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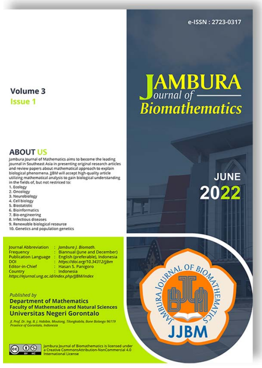

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Research Article

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Bifurcation analysis of phytoplankton-fish model through parametric control by fish mortality rate and food transfer efficiency

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ABSTRACT. *An Algae-zooplankton fish model is studied in this article. First the proposed model is evaluated for positive invariance and boundedness. Then,the Routh-Hurwitz parameters and the Lyapunov function are used to determine the presence of a positive interior steady state and the criteria for plankton model stability (both local and global). Taylor's sequence is also used to discuss Hopf bifurcation and the stability of bifurcated periodic solutions. The model's bifurcation analysis reveals that Hopf-bifurcation can occur when mortality rate and food transfer efficiency are used as bifurcation parameters. Finally, we use numerical simulation to validate the analytical results.*

1. Introduction

There is an all-inclusive custom of burning-through prey hunter models to depict the Algae-zooplankton fish frameworks [1, 2]. Despite the fact that the interface of Algae-zooplankton explains the occurrence of changes, a significant number of the intriguing examples in tiny fish elements in the field are known to be aspiring at any rate partially by predation and by planktivores fish. The conversation about the impacts of fish on microscopic fish was set off by Hrbacek et.al [3], which caused to notice the huge contrasts between the microscopic fish of various lakes relying upon the presence of fish. In lakes with fish, zooplankton comprised chiefly of little bodied species and algal bio mass was high. In lakes without fish, phytoplankton bio mass was low and bigger herbivores overwhelmed the zooplankton. In 1965, Brooks and Dodson [4] noticed the same connections in a portion of the lakes in New England and built up the thus, called size productivity speculation to clarify such moves and they recommended that high densities of planktivorous fish lead to zooplankton networks overwhelmed by little, generally wasteful slow eaters [5–7]. Particular predation by fish eliminates bigger zooplankters, yet certain clones' conduct and life history methodology also change in light of synthetic compounds delivered by fish [8–10]. Lakes with a high fish stock are regularly overwhelmed by huge filamentous blue green growth, which have been appeared to restrain the development of huge bodied species under certain conditions [11–13] implying some beneficial portrayal for their nonattendance in such lakes. Clearly, an intricate transaction of components prompts shifts in size appropriations of zooplankton and green growth in the field. The significance of tiny fish in marine environment is broadly worried by numerous researchers [14–17]. Microscopic fish species are additionally the base of the marine natural way of life. Tiny fish species assumes a fascinating and dynamic part and are ordered into phytoplankton and zooplankton. Phytoplankton are herbivore type in nature, unicellular and minute in size. Zooplankton are omnivore that relay upon phytoplankton and show different elements, which were explored by numerous researchers [18–22]. Hence, zooplankton and phytoplankton structure hunter prey relationship which permits and rouse towards elements and its examination turns out to be seriously intriguing and pulled in by more scientists [23–26]. The issue turns out to be a lot less complicated while thinking about solid top-down control of algal biomass, as seen during the regularly examined spring clear water stage [27, 28] or after the end of fish by normal winterkills [29, 30] or human obstruction [5, 31]. Marten Scheffer, Sergio Rinaldi and Yuri A. Kuznetsov [32] introduced an extension of an old style negligible Daphnia-green growth model to represent the impacts of fish as a top hunter. The investigation recommended that much of the time, the tiny fish should show hysteresis because of predation pressure by fish and in this manner there exist two particular systems, one in which Daphnia is constrained by fish and phytoplankton biomass is high and another

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in which Daphnia is generally unaffected by plankticvores and green growth are constrained by Daphnia. This likewise exhibitions how natural hunter prey variances in the planktonic design can encourage the control to the algal-overwhelmed structure where fish constrain Daphnia. By the incitement of this work, we worked on the indistinguishable model for finding the unfaltering quality concerning the inside harmony point.

The objective of our current work is to analyse the dynamics of An Algae-zooplankton fish model in view of stability and bifurcation. The model consists of Phytoplanton and Zooplankton species with Holling type II and III functional responses. Analytical results are justified with numerical simulations.

The proposed model is evaluated for positive invariance and boundedness. Then, the Routh-Hurwitz parameters and the Lyapunov function are used to determine the presence of a positive interior steady state and the criteria for plankton model stability (both local and global). Taylor's sequence is also used to discuss Hopf bifurcation and the stability of bifurcated periodic solutions. The model's bifurcation analysis reveals that Hopf-bifurcation can occur when mortality rate and food transfer efficiency are used as bifurcation parameters. Finally, we use numerical simulation to validate the analytical results.

This paper is organized as follows. In Section 2 comprises of the mathematical model alongside presumption. Positive invariance and boundedness are analyzed in Section 3.The assessment of balance focuses is given in Section 4. Neighborhood and worldwide security qualities are investigated in Subsection 5.1 and 5.2 separately. Hopf-bifurcation examination is done in Section 6. Mathematical reproduction is done in Section 7. At last closing comments are given in Section 8.

2. Formulation of mathematical model

In this section, we contemplate the deterministic mathematical model, an innovative plankton model combination with two different functional responses. Many researchers studied models with functional responses [33–41]. But this was a typical complex problem modeled as mathematical plankton ecology algae and zooplankton model with Holing type-II and type-III functional responses where the algae densities denoted by x assuming the logistic growth, when the consumer (fish) is absent and also following a simple saturating functional response as a monoid function of algal density with a half saturation value α. The density of zooplankton fishes at a time t is denoted by y with the sigmoidal functional response of fish with a half saturation value of β. With these assumptions, we have the following model as given by

$$x'(t) = rx\left(1-\frac{x}{k}\right) - \frac{axy}{\alpha+x} + i\,(k-x)\,, \tag{1}$$

$$y'(t) = \frac{aexy}{\alpha+x} - my - \frac{by^2}{\beta+y^2}, \tag{2}$$

with initial densities

$$x\,(0) = x_0 \geq 0, y\,(0) = y_0 \geq 0, \tag{3}$$

where x represents density of edible algae, y represents density of large herbivores zooplankton. r denotes the growth rate of algae. k denotes the carrying capacity of algae. Loses of algae are due to grazing by zooplankton following a simple saturating (Type-II) functional response $\frac{axy}{(\alpha+x)}$ which is formulated as a monoid function of algal density with the half saturation value of α and a maximum per capita grazing rate of g . i stands for the diffusive inflow of algae, $i\,(k-x)$ represents stabilizing inflow of algae. e denotes the food transfer efficiency of zooplankton. m denotes mortality rate of zooplankton. f denotes that the capacity of planktivores. The term $\frac{by^2}{(\beta+y^2)}$ is a sigmoidal functional response of a fish with a half saturation value of β (Type- III). We assumed that all the parameters are positive.

