic equation around E^* is

$$\lambda^2 + p_1\lambda + p_2 = 0 \tag{6}$$

where

$$\begin{aligned}
p_1 &= \frac{qy^*}{x^*+c} + x^*\left\{\alpha - \frac{(b+2x^*)py^*}{\left(a+bx^*+x^{*2}\right)^2}\right\}, \\
p_2 &= \frac{qx^*y^*}{x^*+c}\left[\left\{\alpha - \frac{\left(b+2x^*\right)py^*}{\left(a+bx^*+x^{*2}\right)^2}\right\}\right. \\
&\quad \left. + \frac{y^*}{x^*+c}\left\{\frac{rk}{\left(1+ky^*\right)^2} + \frac{p}{a+bx^*+x^{*2}}\right\}\right].
\end{aligned}$$

Now $p_1 > 0$ and $p_2 > 0$ if $\alpha > \frac{(b+2x^*)py^*}{\left(a+bx^*+x^{*2}\right)^2}$. In that case, eq. (6) contains two roots whose real parts are negative. Hence, E^* becomes stable under the assumption of the theorem. □

2.3. *The effect of fear factor*

Now, we shall examine the influence of fear factor on the equilibrium densities. Now we first show the effect of k on x^* and y^*. We have already observed that the coordinates of E^* must satisfy

$$\frac{r}{1+ky^*} - \alpha x^* - \frac{py^*}{a+bx^*+x^{*2}} = 0, \tag{7}$$

$$h - \frac{qy^*}{x^*+c} = 0. \tag{8}$$

Differentiating (8) with respect to k, we have

$$\frac{dy^*}{dk} = \frac{h}{q}\frac{dx^*}{dk}. \tag{9}$$

Differentiating (7) with respect to k, we get

$$-\frac{r}{\left(1+ky^*\right)^2}\left\{y^* + k\frac{dy^*}{dk}\right\} - \alpha\frac{dx^*}{dk} - \frac{p\{\frac{dy^*}{dk}\left(a+bx^*+x^{*2}\right) - y^*\left(b+2x^*\right)\frac{dx^*}{dk}\}}{\left(a+bx^*+x^{*2}\right)^2} = 0. \tag{10}$$

Using (9) in (10), we obtain

$$\frac{dx^*}{dk}\left\{\frac{rkh}{q(1+ky^*)^2} + \alpha + \frac{ph}{q(a+bx^*+x^{*2})} - \frac{py^*(b+2x^*)}{\left(a+bx^*+x^{*2}\right)^2}\right\} = -\frac{ry^*}{\left(1+ky^*\right)^2}.$$

Here we consider two cases.
Case 1 If $\frac{rkh}{q(1+ky^*)^2} + \alpha + \frac{ph}{q(a+bx^*+x^{*2})} > \frac{py^*(b+2x^*)}{(a+bx^*+x^{*2})^2}$ then $\frac{dx^*}{dk} < 0$ which in turn implies that $\frac{dy^*}{dk} < 0$.
Case 2 If $\frac{rkh}{q(1+ky^*)^2} + \alpha + \frac{ph}{q(a+bx^*+x^{*2})} < \frac{py^*(b+2x^*)}{(a+bx^*+x^{*2})^2}$ then $\frac{dx^*}{dk} > 0$ which in turn implies that $\frac{dy^*}{dk} > 0$.
In Case 1, we note that increasing amount of fear decrease the density of prey as well as predator species whereas opposite holds in Case 2. The condition in Case 2 indicates that the positive equilibrium point is unstable.

