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Amelia Tri Rahma Sidik, Hasan S. Panigoro, Resmawan, and Emli Rahmi



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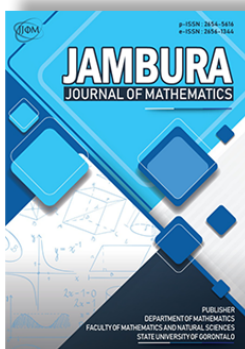
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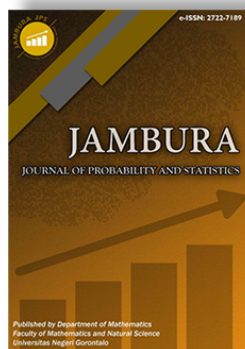
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The existence of Neimark-Sacker bifurcation on a discrete-time SIS-epidemic model incorporating logistic growth and Allee effect

Amelia Tri Rahma Sidik¹, Hasan S. Panigoro^{2,*} , Resmawan³ , and Emli Rahmi⁴ 

^{1,2,3,4}Biomathematics Research Group, Universitas Negeri Gorontalo, Bone Bolango 96554, Indonesia

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ABSTRACT. This article investigates the dynamical properties of a discrete time SIS-Epidemic model incorporating logistic growth rate and Allee effect. The forward Euler discretization method is employed to obtain the discrete-time model. All possible fixed points are identified including their local dynamics. Some numerical simulations by varying the step size parameter are explored to show the analytical findings, the existence of Neimark-Sacker bifurcation, and the occurrence of period-10 and 20 orbits.



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1. Introduction

The spread of infectious diseases is still one of the fundamental problems in population ecology. Through mathematical modeling, the rate of change of the population which is divided into several interacting compartments is given by the epidemic model. Most of these model are define in a system of differential equations, for example [1–3]. One of the simple compartment model is the SIS model which is divided into two compartments: the susceptible compartment (S) and the infected compartment (I). In an SIS model, it is assumed that the individual who recovered from infected compartment has no immune to the disease, and will again transfer to susceptible compartment.

In this work, we assume that the population growth rate following the logistic population model given by Verhulst (1838) as follows.

$$\frac{dP}{dt} = rP \left(1 - \frac{P}{K} \right), \quad (1)$$

where P is the population density, r is the intrinsic growth rate, and K is the carrying capacity of population. Next, we also assume there is a transmission of disease in this population following the SIS model. Therefore, model (1) becomes

$$\begin{aligned} \frac{dS}{dt} &= rS \left(1 - \frac{S+I}{K} \right) - \beta SI + \omega I, \\ \frac{dI}{dt} &= \beta SI - (\delta + \omega)I, \end{aligned} \quad (2)$$

where S and I denote the numbers of susceptible and infected, respectively, β is the transmission rate of disease, δ is the death rate from the disease, and ω is the natural recovery rate.

One of the popular environmental components namely the Allee effect has gain much researchers attention, see [4, 5], for example. The Allee effect refers to a decrease in population growth

rate when the population density is low [6]. Some mechanisms potentially create Allee effect in population, such as environmental conditioning, antipredator strategies, breeding behavior, or mate finding problem. There is two type of the Allee effect terms which usually use to describe this phenomenon, i.e. multiplicative Allee effect [7, 8] and additive Allee effect [9, 10]. If we assume there is Allee effect in the susceptible compartment then we obtain the following system.

$$\begin{aligned} \frac{dS}{dt} &= rS \left(1 - \frac{S+I}{K} \right) (S - m) - \beta SI + \omega I, \\ \frac{dI}{dt} &= \beta SI - (\delta + \omega)I, \end{aligned} \quad (3)$$

where m is the Allee threshold and all parameters are positive.

Currently, some studies about the biological system are using the discrete time model (see for example [11–15]). The main reason is related to the statistical data of the population which is collected in discrete time. Existing research exhibits that the dynamics of the discrete time models are more complex rather than their continuous time models [16, 17].