3. Boundedness and positive invariance

The aim of feasibility or biological positivity studies is to discover the power of a proposed model in a given setting objectively and rationally. Biologically positive populations ensure that the population never goes negative and that the population still survives. The following theorems guarantee the system's positivity and boundedness equation (1) and (2).

> **Theorem 1.** *All solutions $(x(t), y(t))$ of the system (1) and (2) with the initial condition eq. (3) are positive for all $t \geq 0$*

Proof. From equation (1) and (2), it is observed that

$$\begin{aligned} \frac{dx}{x} &= \phi_1(x,y)dt, \\ \frac{dy}{y} &= \phi_2(x,y)dt, \end{aligned}$$

where

$$\begin{aligned} \phi_1(x,y) &= r\left(1-\frac{x}{K}\right) - \frac{ay}{\alpha+x} - i\,(k-x)\,, \\ \phi_2(x,y) &= \frac{aex}{\alpha+x} - m - b\left(\frac{y}{y^2+\beta}\right), \end{aligned}$$

and their solutions in the region $[0,t]$ are

$$\begin{aligned} x(t) &= x(0)\exp\left(\textstyle\int\phi_1(x,y)dt\right) > 0 \ \text{ and} \\ y(t) &= y(0)\exp\left(\textstyle\int\phi_2(x,y)dt\right) > 0 \end{aligned}$$

for all t. Hence, all solutions starting from interior of the first octant (In $\Re^2_+$) remain positive in it for future time. □

> **Theorem 2.** *All of the non-negative model system (1) and (2) solutions that start in $\Re^2_+$ are uniformly bounded.*

Proof. Let $(x(t), y(t))$ be any of the system (1) and (2) solutions. Since, from equation (1),

$$\frac{dx}{dt} \leq x\left(r - \frac{rx}{K}\right),$$

we have

$$\lim_{t\to\infty}\sup x\,(t) \leq \frac{1}{K}.$$

Let $\xi = x + \frac{y}{e}$. Therefore,

$$\frac{d\xi}{dt} = \frac{dx}{dt} + \frac{1}{e}\frac{dy}{dt}. \tag{4}$$

Substituting the equation (1) and (2) in eq. (4), we get

$$\begin{aligned}\frac{d\xi}{dt}+\zeta\xi &= x\left((r+\zeta-i)-\left(\frac{rx}{K}-\frac{ki}{x}\right)\right)+\left(\zeta-\frac{m}{e}\right)y\\ &\quad+\frac{y}{e}\left(\zeta-\frac{b}{y^2+\beta}\right),\\ &\le x\left((r+\zeta-i)-\left(\frac{rx}{K}-\frac{ki}{x}\right)\right)\le\mu.\end{aligned}$$

When we apply Lemma to differential inequalities, we get

$$0\le\xi(x,y)\le\left(\frac{\mu}{\zeta}\right)\left(1-e^{-\zeta t}\right)+\frac{\xi(x(0),y(0))}{e^{\zeta t}},$$

and for $t\to\infty$ we have $0\le\xi(x,y)\le\frac{\mu}{\zeta}$. Thus, all solutions of system (1) and (2) enter into the region

$$\Gamma=\left\{(x,y)\in R_+^2:0\le x\le\frac{1}{K},0\le\xi\le\frac{\mu}{\zeta}+\varepsilon,\forall\varepsilon>0\right\}$$

. □

4. Analysis of steady states

The possible steady states of the system (1) and (2) are $(0,0)$, $(x^{\phi},0)$, $(0,y^{\varphi})$, (x^*,y^*). Now we are interested in studying the system's dynamics about (x^*,y^*). For steady state points, equating the left-hand sides of equation (1) and (2) to zero is required. Then,

$$a_0x^3+a_1x^2+a_2x+a_3=0 \tag{5}$$

and

$$b_0y^2+b_1y+a_2=0 \tag{6}$$

where

$$\begin{array}{ll} a_0=r>0, & a_1=iK+r\alpha-rK,\\ a_2=yK-iK\alpha-iK^2+rK\alpha, & a_3=-iK^2<0,\\ b_0=eax-mx-m\alpha, & b_1=-b(x+\alpha)<0,\\ b_2=\beta b_0. & \end{array}$$

The eq. (5) has a unique positive root $x=x^*$ if any of the following cases is true: $a_1>0$ and $a_2>0$; $a_1<0$ and $a_2<0$; $a_1>0$ and $a_2<0$ and the eq. (6) has a positive root $y=y^*$ if $b_0>0$. Since the eqs. (5) and (6) are implicit equations of x and y, it is possible to find the positive solution (x^*,y^*) numerically using the Newton-Raphson method for nonlinear systems with two unknowns.

5. Stability analysis

5.1. Investigation of local stability

The system (1) and (2) Jacobian matrix evaluated at the interior steady state (x^*,y^*) is given by

$$J(x^*,y^*)=\begin{pmatrix}a_{10} & a_{11}\\ b_{10} & b_{11}\end{pmatrix} \tag{7}$$

where

$$\begin{aligned} a_{10} &= \frac{yax}{(x+\alpha)^2}-\frac{rx}{K}-\frac{ik}{x},\\ a_{11} &= -\frac{ax}{x+\alpha},\\ b_{10} &= \frac{eay\alpha}{(x+\alpha)^2},\\ b_{11} &= \frac{by\left(y^2-\beta\right)}{\left(y^2+\beta\right)^2}.\end{aligned}$$

The characteristic eq. (7) is in the form

$$\lambda^2+P\lambda+Q=0$$

where

$$\begin{aligned} P &= \frac{rx}{K}+\frac{iK}{x}-\frac{yex}{\left(y^2+\beta\right)^2},\\ Q &= \left[\frac{yax}{(x+\alpha)^2}-\frac{rx}{K}-\frac{ik}{x}\right]\left[\frac{by\left(y^2-h_Z^2\right)}{\left(y^2+\beta\right)^2}\right]-\frac{ea^2xy\alpha}{(x+\alpha)^3}\end{aligned}$$

By Routh Hurwitz criteria, if $P>0$ and $Q>0$, then the system (1) and (2) is locally asymptotically stable around the positive equilibrium.