2.4. Local bifurcation analysis

Theorem 4. *System (1) admits a transcritical bifurcation at E_2 related to the parameter r if $r = \dfrac{pch(q+kch)}{aq^2} = \overline{r}$ (say) and $\dfrac{bpch}{qa^2} \neq \alpha + \dfrac{h}{q}\left\{\dfrac{kpch}{a(q+kch)} + \dfrac{p}{a}\right\}$.*

Proof. System (1) is written in the form $\dot{X} = F(X)$ where $X = (x,\ y)^T$ and $F = (f_1,\ f_2)^T$. The variational matrix of system (1) at E_2 is

$$J(E_2) = \begin{pmatrix} \dfrac{rq}{q+kch} - \dfrac{pch}{aq} & 0 \\ \dfrac{h^2}{q} & -h \end{pmatrix}$$

and the corresponding eigenvalues are

$$\lambda_1 = \frac{rq}{q+kch} - \frac{pch}{aq},$$
$$\lambda_2 = -h.$$

If $r = \dfrac{pch(q+kch)}{aq^2}$ then $\lambda_1 = 0$ and λ_2 is always negative. If $v = (v_1, v_2)^T$ and $w = (w_1, w_2)^T$ represent the eigenvectors in respect of the zero eigenvalue of the matrix $J(E_2)$ and $J(E_2)^T$, we get

$$\frac{h}{q}v_1 - v_2 = 0 \text{ and } w = (1,\ 0)^T.$$

Note that $w^T[F_r(E_2, \overline{r})] = (1,\ 0)(0,\ 0)^T = 0$. So saddle-node bifurcation around E_2 cannot occur for the system. Again $w^T[DF_r(E_2, \overline{r})v] = \dfrac{v_1 q}{q+kch} \neq 0$. Further,

$$\begin{aligned} w^T\left[D^2F(E_2, \overline{r})(v,\ v)\right] &= -2\left(\alpha - \frac{pbch}{qa^2}\right)v_1^2 \\ &\quad - 2\left\{\frac{kpch}{a(q+kch)} + \frac{p}{a}\right\}v_1v_2 = \\ &\quad - 2v_1^2\left[\alpha - \frac{pbch}{qa^2} + \frac{h}{q}\left\{\frac{kpch}{a(q+kch)} + \frac{p}{a}\right\}\right], \end{aligned}$$

since $v_2 = \dfrac{h}{q}v_1$. By the assumption of the theorem, it follows that $w^T\left[D^2F(E_2, \overline{r})(v,\ v)\right] \neq 0$. Thus using Sotomayor's theorem [40], we obtain transcritical bifurcation for the system around E_2 with respect to the parameter r. □

2.5. Hopf bifurcation

Set $f(k) = p_1(k)$.

Theorem 5. *Suppose there is a $k = k^*$ such that $f(k^*) = 0$ and $f'(k^*) < 0$ then the interior equilibrium E^* is stable if $k < k^*$ but is unstable for $k > k^*$ and a Hopf bifurcation of periodic solution appears at $k = k^*$.*

Proof. We now follow the technique developed in [41]. We observe that $f(k)$ is a decreasing function in the neighborhood of $k = k^*$ as $f(k) > 0$ for $k < k^*$ then E^* is stable. Also, $f(k) < 0$ for $k > k^*$ and hence E^* is unstable. Applying a result in [42], we find Hopf bifurcation. □

2.6. Nature of limit cycle

To determine the character of the limit cycle, we now compute the first Lyapunov number [40] σ at E^* of system (1). Let $x = u - x^*$, $y = v - y^*$ and then expand in Taylor series we obtain

$$\begin{aligned} \frac{du}{dt} &= a_{10}u + a_{01}v + a_{11}uv + a_{20}u^2 + a_{02}v^2 + a_{30}u^3 \\ &\quad + a_{21}u^2v + a_{12}uv^2 + a_{03}v^3 + P(u,\ v) \\ \frac{dv}{dt} &= b_{10}u + b_{01}v + b_{11}uv + b_{20}u^2 + b_{02}v^2 + b_{30}u^3 \\ &\quad + b_{21}u^2v + b_{12}uv^2 + b_{03}v^3 + Q(u,\ v) \end{aligned}$$

