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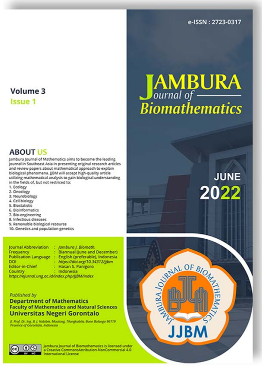

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Research Article

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Dynamical Behavior in Prey-Predator Model with Mutualistic Protection for Prey

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ABSTRACT. *This article reconstructs the model of predator-prey mutualistic protection based on a journal written by Revilla and Křivan (2022). The predator-prey model considers mutualistic protection for the prey. The study focuses on the analysis of equilibrium points and combines an adaptive model to study the influence of both models on predator-prey dynamics. This research continues the stability analysis and numerical simulations of the predator-prey model with mutualistic protection to examine the impact of mutualistic protection on prey dynamics in the model. The research process begins with a literature review, reconstructing the predator-prey model, determining equilibrium points, analyzing stability at the equilibrium points, conducting numerical simulations including bifurcation diagrams and phase portraits of the model solutions, and drawing conclusions. The analysis yields three equilibrium points: the unstable co-extinction of both populations, predator extinction, and the conditionally stable coexistence of both populations. Based on the analysis results, there are changes in the system solutions, with the originally stable* E_3 *becoming unstable. There is also a change in* E_2 *from being unstable to stable. Through numerical continuation with variations in the parameter representing the mutualistic protector's preference for prey resources* (u)*, a transcritical bifurcation (Branch Point) is obtained at* $u = 0.888889$*. The simulation results demonstrate that* (u) *can influence the stability of predator and prey populations.*

1. Introduction

One of the powerful tools to study the existence of one or more populations in an ecosystem is using mathematical modeling with a deterministic approach. The popular topic in ecosystems is studying the interaction between prey and its predator, see [1–4] and references therein. Moreover, the interaction between two populations is arriving in symbiosis ways such as commensalism, parasitism, and mutualism. Mutualism, a fundamental concept in ecology, encapsulates the intricate relationships that can develop between different species in an ecosystem, often leading to mutual benefits [5–7]. Protection mutualism, a distinctive subtype of mutualism, revolves around one species offering protection to another in exchange for resources or other advantages [8–10]. This type of symbiosis holds intriguing implications for ecological dynamics, especially when it involves species such as ants and aphids.

Protection mutualism hinges on a delicate balance of cooperation: ants, renowned for their aggressive behavior, provide a shield for aphids, safeguarding them from natural predators, particularly ladybirds[11–13]. These aphid colonies, in turn, provide ants with a valuable source of sustenance in the form of sweet secretions. This captivating interplay between aphids, ladybirds, and ants forms the core of our study, as we delve into the intricacies of protection mutualism within a predator-prey framework [14–16].

In our investigation, we focus on a specific ecological scenario, where the population participating in protection mutualism comprises prey species and their mutualistic protectors. The primary actors in this ecological phenomena are ladybirds, the predators; aphids, their prey; and ants, the mutualistic protectors of the aphids [17–19]. Recent studies have highlighted the significance of this intricate relationship within various contexts [20, 21]. Together, they constitute an ecosystem where survival, predation, and cooperation intersect.

The relevance of protection mutualism to ecological dynamics has been well-documented in previous research [22–24]. In particular, Revilla and Křivan [14] advanced the field by constructing a mathematical model that explores the intricate dance between protection mutualism and predator-prey dynamics. Their pioneering work laid the foundation for our study, which seeks to build upon and extend their insights.

In the pages that follow, we embark on a journey to reconstruct and analyze the predator-prey model of protection mutualism, drawing from the study established by Revilla and Křivan [14]. Through rigorous stability analysis and numerical simulations, we aim to unravel the nuanced influence of protection mutualism on the intricate dynamics of predator-prey relationships. By systematically varying the parameters governing this interaction, we aim to shed light on the subtle shifts and fluctuations that can occur in ecological systems.