5.2. Investigation of global stability

Theorem 3. *The interior equilibrium point is globally asymptotically stable.*

Proof. Let us consider the following Lyapunov function

$$\begin{aligned} V(x,y) &= x-x^*-x^*\ln\left(\frac{x}{x^*}\right)+l_1\left(y-y^*-y^*\ln\frac{y}{y^*}\right),\\ V'(t) &= \left(\frac{x-x^*}{x}\right)x'(t)+l_1\left(\frac{y-y^*}{y}\right)y'(t).\end{aligned}$$

The derivative of the Lyapunov function evaluated at the interior equilibrium point is

$$\begin{aligned} V'(t) &= -\frac{r}{K}(x-x^*)^2-\frac{a(x-x^*)(y-y^*)}{(x+\alpha)}+\frac{ay^*(x-x^*)^2}{(x+\alpha)(x^*+\alpha)}\\ &\quad-\frac{iK(x-x^*)}{xx^*}+l_1\left(\frac{ea\alpha(x-x^*)(y-y^*)}{(x+\alpha)(x^*+\alpha)}\right)\\ &\quad-l_1\left(\frac{b(\beta-yy^*)(y-y^*)^2}{(y^2+\beta)\left(y^{*2}+\beta\right)}\right),\end{aligned}$$

by choosing $l_1=\frac{x^*+\alpha}{e\alpha}$,

$$V'(t)=-R(x-x^*)^2-T(y-y^*)^2$$

where

$$\begin{aligned} R &= \frac{ay^*}{(x+\alpha)(x^*+\alpha)}-\frac{r}{K}-\frac{iK}{xx^*},\\ T &= \frac{(x^*+\alpha)b(\alpha+yy^*)}{e\alpha(y^2+\beta)\left(y^{*2}+\beta\right)}.\end{aligned}$$

By Lyapunov's theorem, if $R>0$ and $T>0$, then the system (1) and (2) is globally asymptotically stable around the positive steady state. □

6. Hopf-bifurcation analysis

One of the situations discussed in this section is the presence of a Hopf bifurcation. This occurs when equilibrium loses its stability and a limit cycle develops around it. When the Jacobian matrix has a pair of pure imaginary eigenvalues at $E*$, the Hopf bifurcation occurs. i.e., $Tr(J(E^*))=0$ and $\det(J(E^*))>0$. The Jacobean matrix of the system about $E^*=(x^*,y^*)$ is given by

$$J(x^*,y^*)=\begin{pmatrix}a_{10} & a_{11}\\ b_{10} & b_{11}\end{pmatrix} \tag{8}$$

where

$$\begin{aligned} a_{10} &= \frac{yax}{(x+\alpha)^2} - \frac{rx}{K} - \frac{ik}{x}, \\ a_{11} &= -\frac{ax}{x+\alpha}, \\ b_{10} &= \frac{eay\alpha}{(x+\alpha)^2}, \\ b_{11} &= \frac{by\left(y^2-\beta\right)}{\left(y^2+\beta\right)^2}. \end{aligned}$$

To get a pair of pure imaginary eigenvalues for $J\left(x^*, y^*\right)$ we ask for

$$\delta = \frac{rx}{K} + \frac{ik}{x} - \frac{yax}{\left(y^2+\beta\right)^2} = \delta^H \ ; \delta^H \left[\frac{by\left(y^2-\beta\right)}{\left(y^2+\beta\right)^2}\right] - \frac{ea^2xy\alpha}{(x+\alpha)^3} > 0 \quad (9)$$

The no-degenerate condition must be checked to ensure the presence of Hopf bifurcation.

$$\left.\frac{d\left(Tr\left(E^*\right)\right)}{d\left(\delta\right)}\right|_{\delta=\delta^H=-1\neq 0}$$

We use a shift of coordinates to discuss the stability of the limit cycle. Shift of coordinates $u = x - x^*$, $v = y - y^*$ to transform the system (1) and (2) into

$$\begin{aligned} u'(t) &= r(u+x^*)\left(1-\frac{(u+x^*)}{K}\right) - b(v+y^*)\left(\frac{(u+x^*)}{(u+x^*)+\alpha}\right) \\ &\quad + i\left(k-\left((u+x^*)\right)\right), \\ v'(t) &= ea(v+y^*)\left(\frac{(u+x^*)}{(u+x^*)+\alpha}\right) - m(v+y^*) \\ &\quad - b\left(\frac{(v+y^*)^2}{(v+y^*)^2+\beta}\right). \end{aligned} \quad (10)$$

The system (10) is re written as(using Taylor expansion around (0,0))

$$\begin{aligned} u'(t) &= a_{10}u + a_{01}v + a_{20}u^2 + a_{11}uv + a_{30}u^3 + a_{21}u^2v \\ &\quad + Q_1(x,y), \\ v'(t) &= b_{10}u + b_{01}v + b_{20}u^2 + b_{02}v^2 + b_{11}uv + b_{30}u^3 \\ &\quad + b_{21}u^2v + b_{12}uv^2 + Q_2(x,y), \end{aligned}$$

where $a_{10}, a_{01}, b_{10}, b_{01}$ are given by the Jacobian matrix $J\left(E^*\right)$ in eq. (8), Q_1, Q_2 are polynomials in x^i, y^j with $i+j \geq 4$ and

$$\begin{aligned} a_{20} &= \frac{ay}{(x+\alpha)^2} - \frac{2r}{K} - \frac{axy}{(x+\alpha)^3}, & a_{21} &= \frac{ax}{(x+\alpha)^2} - \frac{a}{x+\alpha}, \\ a_{11} &= \frac{-ax}{(x+\alpha)^2}, & a_{30} &= -\frac{2ay}{(x+\alpha)^3} - \frac{ay\left(\alpha-3x\right)}{(x+\alpha)^4}, \\ a_{11} &= \frac{-ax}{(x+\alpha)^2}, & b_{11} &= \frac{ea}{x+\alpha} - \frac{eay}{(x+\alpha)^2}, \\ b_{20} &= \frac{by^2}{(y^2+\beta)^2} - \frac{2b}{y^2+\beta} & b_{21} &= \frac{-ea}{(x+\alpha)^2} + \frac{2eay}{(x+\alpha)^3}, \\ &\quad - \frac{8by^4}{(y^2+\beta)^3}, & & \\ b_{30} &= \frac{b(2y\beta-2y^3)}{(y^2+\beta)^3} + \frac{4yb}{y^2+\beta} & b_{12} &= \frac{-ea}{(x+\alpha)^2}. \\ &\quad - \frac{8b(-2y^5+4y^3\beta)}{(y^2+\beta)^4}, & & \end{aligned}$$

Therefore the system (10) can be expressed as(using matrix notation)

$$\begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix} = J\left(E^*\right)\begin{pmatrix} u \\ v \end{pmatrix} + L\left(u,v\right)$$

with

$$L = \begin{pmatrix} a_{20}u^2 + a_{11}uv + a_{30}u^3 + a_{21}u^2v + Q_1\left(x,y\right) \\ b_{20}u^2 + b_{11}uv + b_{02}v^2 + b_{30}u^3 + b_{21}u^2v + b_{12}uv^2 + Q_2\left(x,y\right) \end{pmatrix}$$