By following the similar way in [18], we apply the forward Euler scheme for the discretization eq. (3) and we obtain the discrete-time SIS model as follows.

$$\begin{aligned} S_{n+1} &= S_n + h \left[\left(rS_n \left(1 - \frac{S_n + I_n}{K} \right) (S_n - m) \right) \right. \\ &\quad \left. - \beta S_n I_n + \omega I_n \right], \\ I_{n+1} &= I_n + h [\beta S_n I_n - (\delta + \omega)I_n]. \end{aligned} \quad (4)$$

where h is the step size, $S(0) > 0$ and $I(0) \geq 0$.

The outline of this work is the following. Section 2 presents the existence and local stability analysis of the fixed point of eq. (4). In Section 3, some numerical results are performed not

*Corresponding Author.

only to confirm the analytical results but also to investigate the dynamics of the model numerically such as the Neimark-Sacker bifurcation. Finally, some conclusions are given in Section 4.

2. Fixed Point and Their Local Stability

In this section, we derive the fixed points of eq. (4) in the closed first quadrant R_+^2 of the (S, I) plane and analyze the stability behavior around its fixed points.

First, we identify the fixed points by solving the following equations:

$$\begin{aligned} S &= S + h \left[rS \left(1 - \frac{S+I}{K} \right) (S-m) - \beta SI + \omega I \right], \\ I &= I + h [\beta SI - (\delta + \omega)I] \end{aligned} \tag{5}$$

In this way, we get three types of fixed points as follows.

- The origin point $E_0 = (0, 0)$ always exists.
- The disease free point $E_1 = (K, 0)$ or $E_2 = (m, 0)$ always exists.
- The endemic point $E^* = (S^*, I^*)$ where

$$\begin{aligned} S^* &= \frac{\delta + \omega}{\beta} \\ I^* &= \frac{mrS^*(\beta m - (\delta + \omega))(\beta K - (\delta + \omega))}{mr(\delta + \omega)(\beta m - (\delta + \omega)) - \beta^2 mK(2\omega + \delta)} \end{aligned}$$

Furthermore, using the next generation matrix the system (4) has the following basic reproduction numbers.

$$\begin{aligned} R_0^m &= \frac{\beta m}{\delta + \omega}, \text{ or} \\ R_0^K &= \frac{\beta K}{\delta + \omega}. \end{aligned} \tag{6}$$

Applying the basic reproduction numbers, $E^* = (S^*, I^*)$ can be reformulate as

$$\begin{aligned} S^* &= \frac{K}{R_0^K} \text{ or } \frac{m}{R_0^m}, \\ I^* &= \frac{mrS^*(R_0^m - 1)(R_0^K - 1)}{mr(R_0^m - 1) - R_0^m R_0^K (2\omega + \delta)}. \end{aligned}$$

Next, we give the existence condition of the endemic point E^* in the following lemma.

Lemma 1. (1) If the basic reproduction numbers satisfy $R_0^m < 1$ and $R_0^K < 1$, then the system (4) has no endemic point.
 (2) If the basic reproduction numbers satisfy
 (i) $R_0^m < 1$ and $R_0^K > 1$, or
 (ii) $R_0^m > 1$ and
 (a) $1 < R_0^K < \frac{mr(R_0^m - 1)}{R_0^m(2\omega + \delta)}$ or,
 (b) $\frac{mr(R_0^m - 1)}{R_0^m(2\omega + \delta)} < R_0^K < 1$,
 then the system (4) has an endemic points.

Now, we study the stability of the fixed points of system (4) by employing Lemma 4.1 in [14].