In this article, we not only explore the mathematical underpinnings of protection mutualism but also offer practical insights

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into how these dynamics may manifest in real-world ecosystems. Our study underscores the critical role of protection mutualism in shaping predator-prey interactions, ultimately contributing to a deeper understanding of the delicate ecological tapestry that binds species together.

2. Methods

The research methodology used is presented in the flowchart in Figure 1.

2.1. Literature Review

The steps taken in the literature review phase are as follows:

a. Literature review on mutualistic protection interactions among populations in food chains in ecosystems. This initial study involves aphids as prey, ladybugs as predators, and ants as mutualistic protectors of the prey.
b. Literature review to represent population interactions in mathematical models. This study includes predator-prey hunting patterns, mutualistic protection patterns on prey, and the identification of parameters influencing the model. Influential parameters include the mutualistic protector's preference for prey sources, the strength of mutualistic protector interference in predation, and the size of the mutualistic protector population.
c. Literature review based on population dynamics concepts to analyze the dynamic solutions of the system. The system analyzed is the dynamics of predator-prey mutualism with protection for prey.

2.2. Reconstruction of Predator-Prey Interaction Model with Mutualistic Protection for Prey

In this phase, the interaction model between predator and prey considering mutualistic protection is reconstructed based on the mathematical model from the study by Revilla and Kr̆ivan (2022) titled "Prey-Predator Dynamics with Adaptive Protection Mutualism". This study proceeds with local stability analysis and numerical simulation of the mutualistic protection predator-prey model to study its influence on predator-prey dynamics. The reconstructed mathematical model represents the populations of prey and predators at time t using $P(t)$ and $H(t)$. This model is then expanded to consider the interaction between aphids as prey, ladybugs as predators, and ants as mutualistic protectors of the prey.

2.3. Determination of Equilibrium Points

Equilibrium points are states where the predator and prey populations are stable, meaning there is no significant change in population numbers over time. Equilibrium points can be found by identifying values of P and H that satisfy the equations $\frac{dP}{dt} = 0$ and $\frac{dH}{dt} = 0$. This condition indicates that there are no changes in predator and prey populations over time, making it an equilibrium point. The next step is to determine the conditions or requirements for the equilibrium points in the reconstructed model to exist.

2.4. Local Stability Analysis

Local stability analysis is performed by determining the type of stability of the system around the equilibrium points. This step can be done by analyzing the eigenvalues of the Jacobian matrix around those points. This step is performed to determine whether the system solutions are stable (all eigenvalues have negative real parts) or unstable (at least one eigenvalue has a positive real part).

2.5. Numerical Simulation

Numerical simulation can be used to confirm the results of the analysis and visualize the dynamic behavior changes of the system solutions using PPlane and MatCont software in MATLAB R2018a. The results of numerical simulations can be illustrated through phase portraits showing the patterns of solution changes in phase space and bifurcation diagrams that depict the structural changes of the system as parameters are varied.

2.6. Conclusion

The conclusion contains the biological interpretation of changes in the predator-prey model found in the local stability analysis phase around equilibrium points. This conclusion is based on the results of local stability analysis and numerical simulations of the model's behavior, particularly in the context of mutualistic protection phenomena for prey in ecology.

3. Results and Discussion

3.1. Predator-Prey Interaction Model with Protection Mutualism for Prey

In this study, the author reconstructed a model of mutualistic protection in predator-prey interactions based on the study by [14]. The predator-prey model the mutualistic protection for the prey. The model focuses on the analysis of equilibrium points and combines an adaptive model to study the effects of both models on the predator-prey dynamics. The study continues with the analysis of local stability and numerical simulations of the mutualistic protection predator-prey model to investigate the influence of mutualistic protection on the dynamics of the model.

The species used in the mutualistic protection predator-prey model are ladybugs as predators, aphids as prey, and ants as mutualistic protectors for the prey. The assumptions used in this study are as follows:

1. In the absence of external limiting factors, the growth rate of the aphid population follows a logistic equation.
2. The growth rate of aphids decreases due to interactions with ladybugs when ladybugs prey on aphids.
3. The growth rate of ladybugs increases due to interactions with aphids when ladybugs consume aphids.
4. The death rate of aphids decreases due to interactions with ants, as ants protect aphids from being eaten by ladybugs.
5. The growth rate of aphids increases due to interactions between ants and ladybugs, which prevents ladybugs from preying on aphids.
6. The growth rate of ladybugs decreases due to an increase in natural death rate.