At $\delta = \delta^H$, matrix $J\left(E^*\right)$ has a pair of pure imaginary eigenvalues, so $a_{10} = b_{01}$. Let $w = \sqrt{\det\left(J\left(E^*\right)\right)} > 0$, changes in coordinates are made as $u = Y_2 v$, $v = wY_1 - \frac{\delta}{a_{12}}Y_2$ obtaining the corresponding system as follows

$$\begin{pmatrix} \frac{dY_1}{dt} \\ \frac{dY_2}{dt} \end{pmatrix} = \begin{pmatrix} 0 & -w \\ w & 0 \end{pmatrix} + \begin{pmatrix} F\left(Y_1,Y_2\right) + Q_3 \\ G\left(Y_1,Y_2\right) + Q_3 \end{pmatrix}$$

with Q_3, Q_4 functions in $Y_1^iY_2^j$ for $i+j \geq 4$ and

$$\begin{aligned} F &= \left(-\frac{a_{21}\delta}{a_{01}} + a_{30}\right)Y_2^3 + \frac{a_{21}Y_1Y_2^2}{a_{01}} + \left(a_{20} - \frac{a_{11}\delta}{a_{01}}\right)Y_2^2 \\ &\quad + \frac{a_{11}wY_1Y_2}{a_{01}}, \\ G &= \left(-\frac{b_{21}\delta}{a_{01}} + \frac{b_{12}\delta^2}{a_{01}^2} + b_{30}\right)Y_2^3 + \left(\frac{b_{21}w}{a_{01}} - \frac{2b_{12}\delta\omega}{a_{01}^2}\right)Y_1Y_2^2 \\ &\quad + \left(-\frac{b_{11}}{a_{01}} + b_{20} + \frac{b_{02}\delta^2}{a_{01}^2}\right)Y_2^2 + \frac{b_{12}\omega^2Y_1^2Y_2}{a_{01}^2} + \left(\frac{b_{11}w}{a_{01}}\right. \\ &\quad \left. - \frac{2b_{02}\omega\delta}{a_{01}^2}\right)Y_1Y_2 + \frac{b_{02}\omega^2Y_1^2}{a_{01}^2} \end{aligned}$$

Using Theorem 4, the following reflecting coefficient is described as

$$\begin{aligned} l &= \left(\frac{a_{21}\omega}{8a_{01}} + \frac{b_{12}\omega^2}{8a_{01}^2} - \frac{3b_{21}\delta}{8a_{01}} + \frac{3b_{12}\delta^2}{8a_{01}^2} + \frac{3b_{30}}{8}\right) \\ &\quad + \frac{1}{16\omega}\left(\frac{a_{11}\omega}{a_{01}}\left(2a_{20} - \frac{2a_{11}\delta}{a_{01}}\right) - \left(\frac{b_{11}\omega}{b_{01}}\right.\right. \\ &\quad \left.\left. - \frac{2b_{02}\delta\omega}{a_{01}^2}\right)\left(\frac{2b_{02}\omega^2}{a_{01}^2} - \frac{2b_{11}\delta}{a_{01}} + \frac{2b_{02}\delta^2}{a_{01}^2} + 2b_{20}\right)\right) \\ &\quad + \frac{1}{16\omega}\left(\left(2a_{20} - \frac{2a_{11}}{a_{01}}\right)\left(-\frac{2b_{11}\delta}{a_{01}} + \frac{2b_{02}\delta^2}{a_{01}^2} + 2b_{20}\right)\right). \end{aligned} \quad (11)$$

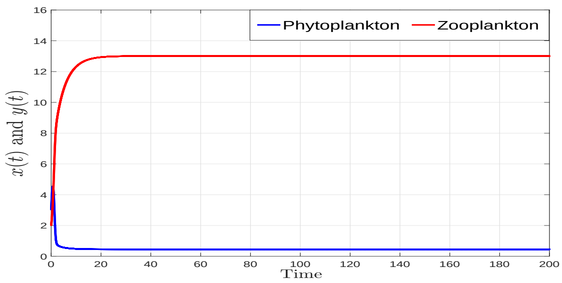
Theorem 4. *Suppose the system (1) and (2) has interior steady state E^*, which satisfy*

$$\delta = \delta^H \text{ and } \delta^H.\left[\frac{by\left(y^2-\beta^2\right)}{\left(y^2+\beta^2\right)^2}\right] - \frac{e\alpha^3xy}{(x+\alpha)^3} > 0$$

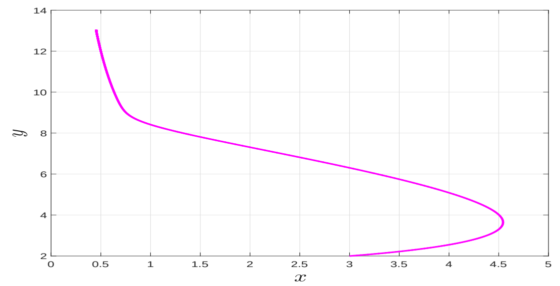
Assume $l \neq 0$ with l defined in eq. (11), the model then undergoes a Hopf bifurcation around E^*, implying that a periodic solution exists on around E^*. Furthermore, the periodic solution is a stable loop if $l > 0$ and that repels if $l < 0$.

7. Numerical simulation

In this section, we perform numerical simulations to validate our empirical findings to solve the following initial value problem (IVP) equation (1) to (3) using Runge-Kutta 4^{th} order method. We take into account the following parameters: $r = 1.0, \alpha = 5.0, i = 0.1, K = 50.0, \beta = 0.2, b = 0.1$, and $a = 5.0$

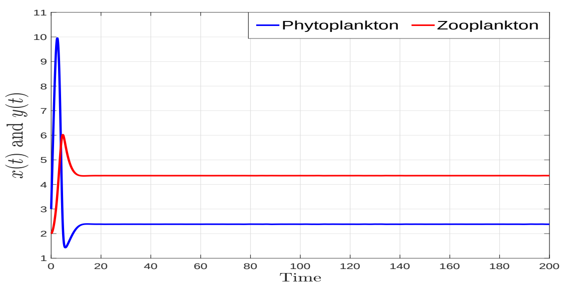


(a) Variations of plankton populations with respect to time



(b) Phase portrait of the populations

Figure 1. Solution of the IVP equation (1) to (3) when $m = 0.2$ and $e = 0.5$ under $m < e$ with parameters: $r = 1.0$, $\alpha = 5.0$, $i = 0.1$, $K = 50.0$, $\beta = 0.2$, $b = 0.1$, and $a = 5.0$.



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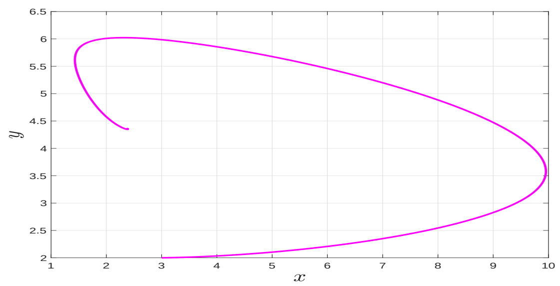
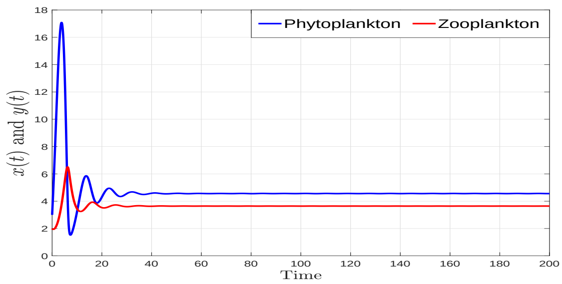
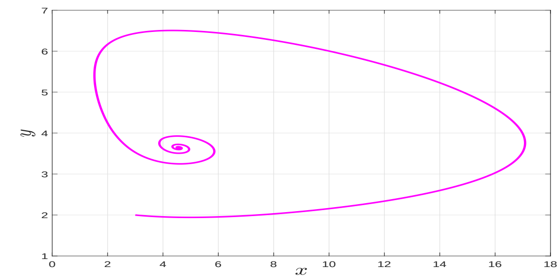


(b) Phase portrait of the populations

Figure 2. Solution of the IVP equation (1) to (3) when $m = 0.3$ and $e = 0.2$ under $m > e$ with parameters: $r = 1.0, \alpha = 5.0, i = 0.1, K = 50.0, \beta = 0.2, b = 0.1$, and $a = 5.0$.