where

$$\begin{aligned} a_{10} &= -x^*\{\alpha - \frac{py^*(b+2x^*)}{(a+bx^*+x^{*2})^2}\}, \\ a_{01} &= -x^*\left\{\frac{rk}{(1+ky^*)^2} + \frac{p}{a+bx^*+x^{*2}}\right\}, \\ a_{11} &= -\left\{\frac{rk}{(1+ky^*)^2} + \frac{p(a-x^{*2})}{(a+bx^*+x^{*2})^2}\right\} \\ a_{20} &= -\alpha + \frac{py^*(3ax^*+ab-x^{*3})}{(a+bx^*+x^{*2})^3}, \\ a_{02} &= \frac{x^*rk^2}{(1+ky^*)^3}, \\ a_{30} &= \frac{py^*(a^2-4abx^*-ab^2-6ax^{*2}-3x^{*4})}{(a+bx^*+x^{*2})^4}, \\ a_{21} &= \frac{p(3ax^*+ab-x^{*3})}{(a+bx^*+x^{*2})^3},\ a_{12} = \frac{rk^2}{(1+ky^*)^3}, \\ a_{03} &= -\frac{rk^3}{(1+ky^*)^4},\ b_{10} = \frac{qy^{*2}}{(x^*+c)^2},\ b_{01} = -h, \\ b_{11} &= \frac{2qy^*}{(x^*+c)^2},\ b_{20} = -\frac{qy^{*2}}{(x^*+c)^3},\ b_{02} = -\frac{q}{x^*+c}, \\ b_{30} &= \frac{qy^{*2}}{(x^*+c)^4},\ b_{21} = -\frac{2qy^*}{(x^*+c)^3}, \\ b_{12} &= \frac{q}{(x^*+c)^2},\ b_{03} = 0, \end{aligned}$$

$P(u,\ v) = \sum_{i+j}^4 a_{ij}u^iv^j$ and $Q(u,\ v) = \sum_{i+j}^4 b_{ij}u^iv^j$.Thus, Lyapunov number σ as developed in [40] is

$$\begin{aligned} \sigma = &-\frac{3\pi}{2a_{01}\Delta^{3/2}}\{[a_{10}b_{10}(a_{11}^2 + a_{11}b_{02} + a_{02}b_{11}) \\ &+ a_{10}a_{01}(b_{11}^2 + a_{20}b_{11} + a_{11}b_{02}) \\ &+ b_{10}^2(a_{11}a_{02} + 2a_{02}b_{02}) - 2a_{10}b_{10}(b_{02}^2 - a_{20}a_{02}) \\ &- 2a_{10}a_{01}(a_{20}^2 - b_{20}b_{02}) - a_{01}^2(2a_{20}b_{20} + b_{11}b_{20}) \\ &+ (a_{01}b_{10} - 2a_{10}^2)(b_{11}b_{02} - a_{11}a_{20})] \\ &- (a_{10}^2 + a_{01}b_{10})[3(b_{10}b_{03} - a_{01}a_{30}) \end{aligned}$$

$$+2a_{10}\left(a_{21}+b_{12}\right)+\left(b_{10}a_{12}-a_{01}b_{21}\right)]\}$$

where $\Delta = a_{10}b_{01} - a_{01}b_{10}$. The sign of σ indicates the nature of limit cycle.

2.7. Persistence

Biologically, persistence indicates that none of the species facing extinction in the long run. Mathematically, it ensures the occurrence of a compact set in the interior of $\mathbb{R}^2_+$ in which all populations must lie ultimately, Now we investigate the uniform persistence of system (1).

Theorem 6. *Consider that* $r > \dfrac{pch(q+kch)}{aq^2}$. *Then system (1) is uniformly persistent.*

Proof. We establish the result by the method developed in [43]. Consider the average Lyapunov function

$$\rho(X) = x^{c_1}y^{c_2}$$

where each $c_i > 0,\ i = 1,2$. In the interior of $\mathbb{R}^2_+$, one finds

$$\begin{aligned}
\frac{1}{\rho(X)}\frac{d\rho(X)}{dt} &= \gamma(X)\\
&= \frac{c_1}{x}\frac{dx}{dt} + \frac{c_2}{y}\frac{dy}{dt}\\
&= c_1\left(\frac{r}{1+ky} - \alpha x - \frac{py}{a+bx+x^2}\right)\\
&\quad + c_2\left(h - \frac{qy}{x+c}\right).
\end{aligned}$$