Based on the above descriptions and assumptions, the predator-prey model with mutualistic protection for the prey is as follows.

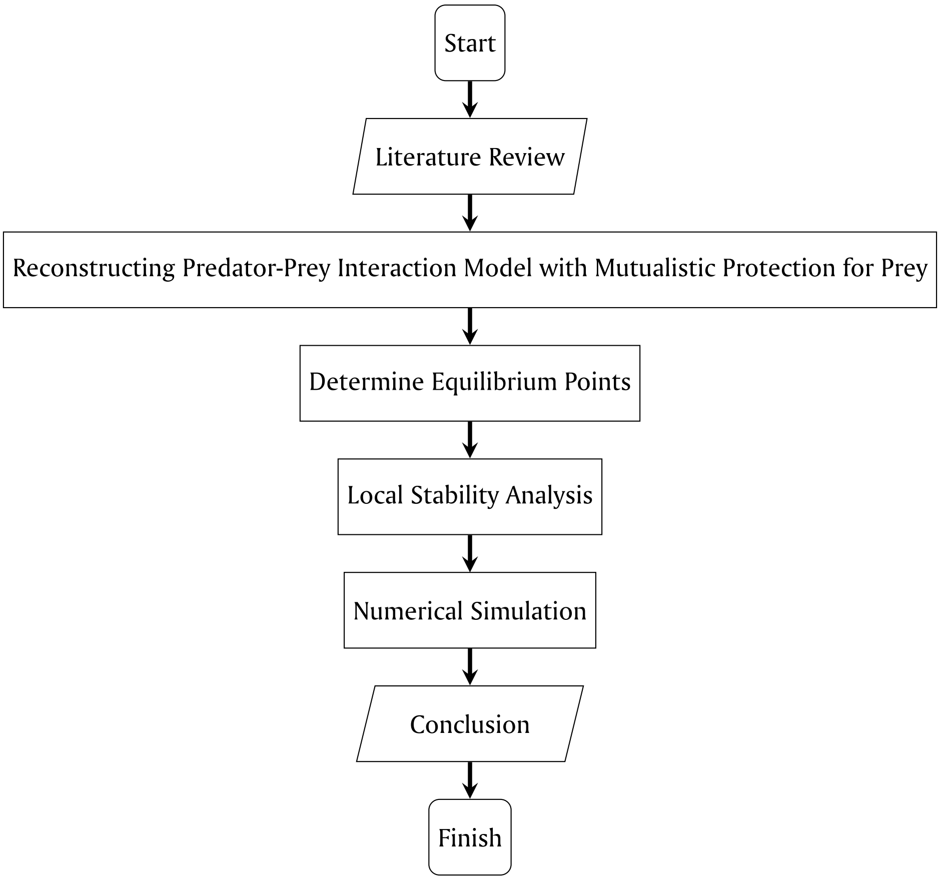

Figure 1. Research Flowchart

a. Aphid population as prey (P)
The growth of aphids is assumed to follow a logistic growth model in the absence of interactions between aphids and ladybugs. This model involves two important factors: the carrying capacity of the environment (K) and the intrinsic growth rate of aphids (r). The growth of aphids is described by the equation $rP\left(1-\frac{P}{K}\right)$, where P is the population of aphids.
When there are interactions between aphids and ladybugs, the growth of aphids is affected. This interaction can be described by the function $f(P)H$, which represents the response of ladybugs to the population of aphids. This function is the product of the ladybug population (H), which represents the consumption rate of ladybugs on aphids. Therefore, $f(P)H$ can be expressed as $\frac{aPH}{1+Mqu}$, where a is the predation rate of ladybugs on aphids. The variables M, q, and u play an important role in explaining the interactions between aphids, ladybugs, and ants. The variable M represents the size of the ant colony. The larger the value of M, the more ants are present in the colony. The variable q represents the strength of ant interference in the predation of ladybugs on aphids. The larger the value of q, the greater the influence of ants on ladybug predation. Finally, the variable u represents the preference of ants for aphid resources. The larger the value of u, the higher the preference of ants for aphids.
The parameters r, K, a, M, q, and u are positive values. If the population density of aphids at each unit of time is denoted by $P(t)$, then the growth rate of aphids is given by