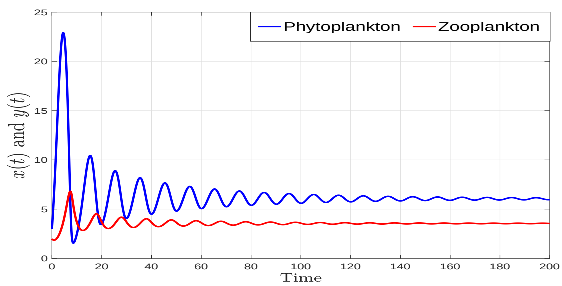


(a) Variations of plankton populations with respect to time

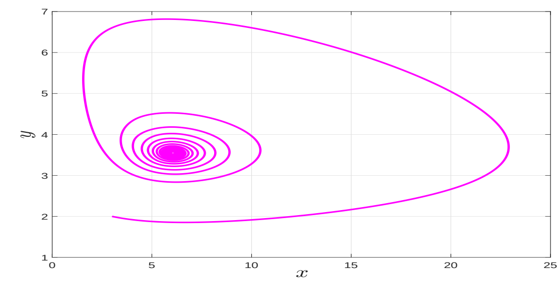


(b) Phase portrait of the populations

Figure 3. Solution of the IVP equation (1) to (3) when $m = 0.45$ and $e = 0.2$ under $m > e$ with parameters: $r = 1.0, \alpha = 5.0, i = 0.1, K = 50.0, \beta = 0.2, b = 0.1$, and $a = 5.0$.

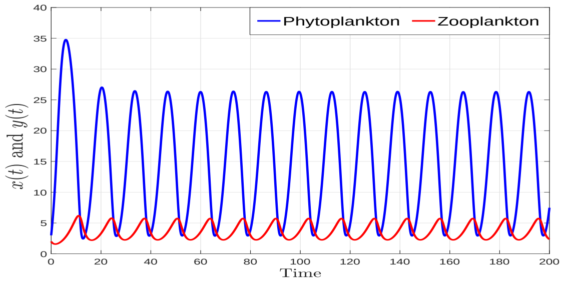


(a) Variations of plankton populations with respect to time

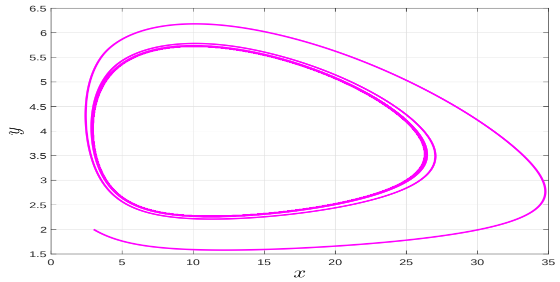


(b) Phase portrait of the populations

Figure 4. Solution of the IVP (1) to (3) when $m = 0.52$ and $e = 0.2$ under $m > e$ with parameters: $r = 1.0, \alpha = 5.0, i = 0.1, K = 50.0, \beta = 0.2, b = 0.1$, and $a = 5.0$.

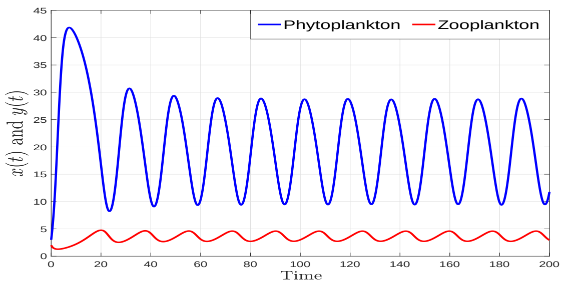


(a) Variations of plankton populations with respect to time

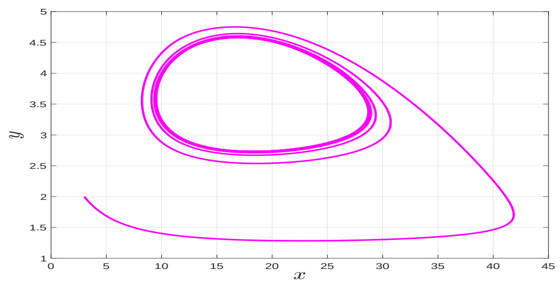


(b) Phase portrait diagram of population where $m < e$

Figure 5. Solution of the IVP (1) to (3) when $m = 0.65$ and $e = 0.2$ under $m > e$ with parameters: $r = 1.0, \alpha = 5.0, i = 0.1, K = 50.0, \beta = 0.2, b = 0.1$, and $a = 5.0$.

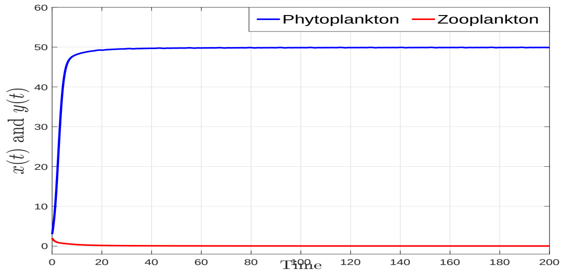


(a) Variations of plankton populations with respect to time

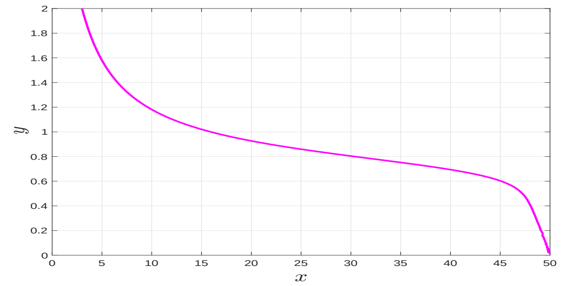


(b) Phase portrait of the populations

Figure 6. Solution of the IVP (1) to (3) when $m = 0.75$ and $e = 0.2$ under $m > e$ with parameters: $r = 1.0, \alpha = 5.0, i = 0.1, K = 50.0, \beta = 0.2, b = 0.1$, and $a = 5.0$.