If $\gamma(X) > 0$ for all feasible equilibria $X \in \mathbb{R}^2_+$, for an appropriate selection of $c_i > 0,\ i = 1,2$, then (1) will be uniformly persistent. So, we have to check the following constraints on the boundary steady states $E_0,\ E_1$ and E_2:

$$E_0 : c_1 r + c_2 h > 0, \tag{11}$$
$$E_1 : c_2 h > 0, \tag{12}$$
$$E_2 : c_1\left(\frac{qr}{q+kch} - \frac{pch}{qa}\right) > 0. \tag{13}$$

As $c_i > 0,\ i = 1,2$, the inequalities (11) and (12) are automatically satisfied. Since $r > \dfrac{pch(q+kch)}{aq^2}$ positivity of (13) is obvious. This completes the proof. □

3. Global Stability

We now examine global stability of the positive equilibrium point E^*. Lyapunov function will be used to prove global stability. From Lemma 2, we note that $x(t) \le \dfrac{r}{\alpha}$ and $y(t) \le \mu$.

Theorem 7. *Let the assumptions of Theorem 3 be satisfied. Further assume that* $r > \dfrac{pch(q+kch)}{aq^2}$ *and*

$$\frac{4q}{x^*+c}\left\{\alpha - \frac{p\mu(b\alpha+2r)}{\alpha(a+bx^*+x^{*2})}\right\} > \left\{\frac{rk}{1+ky^*} + \frac{p}{a+bx^*+x^{*2}} + \frac{q\mu}{c(x^*+c)}\right\}^2$$

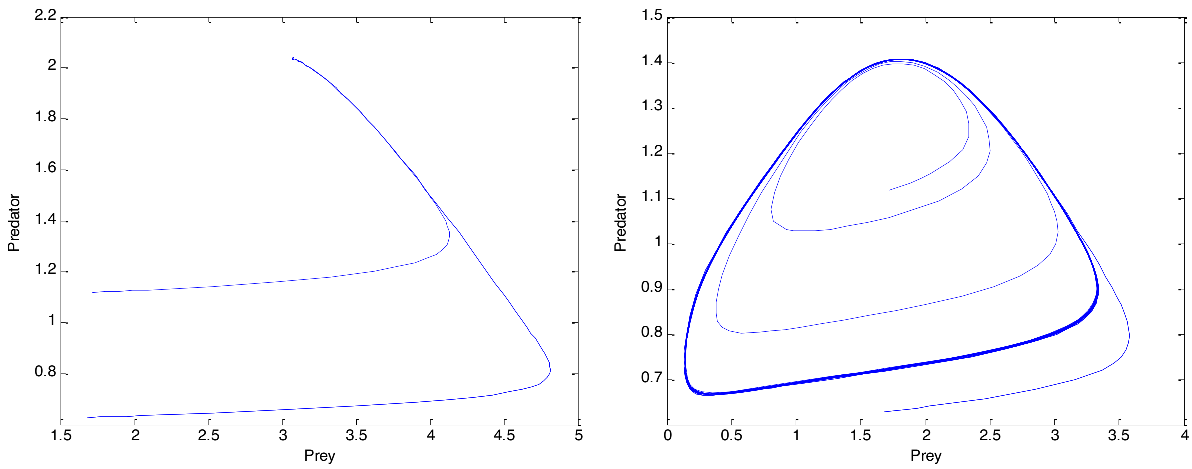
Then E^* is globally asymptotically stable.

Proof. First we note that E_0 and E_1 are always unstable. Again the condition $r > \dfrac{pch(q+kch)}{aq^2}$ indicates that E_2 is unstable. Consider a function $V : \mathbb{R}^2_+ \to \mathbb{R}_+$ defined by

$$V(x,\ y) = \left(x - x^* - x^*\ln\frac{x}{x^*}\right) + \left(y - y^* - y^*\ln\frac{y}{y^*}\right).$$