$$P(t)=\frac{dP}{dt}=rP\left(1-\frac{P}{K}\right)-\frac{aPH}{1+Mqu}. \tag{1}$$

b. Ladybug population as predator (H)
The growth of ladybugs is assumed to increase when there are aphids present. The interaction between ladybugs and aphids leads to the absorption of aphid biomass by ladybugs. This can be described by the function $f(P)H$, where $f(P)$ is the ladybug's response function to the population of aphids, and H is the population of ladybugs. This response function is expressed as $\frac{aePH}{1+Mqu}$, where the parameter e represents the conversion rate of aphid biomass into ladybug biomass. The larger the value of e, the more efficient ladybugs are in converting aphid biomass into ladybug biomass. The decrease in ladybug growth is caused by natural death in ladybugs. This factor is described by $-mH$, where m is the natural death rate of ladybugs. The larger the value of m, the higher the natural death rate in ladybugs.
The parameters e and m are positive values. If the population density of ladybugs at each unit of time is denoted by $H(t)$, then the growth rate of the ladybug population is given by

$$H(t)=\frac{dH}{dt}=\frac{aePH}{1+Mqu}-mH. \tag{2}$$

Based on eqs. (1) and (2), the mathematical system of equations for the predator-prey model with mutualistic protection for the prey as follows:

$$\begin{aligned}\frac{dP}{dt} &= rP\left(1-\frac{P}{K}\right)-\frac{aPH}{1+Mqu},\\ \frac{dH}{dt} &= \frac{aePH}{1+Mqu}-mH.\end{aligned} \tag{3}$$

3.2. Equilibrium Points

The equilibrium points are obtained from the conditions $\frac{dP}{dt}=0$ and $\frac{dH}{dt}=0$ of eqs. (1) and (2), resulting in the following equations:

$$\begin{aligned}rP\left(1-\frac{P}{K}\right)-\frac{aPH}{1+Mqu}&=0,\\ \frac{aePH}{1+Mqu}-mH&=0.\end{aligned} \tag{4}$$

The system of eq. (4) can be solved to find P and H, yielding three equilibrium points as follows:

1. Equilibrium point E_1 exists at $(0,0)$, indicating the extinction of aphids and ladybugs.
2. Equilibrium point E_2 exists at $(K,0)$, indicating the extinction of ladybugs.
3. Suppose that:

$$\begin{aligned}\eta_1 &= Kae,\\ \eta_2 &= m(Mqu+1).\end{aligned}$$

Equilibrium point E_3 exists at $\left(\frac{\eta_2}{ae},\frac{r\left[\eta_1(Mqu+1)-m(1+M^2q^2u^2+2Mqu)\right]}{\eta_1 e}\right)$ with the condition $\eta_1(Mqu+1)>m(1+M^2q^2u^2+2Mqu)$, indicating the coexistence of ladybugs and aphids.

3.3. Local Stability Analysis

The first step in determining the stability of each equilibrium point is linearization. The stability of the system at each equilibrium point can be determined by finding the eigenvalues or roots of the characteristic equation.

The Jacobian matrix of the predator-prey system can be obtained by taking partial derivatives of eqs. (1) and (2), resulting in the following Jacobian matrix:

$$J(P,H)=\begin{bmatrix} r\left(1-\frac{2P}{K}\right)-\frac{aH}{Mqu+1} & -\frac{aP}{Mqu+1}\\ \frac{aeH}{Mqu+1} & \frac{aeP}{Mqu+1}-m\end{bmatrix}. \tag{5}$$

The Jacobian matrix (5) is used to perform local stability analysis of the system by obtaining the characteristic values from its determinant.