(a) Variations of plankton populations with respect to time



(b) Phase portrait of the populations

Figure 7. Solution of the IVP (1) to (3) when $m = 0.9$ and $e = 0.2$ under $m > e$ with parameters: $r = 1.0, \alpha = 5.0, i = 0.1, K = 50.0, \beta = 0.2, b = 0.1$, and $a = 5.0$.

under the initial population densities $x_0 = 3.0$ and $y_0 = 2.0$. We first consider the case $m < e$ by choosing $m = 0.2$ and $e = 0.5$ and subsequently solve the IVP equation (1) to (3). Figure 1a depicts variations of phytoplankton and zooplankton over time, while Figure 1b depicts a phase portrait diagram among organisms. The solution of the IVP equation (1) to (3), $(x, y)(t)$ converges to the steady state (x^*, y^*) after a large time. Although both the planktons survive, the density of phytoplankton is relatively very low. We next change the parameter $m = 0.3$ and $e = 0.2$ so that $m > e$ and observe that the solution of the IVP equation (1) to (3), $(x, y)(t)$ converges to the steady state (x^*, y^*) after large time, that is, both the planktons coexist, as shown in Figure 2. Under such an assumption $m > e(e = 0.2)$, we fix $m = 0.45$ that results the coexistence of both phytoplankton and the zooplankton where the density of phytoplankton is relatively higher than that for the zooplankton, as depicted in Figure 3. Figure 3b demonstrates a spiral in the phase portrait of $x(t)$ and $y(t)$. For $m = 0.52$, similar behavior appears exhibiting spiral for $x(t)$ and $y(t)$, as presented in Figure 4. We next choose $m = 0.65$ under the same assumption for which oscillatory behavior of the density of phytoplankton and zooplankton occurs, as demonstrated in 5. It indicates the occurrence of a Hopf bifurcation. Figure 6 shows that both the planktons exhibit oscillatory behavior when $m = 0.75$. Furthermore for $m = 0.9$, the solution of the IVP equation (1) to (3) converges to the steady state $(k, 0)$, that is, the zooplankton extinct while the phytoplankton attains its carrying capacity. More specifically, higher mortality rate of the zooplankton causes the extinction of the zooplankton, as shown in Figure 7. In order to confirm the occurrence of

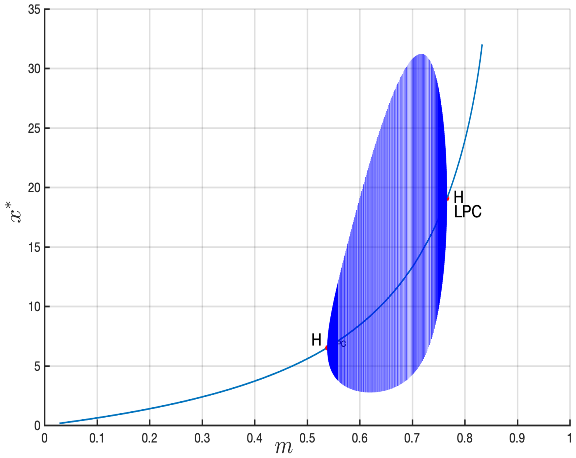


Figure 8. Bifurcation diagram of the system when m is a free parameter with parameters: $r = 1.0, \alpha = 5.0, i = 0.1, K = 50.0, \beta = 0.2, b = 0.1$, and $a = 5.0$.

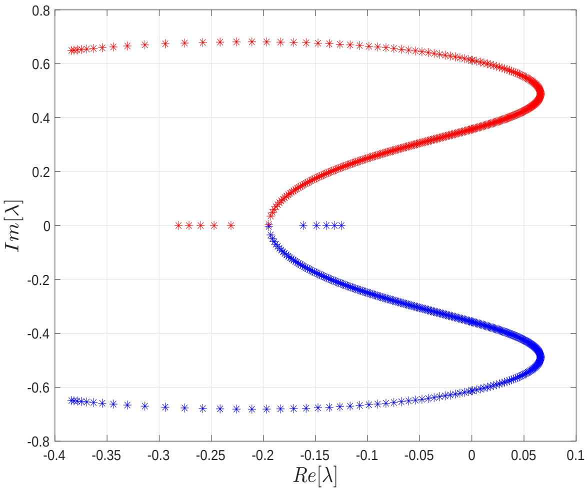


Figure 9. Eigenvalues of the Jacobian when m is a free parameter with parameters: $r = 1.0, \alpha = 5.0, i = 0.1, K = 50.0, \beta = 0.2, b = 0.1$, and $a = 5.0$.

Hopf bifurcation, we perform a numerical bifurcation analysis of the system (1) and (2) considering m as a free parameter. Numerical bifurcation analysis reveals that a Hopf bifurcation occurs at $m = 0.5385$ as a primary bifurcation and a secondary bifurcation of Hopf type occurs at $m = 0.7662$ when m increases under such a parameter setting. A bifurcation diagram is demonstrated in Figure 8. In the regime of m, $0.5385 < m < 0.7662$, a periodic time series of $(x, y)(t)$ appear due to the instability of the steady state (x^*, y^*), as presented in Figures 5 and 6. Real and imaginary parts of the eigenvalues of the Jacobian associated to the linear system (1) and (2) around the equilibrium (x^*, y^*) are presented in Figure 9 which has a significant agreement with the bifurcation diagram when m varies globally. Moreover, Figure 10 depicts real part of the eigenvalues of the Jacobian with respect to the free parameter m which strongly supports the occurrence of Hopf bifurcations.

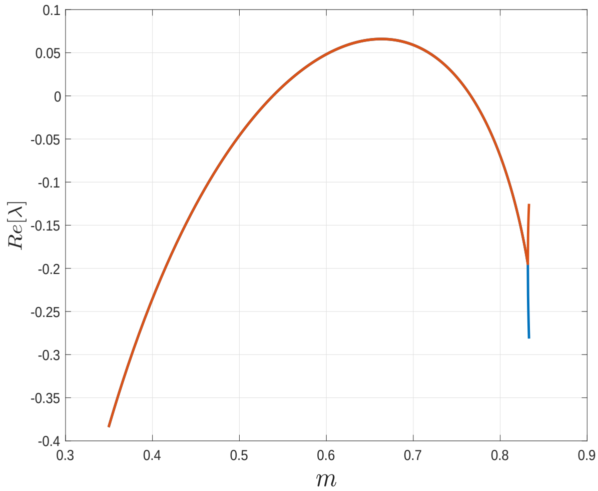


Figure 10. Real part of eigenvalues with respect to the parameter m with parameters: $r = 1.0, \alpha = 5.0, i = 0.1, K = 50.0, \beta = 0.2, b = 0.1$, and $a = 5.0$.