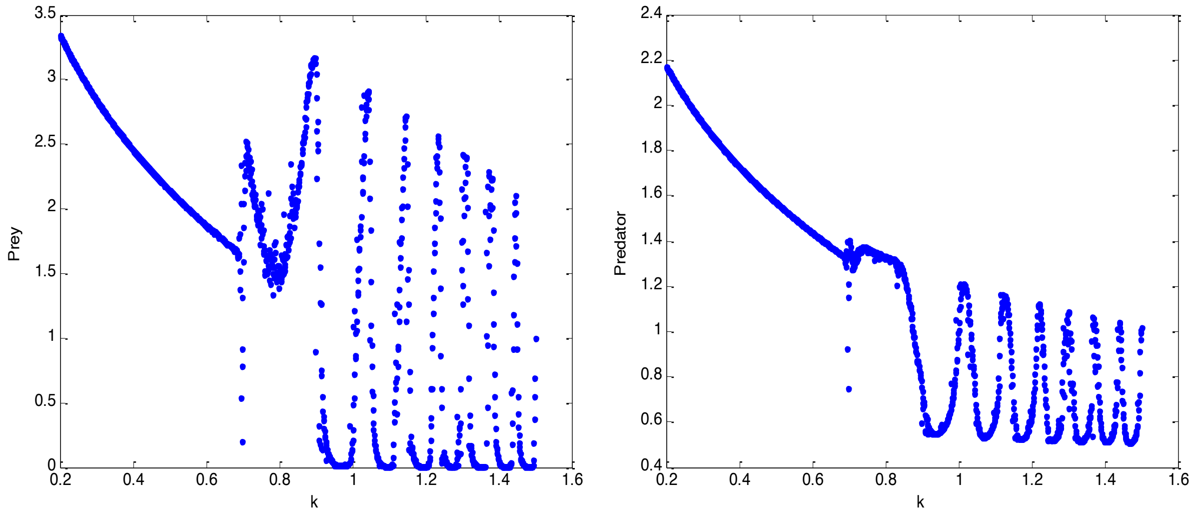
Differentiating V with respect to time, we have

$$\begin{aligned}
\frac{dV}{dt} &= (x-x^*)\frac{\dot{x}}{x} + (y-y^*)\frac{\dot{y}}{y}\\
&= (x-x^*)\left(\frac{r}{1+ky} - \alpha x - \frac{py}{a+bx+x^2}\right)\\
&\quad + (y-y^*)\left(h - \frac{qy}{x+c}\right)\\
&= (x-x^*)\left(\frac{r}{1+ky} - \frac{r}{1+ky^*} - \alpha x + \alpha x^*\right.\\
&\quad \left. - \frac{py}{a+bx+x^2} + \frac{py^*}{a+bx^*+x^{*2}}\right)\\
&\quad + (y-y^*)\left(\frac{qy^*}{x^*+c} - \frac{qy}{x+c}\right)\\
&= (x-x^*)\left[\frac{rk(y^*-y)}{(1+ky)(1+ky^*)} - \alpha(x-x^*)\right.\\
&\quad + p\left\{\frac{y^*-y}{a+bx^*+x^{*2}}\right.\\
&\quad \left.\left. + \frac{y(x-x^*)(b+x+x^*)}{(a+bx+x^2)(a+bx^*+x^{*2})}\right\}\right]\\
&\quad + q(y-y^*)\left\{\frac{y^*-y}{x^*+c} + \frac{y(x-x^*)}{(x^*+c)(x+c)}\right\}\\
&= -(x-x^*)^2\left\{\alpha - \frac{py(b+x+x^*)}{(a+bx+x^2)(a+bx^*+x^{*2})}\right\}\\
&\quad + (x-x^*)(y-y^*)\left\{\frac{qy}{(x^*+c)(x+c)}\right.\\
&\quad \left. - \frac{rk}{(1+ky)(1+ky^*)} - \frac{p}{a+bx^*+x^{*2}}\right\}\\
&\quad - \frac{q(y-y^*)^2}{x^*+c}\\
&\le -(x-x^*)^2\left\{\alpha - \frac{p\mu(b\alpha+2r)}{\alpha(a+bx^*+x^{*2})}\right\}\\
&\quad + |x-x^*|\,|y-y^*|\left\{\frac{q\mu}{c(x^*+c)} + \frac{rk}{1+ky^*}\right.\\
&\quad \left. + \frac{p}{a+bx^*+x^{*2}}\right\} - \frac{q(y-y^*)^2}{x^*+c}\\
&= -X^TMX
\end{aligned}$$



(a) (b)

Figure 1. (a) represents the phase diagram of (1) when $k = 0.25$, (b) represents the phase diagram of (1) when $k = 0.7, r = 18, \alpha = 3, p = 18, a = 1, b = 1, h = 1, q = 2, c = 1$.