3.3. Stability of Equilibrium Point E_1

By evaluating the Jacobian matrix (5) at equilibrium point $E_1(0,0)$, we obtain

$$J_{E_1}=\begin{bmatrix} r & 0\\ 0 & -m\end{bmatrix}. \tag{6}$$

The Jacobian matrix (6) can be used to find the eigenvalues by solving the characteristic equation $det\left(J_{E_1}-\lambda I\right)=0$, yielding

$$\begin{aligned}det\left(\begin{bmatrix} r & 0\\ 0 & -m\end{bmatrix}-\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right)&=0,\\ det\begin{bmatrix} r-\lambda & 0\\ 0 & -m-\lambda\end{bmatrix}&=0,\\ (\lambda-r)(\lambda+m)&=0,\end{aligned}$$

and the eigenvalues are $\lambda_1=r$ and $\lambda_2=-m$. As a result, the equilibrium point E_1 is an unstable saddle point because $\lambda_1>0$ and $\lambda_2<0$.

3.3. Stability of Equilibrium Point E_2

By evaluating the Jacobian matrix (5) at equilibrium point $E_2(K,0)$, we obtain

$$J_{E_2}=\begin{bmatrix} -r & -\frac{aK}{Mqu+1}\\ 0 & \frac{\eta_1}{Mqu+1}-m\end{bmatrix}. \tag{7}$$

The Jacobian matrix (7) can be used to find the eigenvalues by solving the characteristic equation $det\left(J_{E_2}-\lambda I\right)=0$, yielding

$$\begin{aligned}det\left(\begin{bmatrix} -r & -\frac{aK}{Mqu+1}\\ 0 & \frac{\eta_1}{Mqu+1}-m\end{bmatrix}-\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right)&=0,\\ det\begin{bmatrix} -r-\lambda & -\frac{aK}{Mqu+1}\\ 0 & \frac{\eta_1}{Mqu+1}-m-\lambda\end{bmatrix}&=0,\end{aligned}$$

and the eigenvalues are $\lambda_1=-r$ and $\lambda_2=\frac{\eta_1-\eta_2}{Mqu+1}$.

As a result, there are two possibilities for the stability of equilibrium point E_2:

1. If $\eta_1<\eta_2$, the equilibrium point E_2 is a stable asymptotic nodal sink.
2. If $\eta_1>\eta_2$, the equilibrium point E_2 is an unstable saddle point.

3.3. Stability of Equilibrium Point E_3

By evaluating the Jacobian matrix (5) at equilibrium point $E_3=\left(\frac{\eta_2}{ae},\frac{r\left[\eta_1(Mqu+1)-m(1+M^2q^2u^2+2Mqu)\right]}{\eta_1 e}\right)$, we obtain

$$J_{E_3}=\begin{bmatrix} l_{11} & l_{12}\\ l_{21} & 0\end{bmatrix}. \tag{8}$$

where

$$\begin{aligned}l_{11} &= r\left(1-\frac{2\eta_2}{\eta_1}\right)-\frac{r[\eta_1(Mqu+1)-m(1+M^2q^2u^2+2Mqu)]}{\eta_1(Mqu+1)},\\ l_{12} &= -\frac{m}{e},\\ l_{21} &= \frac{r[\eta_1(Mqu+1)-m(1+M^2q^2u^2+2Mqu)]}{aK(Mqu+1)}.\end{aligned}$$

The Jacobian matrix (8) can be used to find the eigenvalues by solving the characteristic equation $det\left(J_{E_3}-\lambda I\right)=0$, yielding

$$\begin{aligned}det\left(\begin{bmatrix} l_{11} & l_{12}\\ l_{21} & 0\end{bmatrix}-\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right)&=0,\\ det\begin{bmatrix} l_{11}-\lambda & l_{12}\\ l_{21} & -\lambda\end{bmatrix}&=0,\\ \eta_1\lambda^2+r\eta_2\lambda+mr(\eta_1-\eta_2)&=0,\end{aligned}$$

Table 1. Parameter Values in the Predator-Prey Model with Mutualism Protection for Prey