Theorem 1. The origin point $E_0 = (0, 0)$ is

- (i) A sink if $h < 2 \min\{\frac{1}{mr}, \frac{1}{\delta + \omega}\}$,
- (ii) A source if $h > 2 \max\{\frac{1}{mr}, \frac{1}{\delta + \omega}\}$,
- (iii) A saddle if $\frac{2}{mr} < h < \frac{2}{\delta + \omega}$ or $\frac{2}{\delta + \omega} < h < \frac{2}{mr}$,
- (iv) A non-hyperbolic if $h = \frac{2}{mr}$ or $h = \frac{2}{\delta + \omega}$.

Proof. By evaluating the Jacobian matrix of system (4) at $E_0 = (0, 0)$, we acquire

$$J(E_0) = \begin{bmatrix} 1 - hmr & h\omega \\ 0 & 1 - h(\delta + \omega) \end{bmatrix}, \tag{7}$$

The Jacobian matrix (7) gives two eigenvalues as follows.

$$\begin{aligned} \lambda_1 &= 1 - hmr, \\ \lambda_2 &= 1 - h(\delta + \omega). \end{aligned} \tag{8}$$

If $h < \frac{2}{mr}$ then $|\lambda_1| < 1$, if $h > \frac{2}{mr}$ then $|\lambda_1| > 1$, if $h = \frac{2}{mr}$ then $|\lambda_1| = 1$, if $h < \frac{2}{\delta + \omega}$ then $|\lambda_2| < 1$, if $h > \frac{2}{\delta + \omega}$ then $|\lambda_2| > 1$, and if $h = \frac{2}{\delta + \omega}$ then $|\lambda_2| = 1$. By applying Lemma 4.1 in [14], the results of (i) – (iv) can be proven. \square

Theorem 2. Let $K > m$ and $R_0^K < 1$. The disease free point E_1 is

- (i) A sink if $h < 2 \min\{\frac{1}{r(K-m)}, \frac{1}{(1-R_0^K)(\delta + \omega)}\}$,
- (ii) A source if $h > 2 \max\{\frac{1}{r(K-m)}, \frac{1}{(1-R_0^K)(\delta + \omega)}\}$,
- (iii) A saddle if $\frac{2}{r(K-m)} < h < \frac{2}{(1-R_0^K)(\delta + \omega)}$ or $\frac{2}{(1-R_0^K)(\delta + \omega)} < h < \frac{2}{r(K-m)}$,
- (iv) A non-hyperbolic if $h = \frac{2}{r(K-m)}$ or $h = \frac{2}{(1-R_0^K)(\delta + \omega)}$.

Proof. We first compute the Jacobian matrix of system (4) at fixed point E_1 as follows.

$$J(E_1) = \begin{bmatrix} 1 - hr(K-m) & -h[r(K-m) - \omega + R_0^K(\delta + \omega)] \\ 0 & 1 - h(1 - R_0^K)(\delta + \omega) \end{bmatrix}, \tag{9}$$

which gives eigenvalues $\lambda_1 = 1 - hr(K-m)$ and $\lambda_2 = 1 - h(1 - R_0^K)(\delta + \omega)$. According to Lemma 4.1 in [14] the statement (i) – (iv) in Theorem 2 can be achieved. \square

Theorem 3. Let $K < m$ and $R_0^m < 1$. The disease free point E_2 is

- (i) A sink if $h < 2 \min\{\frac{K}{mr(m-K)}, \frac{1}{(1-R_0^m)(\delta + \omega)}\}$,
- (ii) A source if $h > 2 \max\{\frac{K}{mr(m-K)}, \frac{1}{(1-R_0^m)(\delta + \omega)}\}$,
- (iii) A saddle if $\frac{2K}{mr(m-K)} < h < \frac{2}{(1-R_0^m)(\delta + \omega)}$ or $\frac{2}{(1-R_0^m)(\delta + \omega)} < h < \frac{2K}{mr(m-K)}$,
- (iv) A non-hyperbolic if $h = \frac{2K}{mr(m-K)}$ or $h = \frac{2}{(1-R_0^m)(\delta + \omega)}$.