(a) (b)

Figure 2. Bifurcation diagram for (a) prey population x, (b) predator population y with respect to the parameter k when $r = 18, \alpha = 3, p = 18, a = 1, b = 1, h = 1, q = 2, c = 1$.

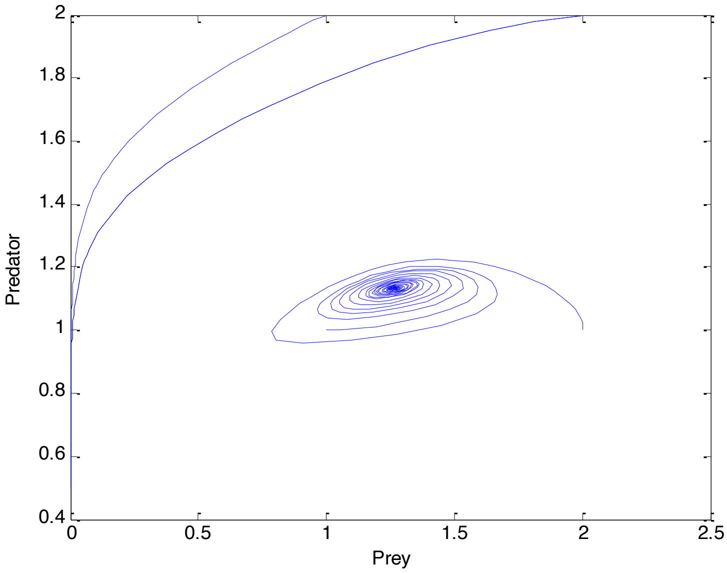


Figure 3. Phase diagram of (1) when $r = 10.125, \alpha = 3, p = 18, a = 1, b = 1.88, k = 0.25, h = 1, q = 2, c = 1$.

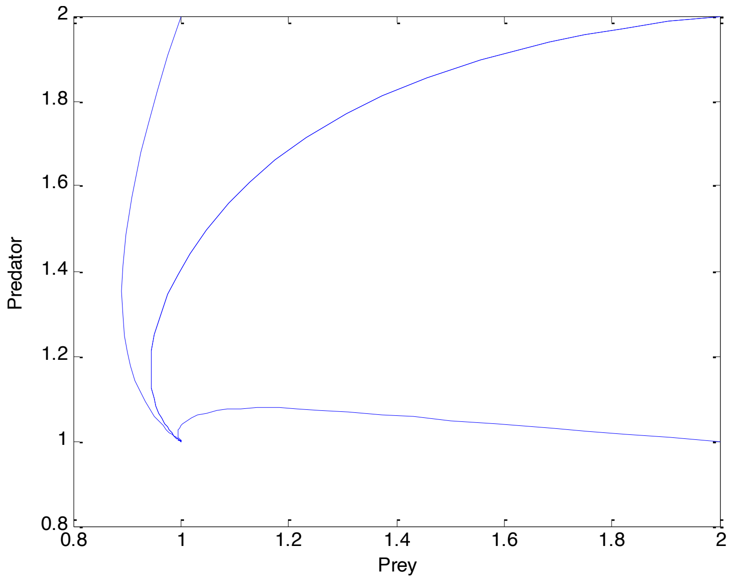


Figure 4. Phase diagram of (1) $k = 19/41,\ r = 3,\ \alpha = 2,\ p = 1,\ a = 18,\ b = 1,\ h = 1,\ q = 2\ ,\ c = 1$ showing global stability of the equilibrium point $E^* = (1,\ 1)$.