Variable Parameter	Interpretation	Value Parameter
r	Intrinsic growth rate of aphids as prey	0.1 per day [14]
K	Carrying capacity of the aphid population as prey	50 individuals [14]
a	Predation rate of ladybirds as predators on aphids as prey	0.01 per day [14]
e	Conversion rate of aphid biomass as prey to ladybirds as predators	0.5 per day [14]
m	Natural death rate of ladybirds as predators	0.05 per day [14]
M	Size of ant colony as mutualistic protectors	150 individuals [14]
q	Interference strength of ants as mutualistic protectors against predation	0.03 [14]
u	Preference of ants as mutualistic protectors for aphid resources as prey	$0 \leq u \leq 1$ [14]

and the eigenvalues are given by

$$\lambda_{1,2} = \frac{-r\eta_2 \pm \sqrt{(r\eta_2)^2 - 4(\eta_1)\, mr\,(\eta_1 - \eta_2)}}{2\eta_1}.$$

Therefore, there are two possibilities for the stability of equilibrium point E_3:

1. If $\eta_1 > \eta_2$, the equilibrium point E_3 is a stable asymptotic spiral sink.
2. If $\eta_1 < \eta_2$, the equilibrium point E_3 is an unstable saddle point.

3.4. Numerical Simulation

Numeric simulation is used to demonstrate the consistency of the results of the local stability analysis of the predator-prey model using PPlane and MatCont in MATLAB R2018a software. The variables and parameter values used in conducting the numerical simulation are following [14] and are presented in Table 1. A numerical continuation of one of the parameter values is performed to analyze the effect of mutualistic protection on the dynamics of the predator-prey population. The parameter in question is u, which represents the preference of ants as mutualistic protectors for aphid resources as the main prey, with a range of values $0 \leq u \leq 1$. When $u = 0$, ants specialize in alternative prey or prey other than aphids, such as plants. When $0 < u < 1$, ants do not specialize in a single type of prey resource but utilize other resources as well. When $u = 1$, ants specialize in aphid resources as the primary prey.

Based on the parameter values used in Table 1, there are three existing equilibrium points, namely E_1, E_2, and E_3. The behavior of the solution of the system of equations (3) is demonstrated through several simulations. The simulations show phase portraits for varying values of the parameter u.

In the phase portrait displayed in Figure 2, we examine the dynamics of an ecological system involving ants, aphids, and ladybirds. When the value of parameter u is set to 0.4, the phase portrait reveals the existence of three equilibrium points: $E_1(0,0)$, $E_2(50,0)$, and $E_3(28,12.32)$. These equilibrium points are critical in understanding the system's behavior.

$E_1(0,0)$ and $E_2(50,0)$ represent unstable saddle points. This means that if the population densities of aphids and ladybirds were initially at these points, small perturbations could lead to dramatic changes in the populations, potentially destabilizing the ecosystem. $E_3(28,12.32)$ is identified as a stable asymptotic spiral. It signifies a stable coexistence state for aphids and ladybirds. When the populations of these two species are near this equilibrium point, they tend to remain there over time, indicating a balanced and stable ecosystem.

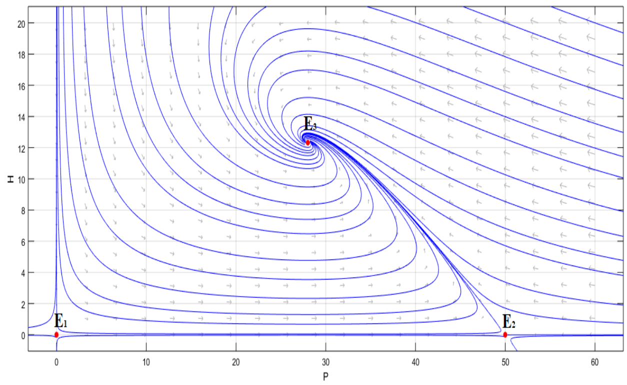

Figure 2. Phase portrait when $u = 0.4$, the solution of the system approaches the equilibrium point E_3

The phase portrait in Figure 2 analysis reveals that in this ecological system, when ants do not specialize in a single type of prey resource, a stable coexistence between aphids and ladybirds can occur. This insight is vital for understanding the dynamics and long-term stability of the ecosystem.