8. Concluding remarks and future research

In this paper, we discussed theoretical analysis of an algae zooplankton fish species model. Initially, we examined the positive invariance and boundedness of the proposed model.Using Routh-Hurwitz criteria and Lyapunov function, we found the existence of the positive interior steady state and the criteria for stability(both local and global) of the plankton model. Taylor's sequence was also used to discuss Hopf bifurcation and the stability of bifurcated periodic solutions. The model's bifurcation analysis reveals that Hopf bifurcation can occur when mortality rate and food transfer efficiency are used as bifurcation parameters. From Figures 5 and 6, it is concluded that when $m > e$, i.e., mortality rate of zooplankton is greater than its efficiency of food transfer then the system is highly oscillatory and no more stable nature when m lies in the interval $0.5385 < m < 0.7662$. On the other hand for $m < e$, that is, the mortality rate of zooplankton is less than its efficiency of food transfer then the system attains stable nature completely, that is, both phytoplankton and zooplankton survive when the remaining parameters are fixed. However, for a relatively larger value of m under the assumption $m > e$, the zooplankton is extinct while the phytoplankton attains its carrying capacity. In order to understand such complex dynamics of the system, we performed a numerical bifurcation analysis and subsequently generated bifurcation diagram considering m as a free parameter using bifurcation analysis. Numerical bifurcation analysis also justifies the occurrence of Hopf bifurcation at $m = 0.5385$ and at $m = 0.7662$, as a primary and secondary bifurcation, respectively. We also note that the parameter m (zooplankton death rate) has a destabilizing impact on the system's dynamics. Finally, the survival and extinction of the phytoplankton and zooplankton can be controlled under a suitable choice of the mortality rate of zooplankton (m) and efficiency of food transfer (e) that would really be supportive to assure sustainable balance of the ecosystems.

Predator-prey systems have grown in popularity in recent years and given rise to systems that more effectively depict various biological challenges that arise in the setting of interacting species. It can be extended to study the dynamics of species like zooplankton and phytoplankton systems by incorporating some intriguing features like the Allee effect, the fear effect, cannibalism, and immigration. We can also extend this work to study the stochasticity and movement of species with reference to time and space by considering the noise and cross diffusion effects and also the presence of limit cycles.

Author Contributions. **Das, K.**: Conceptualization, methodology, software, formal analysis, investigation, resources, data curation, writing-original draft preparation. **Madhusudanan, V.**: Conceptualization, methodology, validation, writing—review and editing, visualization, supervision. **Srinivas, M.N.**: Conceptualization, methodology, validation, writing—review and editing, visualization, supervision. **Kabir, H.**: validation, review and editing. **Gani, O.**: methodology, review and editing. **Sardar, M. N. I.**: visualization, supervision.

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References

[1] G. Caughley and J. H. Lawton, *Plant-herbivore systems in May, RM: Theoretical Ecology*. Blcakwell Scientific publications, 1981.
[2] M. L. Rosenzweig, "Paradox of enrichment: Destabilization of exploitation ecosystems in ecological time," *Science*, vol. 171, no. 3969, pp. 385–387, 1971. DOI:10.1126/science.171.3969.385
[3] J. Hrbáček, M. Dvořakova, V. Kořínek, and L. Procházkóva, "Demonstration of the effect of the fish stock on the species composition of zooplankton and the intensity of metabolism of the whole plankton association," *SIL Proceedings, 1922-2010*, vol. 14, no. 1, pp. 192–195, 1961. DOI:10.1080/03680770.1959.11899269
[4] J. L. Brooks and S. I. Dodson, "Predation, body size, and composition of plankton," *Science*, vol. 150, no. 3692, pp. 28–35, 1965. DOI:10.1126/science.150.3692.28
[5] J. Shapiro and D. I. Wright, "Lake restoration by biomanipulation: Round lake, minnesota, the first two years," *Freshwater Biology*, vol. 14, no. 4, pp. 371–383, 1984. DOI:10.1111/j.1365-2427.1984.tb00161.x
[6] K. D. Hambright, "Can zooplanktivorous fish really affect lake thermal dynamics?" *Archiv fr Hydrobiologie*, vol. 130, no. 4, pp. 429–438, 1994. DOI:10.1127/archiv-hydrobiol/130/1994/429
[7] J. Sedal and A. Duncan, "Low fish predation pressure in London reservoirs: II. Consequences to zooplankton community structure," *Hydrobiologia*, vol. 291, no. 3, pp. 179–191, 1994. DOI:10.1007/BF00014707
[8] S. Dodson, "The ecological role of chemical stimuli for the zooplankton: Predator-avoidance behavior in Daphnia," *Limnology and Oceanography*, vol. 33, no. 6part2, pp. 1431–1439, 1988. DOI:10.4319/lo.1988.33.6part2.1431
[9] L. D. Meester, L. J. Weider, and R. Tollrian, "Alternative antipredator defences and genetic polymorphism in a pelagic predator–prey system," *Nature*, vol. 378, no. 6556, pp. 483–485, 1995. DOI:10.1038/378483a0
[10] G. Stirling, "Daphnia Behaviour as a Bioassay of Fish Presence or Predation," *Functional Ecology*, vol. 9, no. 5, pp. 778–784, 1995. DOI:10.2307/2390252
[11] Z. M. Gliwicz, "Why do cladocerans fail to control algal blooms?," *Hydrobiologia*, vol. 200, no. 1, pp. 83–97, 1990. DOI:10.1007/BF02530331
[12] Z. M. Gliwicz and W. Lampert, "Food Thresholds in Daphnia Species in the Absence and Presence of Blue-Green Filaments," *Ecology*, vol. 71, no. 2, pp. 691–702, 1990. DOI:10.2307/1940323
[13] J. Chattopadhyay, R. R. Sarkar, and A. el Abdllaoui, "A delay differential equation model on harmful algal blooms in the presence of toxic substances," *Mathematical Medicine and Biology*, vol. 19, no. 2, pp. 137–161, 2002. DOI:10.1093/imammb/19.2.137
[14] R. Sarkar and J. Chattopadhayay, "Occurrence of planktonic blooms under environmental fluctuations and its possible control mechanism—mathematical models and experimental observations," *Journal of Theoretical Biology*, vol. 224, no. 4, pp. 501–516, 2003. DOI:10.1016/S0022-5193(03)00200-5
[15] S. Chakraborty, S. Roy, and J. Chattopadhyay, "Nutrient-limited toxin production and the dynamics of two phytoplankton in culture media: A mathematical model," *Ecological Modelling*, vol. 213, no. 2, pp. 191–201, 2008. DOI:10.1016/j.ecolmodel.2007.12.008
[16] K. Das and S. Ray, "Effect of delay on nutrient cycling in phytoplankton–zooplankton interactions in estuarine system," *Ecological Modelling*, vol. 215, no. 1-3, pp. 69–76, 2008. DOI:10.1016/j.ecolmodel.2008.02.019
[17] S.-J. Jang, J. Baglama, and J. Rick, "Nutrient-phytoplankton-zooplankton models with a toxin," *Mathematical and Computer Modelling*, vol. 43, no. 1-2, pp. 105–118, 2006. DOI:10.1016/j.mcm.2005.09.030
[18] J. Chattopadhayay, R. Sarkar, and S. Mandal, "Toxin-producing plankton may act as a biological control for planktonic blooms—field study and mathematical modelling," *Journal of Theoretical Biology*, vol. 215, no. 3, pp. 333–344, 2002. DOI:10.1006/jtbi.2001.2510
[19] T. Saha and M. Bandyopadhyay, "Dynamical analysis of toxin producing phytoplankton–zooplankton interactions," *Nonlinear Analysis: Real World Applications*, vol. 10, no. 1, pp. 314–332, 2009. DOI:10.1016/j.nonrwa.2007.09.001
[20] Y. Wang, W. Jiang, and H. Wang, "Stability and global Hopf bifurcation in toxic phytoplankton–zooplankton model with delay and selective harvesting," *Nonlinear Dynamics*, vol. 73, no. 1-2, pp. 881–896, 2013. DOI:10.1007/s11071-013-0839-2
[21] R. K. Upadhyay, P. Roy, and J. Datta, "Complex dynamics of ecological systems under nonlinear harvesting: Hopf bifurcation and Turing instability," *Nonlinear Dynamics*, vol. 79, no. 4, pp. 2251–2270, 2015. DOI:10.1007/s11071-014-1808-0
[22] F. Bian, W. Zhao, Y. Song, and R. Yue, "Dynamical Analysis of a Class of Prey-Predator Model with Beddington-DeAngelis Functional Response, Stochastic Perturbation, and Impulsive Toxicant Input," *Complexity*, vol. 2017, pp. 1–18, 2017. DOI:10.1155/2017/3742197
[23] L. Liu and X. Meng, "Optimal harvesting control and dynamics of two-species stochastic model with delays," *Advances in Difference Equations*, vol. 2017, no. 1, pp. 1–18, 2017. DOI:10.1186/s13662-017-1077-6
[24] Y. Pei, Y. Lv, and C. Li, "Evolutionary consequences of harvesting for a two-zooplankton one-phytoplankton system," *Applied Mathematical Modelling*, vol. 36, no. 4, pp. 1752–1765, 2012. DOI:10.1016/j.apm.2011.09.015
[25] T. Zhang, W. Ma, and X. Meng, "Global dynamics of a delayed chemostat model with harvest by impulsive flocculant input," *Advances in Difference Equations*, vol. 2017, no. 1, pp. 1–17, 2017. DOI:10.1186/s13662-017-1163-9
[26] J. Wang, H. Cheng, H. Liu, and Y. Wang, "Periodic solution and control optimization of a prey-predator model with two types of harvesting," *Advances in Difference Equations*, vol. 2018, no. 1, pp. 1–14, 2018. DOI:10.1186/s13662-018-1499-9
[27] W. Lampert, W. Fleckner, H. Rai, and B. E. Taylor, "Phytoplankton control by grazing zooplankton: A study on the spring clear-water phase1," *Limnology and Oceanography*, vol. 31, no. 3, pp. 478–490, 1986. DOI:10.4319/lo.1986.31.3.0478
[28] C. Luecke, M. J. Vanni, J. J. Magnuson, J. F. Kitchell, and P. T. Jacobson, "Seasonal regulation of daphnia populations by planktivorous fish: Implications for the spring clear-water phase," *Limnology and Oceanography*, vol. 35, no. 8, pp. 1718–1733, 1990. DOI:10.4319/lo.1990.35.8.1718
[29] O. Sarnelle, "Herbivore Effects on Phytoplankton Succession in a Eutrophic Lake," *Ecological Monographs*, vol. 63, no. 2, pp. 129–149, 1993. DOI:10.2307/2937177
[30] D. Schindler, "The dependence of primary production upon physical and chemical factors in a small, senescing lake, including the effects of complete winter oxygen depletion," *Arch. Hydrobiol.*, vol. 69, pp. 413–451, 1972.
[31] E. van Donk, M. P. Grimm, R. D. Gulati, and J. P. G. Klein Breteler, "Whole-lake food-web manipulation as a means to study community interactions in a small ecosystem," *Hydrobiologia*, vol. 200, no. 1, pp. 275–289, 1990. DOI:10.1007/BF02530346
[32] M. Scheffer, S. Rinaldi, and Y. A. Kuznetsov, "Effects of fish on plankton dynamics: a theoretical analysis," *Canadian Journal of Fisheries and Aquatic Sciences*, vol. 57, no. 6, pp. 1208–1219, 2000. DOI:10.1139/f00-018