where $M = \{|x - x^*|, |y - y^*|\}$ and $M = [m_{ij}]_{2\times 2}$. The components of the matrix M are

$$m_{11} = \alpha - \frac{py\,(b + x + x^*)}{(a + bx + x^2)\,\left(a + bx^* + x^{*2}\right)},$$

$$m_{12} = m_{21}$$

$$= -\frac{1}{2}\left\{\frac{q\mu}{c\,(x^* + c)} + \frac{rk}{1 + ky^*} + \frac{p}{a + bx^* + x^{*2}}\right\},$$

$$m_{22} = \frac{q}{x^* + c}.$$

Hence M is positive definite if

$$\frac{4q}{x^* + c}\left\{\alpha - \frac{p\mu(b\alpha + 2r)}{\alpha(a + bx^* + x^{*2})}\right\} > \left\{\frac{rk}{1 + ky^*} + \frac{p}{a + bx^* + x^{*2}} + \frac{q\mu}{c(x^* + c)}\right\}^2.$$

Conditions of the theorem establish that M is positive definite which in turn implies that $\frac{dV}{dt} < 0$ and hence E^* is globally asymptotically stable. □

4. Numerical Analysis

In this part, we present our findings through numerical computations. We mainly discuss the role of fear factor on system dynamics.

Example 1. Suppose $r = 18,\ \alpha = 3,\ p = 18,\ a = 1,\ b = 1,\ h = 1,\ q = 2,\ c = 1$. Then it follows from Theorem 5, that a Hopf bifurcation of periodic solution occurs at $k = k^* = 0.695$. when $k = 0.25 < k^*$, the system becomes stable (see Figure 1(a)). when $k = 0.7 > k^*$, the system becomes unstable (see Figure 1(b)). bifurcation diagram with respect to the parameter k is shown in Figure 2.

Example 2. Suppose $k = 0.25,\ \alpha = 3,\ p = 18,\ a = 1,\ b = 1.88,\ h = 1,\ q = 2,\ c = 1$. Then it follows from Theorem 4, that a transcritical Hopf bifurcation around the equilibrium point $E_2(0,\ 0.5)$ at $r = \overline{r} = 10.125$ (see Figure 3).

Example 3. Suppose $k = \frac{19}{41}, r = 3,\ \alpha = 2,\ p = 1,\ a = 18,\ b = 1,\ h = 1,\ q = 2,\ c = 1$. Then it follows from Theorem 7, that all solutions converge globally to the equilibrium point $E^* = (1, 1)$ (see Figure 4).

5. Discussion

In this paper, we primarily concentrated on the effect of fear in a Leslie-Gower predator-prey system using Holling type IV functional response. The inclusion of fear effect and extra food resource to the predator makes the system more realistic that the investigation may enable one to point out that the usual analysis regarding direct killing cannot address the natural ecosystem. As group defense ability of prey is considered in the model, it has an significant role in the reproduction of the prey population.

Positivity and boundedness of solutions are shown to justify the biological validity of the model. Four biologically reasonable equilibrium points are obtained. The population free equilibrium point always exists and unstable. This indicates that the system cannot crash for any parameter values. The predator free equilibrium point always exists and is unstable, which in turns implies that predator population can go to extinction, whereas, the prey attains a maximum population size in the environment. The prey free equilibrium point always exists, but attains stability or instability, according as the intrinsic growth rate of prey is below or above than a fixed value. Existence of unique or multiple positive equilibrium points is discussed in Theorem 2. The criterion for local stability of interior steady state is derived. By using Sotomayor's theorem, we derived conditions for the occurrence of transcritical bifurcation around the boundary equilibrium point. When the fear effect is low, the system is under controlled. If the fear effect is increased, the system admits stable oscillation and limit cycle appears surrounding the positive equilibrium point. We observed that when the intrinsic growth rate of prey crosses a certain threshold value, all the populations survive in long run. Furthermore, we have shown the impact of fear factor on equilibrium densities of the populations.

The results in this paper improve the previous related works [20, 25, 26]. The analytical results obtained in this article may encourage the experimental ecologists to carry out some experimental investigations which in turn improves the population biology to some degree. As our system is not a case study, so it is hard to determine parameter values from quantitative assessment. To check the results, a hypothetical set of parameter values is utilized. In future, it will be better performing numerical experiments by employing realistic data to measure the parameters and reformulate the model accordingly.

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