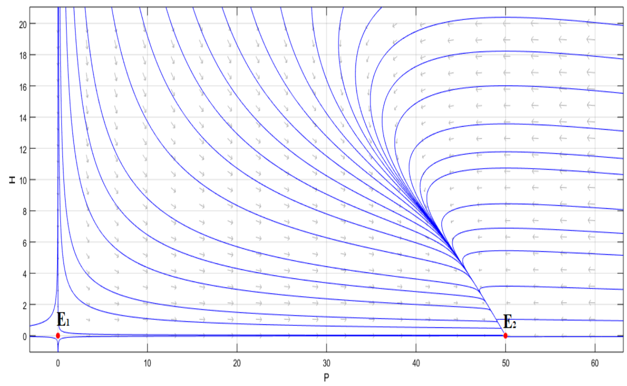

Figure 3. Phase portrait when $u = 0.888889$, the solution of the system approaches the equilibrium point $E_2 = E_3$

In the phase portrait depicted in Figure 3, we explore the dynamics of an ecological system under the parameter setting $u = 0.888889$. Within this context, we observe two equilibrium points: $E_1(0,0)$ and $E_2 = E_3(50,0)$. These equilibrium points play a crucial role in understanding the behavior of the system.

$E_1(0,0)$ represents an unstable saddle point. This means that if the initial populations of aphids and ladybirds were at this point, even minor disturbances could lead to significant fluctuations in these populations, potentially destabilizing the ecosystem. $E_2(50,0)$ is identified as a stable asymptotic node. In this scenario, it signifies a stable state in which aphids thrive, while the ladybird population remains extinct. This indicates that, under the given conditions, the stability of the ecosystem occurs when the ladybird population goes extinct.

The phase portrait in Figure 3 analysis reveals that in this ecological system, a stable state is achieved when the ladybird population is extinct, and aphids exist without any predators. This insight is essential for understanding the dynamics and equilibrium points in the ecosystem, shedding light on scenarios where the ladybird population does not play a role in maintaining balance.

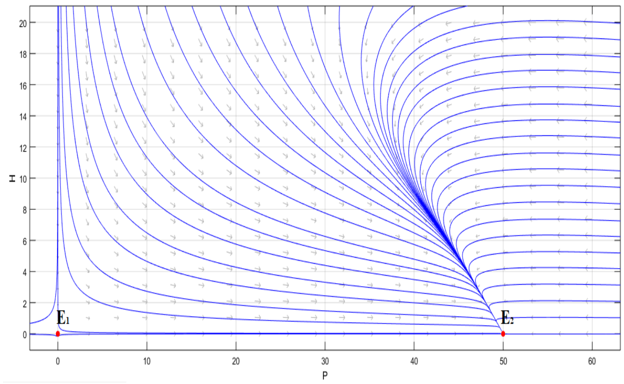

Figure 4. Phase portrait when $u = 1$, the solution of the system approaches the equilibrium point E_2

In the phase portrait depicted in Figure 4, we explore the dynamics of an ecological system with a parameter setting of $u = 1$. Within this context, we observe two equilibrium points: $E_1(0, 0)$ and $E_2(50, 0)$. These equilibrium points are pivotal in understanding the system's behavior.

$E_1(0, 0)$ represents an unstable saddle point, indicating that minor disturbances in the initial populations of aphids and ladybirds can lead to significant fluctuations, potentially destabilizing the ecosystem. $E_2(50, 0)$ is identified as a stable asymptotic node. This stable point in the phase portrait suggests that, under the condition where ants specialize primarily in aphid resources as their prey, the ecosystem exhibits stability. Specifically, it indicates that the ladybird population, acting as predators, goes extinct. The point $E_3(50, -5.5)$ is a theoretical equilibrium point, but it holds no biological significance because it predicts a negative ladybird population, which is biologically undefined. Therefore, for practical purposes, $E_3(50, -5.5)$ is not considered in our analysis.

The phase portrait in Figure 4 analysis reveals that when ants specialize in aphid resources as their primary prey, the stability of the ecosystem occurs when the ladybird population as predators goes extinct. Our focus remains on the biologically meaningful equilibrium points, $E_1(0, 0)$ and $E_2(50, 0)$.