Proof. For $E_2 = (m, 0)$, we obtain the Jacobian matrix as follows.

$$J(E_2) = \begin{bmatrix} 1 - hmr(\frac{m}{K} - 1) & h[\omega - R_0^m(\delta + \omega)] \\ 0 & 1 - h[(1 - R_0^m)(\delta + \omega)] \end{bmatrix}, \tag{10}$$

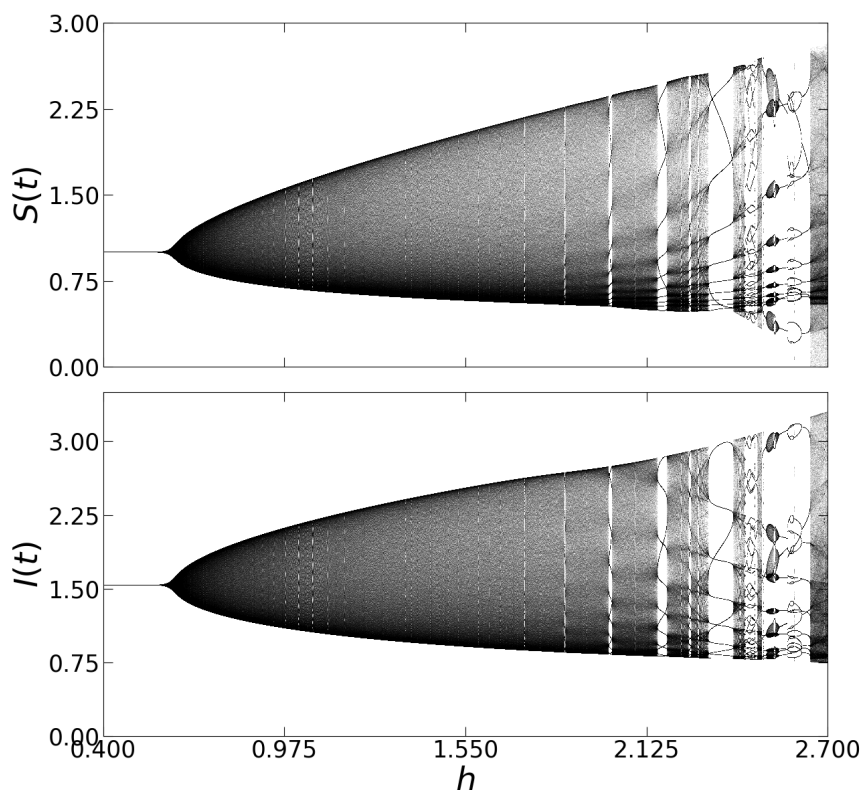


Figure 1. The Neimark-Sacker bifurcation diagram of system (4) corresponding to the bifurcation parameter h and other parameters are given in (14)

From the Jacobian matrix in 10, we obtain two eigen values which is $\lambda_1 = 1 - hmr \left(\frac{m}{K} - 1\right)$ and $\lambda_2 = 1 - h[(1 - R_0^m)(\delta + \omega)]$. Since $K < m$ and $R_0^m > 1$ we have $h < \frac{2K}{mr(m-K)}$, $h > \frac{2K}{mr(m-K)}$ and $h = \frac{2K}{mr(m-K)}$ when $|\lambda_1| < 1$, $|\lambda_1| > 1$ and $|\lambda_1| = 1$, respectively. Furthermore, we have $h < \frac{2}{(1-R_0^m)(\delta+\omega)}$, $h > \frac{2}{(1-R_0^m)(\delta+\omega)}$ and $h = \frac{2}{(1-R_0^m)(\delta+\omega)}$ when $|\lambda_2| < 1$, $|\lambda_2| > 1$ and $|\lambda_2| = 1$, respectively. Thus, again by utilizing Lemma 4.1 in [14], we have the complete dynamics given by Theorem 3. \square