[33] D. Savitri, N. W. Hidajati, and H. S. Panigoro, "Implementasi algoritma genetika dalam mengestimasi kepadatan populasi jackrabbit dan coyote," *Jambura Journal of Biomathematics (JJBM)*, vol. 3, no. 1, pp.23–28, 2022. DOI:10.34312/jjbm.v3i1.11935

[34] A. L. Firdiansyah and N. Nurhidayati, "Dynamics in two competing predators-one prey system with two types of Holling and fear effect," *Jambura Journal of Biomathematics (JJBM)*, vol. 2, no. 2, pp. 58–67, 2021. DOI:10.34312/jjbm.v2i2.11264

[35] R. Ibrahim, L. Yahya, E. Rahmi, and R. Resmawan, "Analisis dinamik model predator-prey tipe Gause dengan wabah penyakit pada prey," *Jambura Journal of Biomathematics (JJBM)*, vol. 2, no. 1, pp. 20–28, 2021. DOI:10.34312/jjbm.v2i1.10363

[36] X. Feng, Y. Song, and X. An, "Dynamic behavior analysis of a prey-predator model with ratio-dependent monod-haldane functional response," *Open Mathematics*, vol. 16, no. 1, pp. 623–635, 2018. DOI:10.1515/math-2018-0060

[37] H. Liu and H. Cheng, "Dynamic analysis of a prey–predator model with state-dependent control strategy and square root response function," *Advances in Difference Equations*, vol. 2018, no. 1, pp. 1–13, 2018. DOI:10.1186/s13662-018-1507-0

[38] X. Feng, Y. Song, J. Liu, and G. Wang, "Permanence, stability, and coexistence of a diffusive predator–prey model with modified Leslie–Gower and B–D functional response," *Advances in Difference Equations*, vol. 2018, no. 1, pp. 1–17, 2018. DOI:10.1186/s13662-018-1735-3

[39] T. Zhou, X. Zhang, M. Xiang, and Z. Wu, "Permanence and almost periodic solution of a predator–prey discrete system with holling iv functional response," *International Journal of Biomathematics*, vol. 9, no. 3, p. 1650035, 2016. DOI:10.1142/S1793524516500352

[40] J. Huang, X. Xia, X. Zhang, and S. Ruan, "Bifurcation of codimension 3 in a predator–prey system of leslie type with simplified holling type iv functional response," *International Journal of Bifurcation and Chaos*, vol. 26, no. 02, p. 1650034, 2016. DOI:10.1142/S0218127416500346

[41] A. Sharma, A. K. Sharma, and K. Agnihotri, "Analysis of a toxin producing phytoplankton–zooplankton interaction with Holling IV type scheme and time delay," *Nonlinear Dynamics*, vol. 81, no. 1-2, pp. 13–25, 2015. DOI:10.1007/s11071-015-1969-5