Analysis of the phase portraits reveals a notable shift in system stability as the parameter u varies. When $u = 0.4$, the equilibrium point E_3 exhibits stable asymptotic behavior, while at $u = 0.888889$ and $u = 1$, stability shifts to E_2. This transformation in stability due to parameter u is depicted in the bifurcation diagram shown in Figure 5.

In this diagram, a transcritical bifurcation occurs precisely at $u = 0.888889$, marked as the bifurcation point (BP). Before

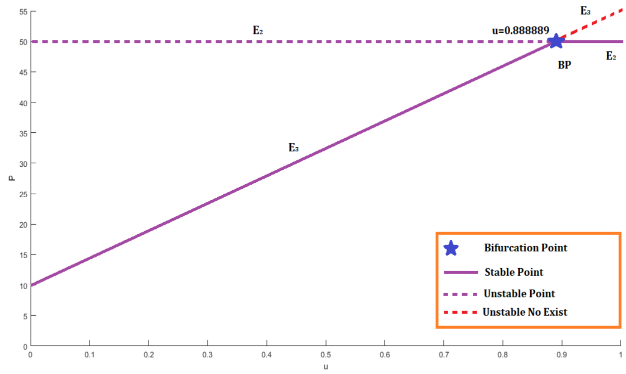

Figure 5. Bifurcation Diagram of the system of equations (3). Transcritical bifurcation occurs at $u = 0.888889$ (bifurcation point).

reaching this point ($0 \leq u < 0.888889$), E_2 functions as an unstable saddle point, while E_3 maintains its stability as an asymptotic spiral. Beyond $u = 0.888889$ ($u > 0.888889$), the stability of E_2 transitions to that of a stable asymptotic node, while E_3 undergoes a shift towards an unstable saddle point. The equilibrium point E_3 loses biological relevance beyond this bifurcation due to a negative ladybird population.

These findings underscore the critical role of the interaction between ladybirds and aphids in maintaining population balance within the ecosystem. They highlight how variations in population dynamics, influenced by factors such as ant preference for aphid resources (u), can significantly impact the stability of predator-prey interactions in ecological systems.

4. Conclusion

The analysis of the predator-prey model with mutualistic protection for prey has provided valuable insights into the dynamics of this ecological system. The key findings are as follows:

Firstly, we reconstructed a model based on the framework proposed in a previous study [14]. This model describes the interactions between aphids (prey), ladybirds (predators), and ants (mutualistic protectors). It captures the population dynamics of these species over time.

Secondly, through a local stability analysis, we identified three equilibrium points within the system: E_1, E_2, and E_3. These equilibrium points serve as critical reference points for understanding the system's behavior. Specifically:

1. $E_1(0, 0)$ was found to be an unstable saddle point, indicating that small perturbations in the initial populations of aphids and ladybirds could lead to significant population fluctuations.
2. $E_2(K, 0)$ showed different behaviors depending on parameter values. It behaves as a stable asymptotic node if certain conditions are met, signifying a stable coexistence of aphids and ladybirds. However, under different conditions, it can transform into an unstable saddle point, suggesting a potential instability.
3. E_3 is defined by a set of complex equations, but its stability also depends on parameter values. It can be a stable asymptotic spiral or an unstable saddle point, contingent upon specific conditions.

Thirdly, the most intriguing aspect of our findings is the impact of the parameter u, representing ant preference for aphid resources. We observed a significant change in the stability of the equilibrium points E_2 and E_3 when u reached the critical value of 0.888889. This transition was identified as a transcritical bifurcation or branch point. Before $u = 0.888889$, all three equilibrium points, E_1, E_2, and E_3, coexisted, with only E_3 being stable. However, after crossing this threshold, the system exhibited a different behavior. Two equilibrium points, E_1 and E_2, remained, with E_2 transitioning from instability to stability. Importantly, equilibrium point E_3 ceased to have biological relevance due to a negative ladybird population, underscoring the ecological significance of this transition. In conclusion, our simulations emphasize the pivotal role of ant preference (u) in shaping the stability and interactions between ladybirds and aphids in this complex ecosystem. These insights contribute to our understanding of predator-prey dynamics and can inform ecological management strategies in real-world scenarios.

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