Theorem 4. Suppose that

$$\Omega = \frac{mr}{R_0^m R_0^K} ((R_0^m - 1)(R_0^K - 2) + (1 - R_0^K))$$

$$+ I^* \left(\beta - \frac{mr}{K} \right)$$

$$\Delta = \Omega^2 - 4\beta I^* \left(\frac{r(R_0^m - 1)}{R_0^m R_0^K} + \delta \right)$$

$$h_1 = \frac{4}{\Omega + \sqrt{\Delta}}, h_2 = \frac{4}{\Omega - \sqrt{\Delta}}, h_3 = \frac{4\Omega}{\Omega^2 - \Delta}$$

Let $R_0^m > 1$. The endemic point E^* is

- A sink if $\Delta \geq 0$ and $0 < h < h_1$, or $\Delta < 0$ and $0 < h < h_3$.
- A source if $\Delta \geq 0$ and $h > h_2$, or $\Delta < 0$ and $h > h_3$.
- A saddle if $\Delta \geq 0$ and $h_1 < h < h_2$.
- A non-hyperbolic if $\Delta \geq 0$ and $h = h_1$ or h_2 , or $\Delta < 0$ and $h = h_3$.

Proof. By computing the Jacobian matrix around $E^* = (S^*, I^*)$, we achieve

$$J(E^*) = \begin{bmatrix} 1 - h\Omega & -h \left[\frac{r(R_0^m - 1)}{R_0^m R_0^K} + \delta \right] \\ h\beta I^* & 1 \end{bmatrix}, \quad (11)$$

which gives a quadratic polynomial characteristic $\lambda^2 - Tr(J(E^*))\lambda + Det(J(E^*)) = 0$, where

$$\begin{aligned} Tr(J(E^*)) &= 2 - h\Omega \\ Det(J(E^*)) &= 1 - h\Omega + h^2\beta I^* \left(\frac{r(R_0^m - 1)}{R_0^m R_0^K} + \delta \right) \end{aligned} \quad (12)$$

Therefore, we obtain two eigen values as follows:

$$\lambda_{1,2} = 1 - \frac{h\Omega}{2} \pm \frac{h\sqrt{\Delta}}{2} \quad (13)$$

By utilizing Lemma 4.1 in [14], the Theorem 4 is completely proven. \square

From the analytical results in Theorem 1 to 4, it is clear that the step size (h) has greatly influences the stability of each fixed point.

3. Numerical Simulation

In this part, some numerical simulations of the system (4) are illustrated. For the sake of simulation needs, we use hypothetical parameter values because of the absence of field data.

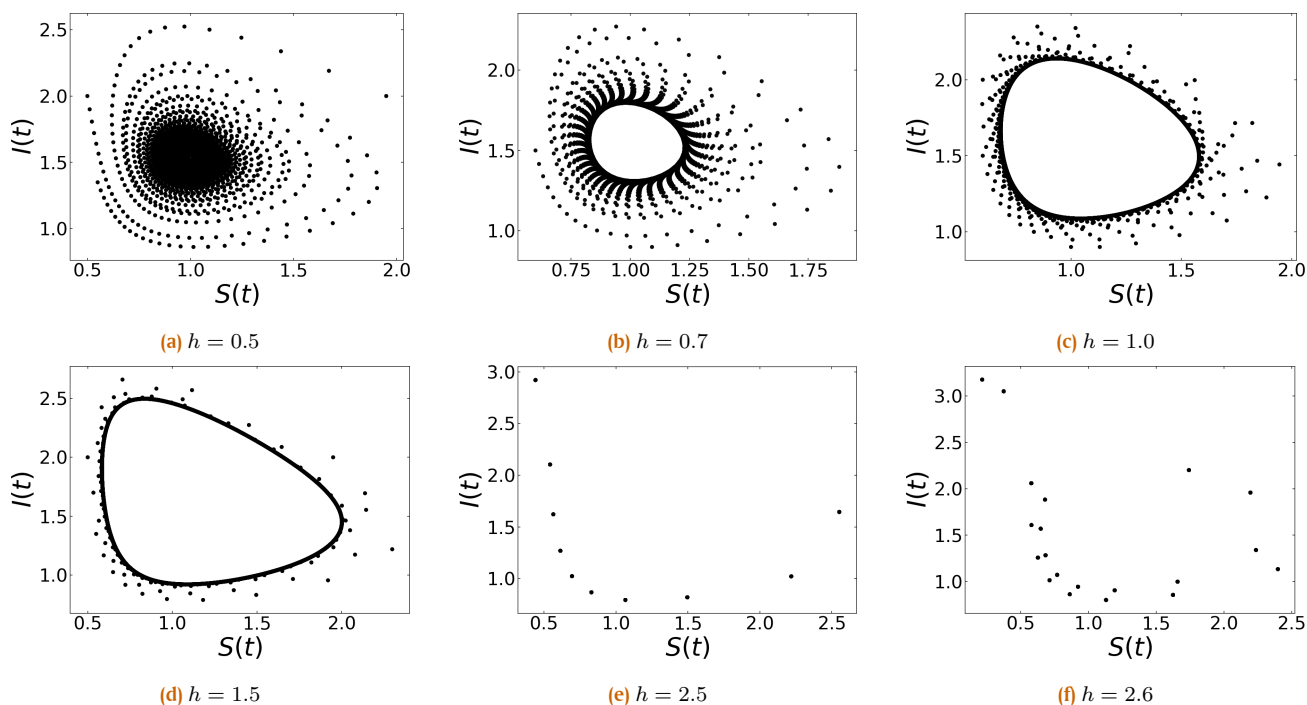


Figure 2. The phase portraits of system (4) for various h corresponding to Figure 1

We set some parameter values as follows,

$$\beta = 0.2, \omega = 0.1, \delta = 0.1, r = 0.6, m = 0.3, \text{ and } K = 4. \quad (14)$$

According to Lemma 1, system (4) has an endemic point $E^* = (1, 1.536)$ since it satisfies $R_0^m = 0.3 < 1$ and $R_0^K = 4 > 1$. Furthermore, from Theorem 4 we also have a switching condition from sink to source behavior. The endemic point $E^* = (1, 1.536)$ has a discriminant value, $\Delta = -0.2505 < 0$. Therefore, it has a pair of complex conjugate eigen values, i.e. $\lambda_{1,2} = 1 - 0.0196h \pm 0.2502hi$. For this reason, we will show numerically the existence of Neimark-Sacker bifurcation around the endemic point which corresponds to the step size (h) parameter.

By varying the step-size (h) in range $0.4 \leq h \leq 2.7$ and using parameter in (14) with initial conditions $(S_0, I_0) = (1, 1.4)$, we portray the occurrence of Neimark-Sacker bifurcation around the endemic point $E^* = (1, 1.536)$ in Figure 1. From Figure 1, we see that E^* is stable when $h < h^* \approx 0.62$, and then loses its stability with the emergence of an invariant closed curve when $h > h^* \approx 0.62$. For better visualization of this phenomenon, we choose six different values of h ($h = 0.5, 0.7, 1, 1.5, 2.5, 2.6$) and illustrate the phase portrait diagram in Figure 2. Moreover, the radius of the invariant closed curve becomes larger as the value of h increases as we can see in Figure 2(b-d). We also show the appearance of periodic-10 and 20 orbits in Figure 2(e-f), respectively.

4. Conclusion

A new discrete time SIS-epidemic model obtained by Euler method has been studied in this article. From the analytical results, it has found that the model has three types of fixed points along with their respective local stability conditions. Each of fixed points could be a sink, source, saddle, or non-hyperbolic

point depend on the step size of discretization. We also show numerically the emergence of Neimark-Sacker bifurcation, period-10 and 20 orbits driven by the step size (h) parameter.

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Conflict of interest. The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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