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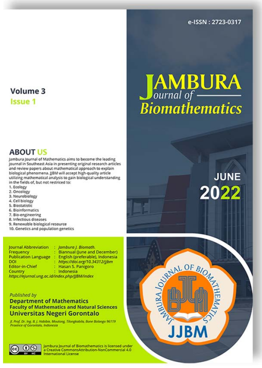
Volume 5, Issue 1, Pages 1–9, June 2024

Received 13 April 2024, Revised 11 May 2024, Accepted 26 May 2024, Published Online 8 June 2024

To Cite this Article : R. R. Musafir et al.,"Comparison of Fractional-Order Monkeypox Model with Singular and Non-Singular Kernels", *Jambura J. Biomath*, vol. 5, no. 1, pp. 1–9, 2024, *https://doi.org/10.37905/jjbm.v5i1.24920*

JOURNAL INFO • JAMBURA JOURNAL OF BIOMATHEMATICS



Homepage	:	http://ejurnal.ung.ac.id/index.php/JJBM/index
Journal Abbreviation	:	Jambura J. Biomath.
Frequency	:	Biannual (June and December)
Publication Language	:	English (preferable), Indonesia
DOI	:	https://doi.org/10.37905/jjbm
Online ISSN	:	2723-0317
Editor-in-Chief	:	Hasan S. Panigoro
Publisher	:	Department of Mathematics, Universitas Negeri Gorontalo
Country	:	Indonesia
OAI Address	:	http://ejurnal.ung.ac.id/index.php/jjbm/oai
Google Scholar ID	:	XzYgeKQAAAAJ
Email	:	editorial.jjbm@ung.ac.id

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Jambura Journal of Biomathematics, Volume 5, Issue 1, Pages 1–9, June 2024 https://doi.org/10.37905/jjbm.v5i1.24920

Research Article

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Comparison of Fractional-Order Monkeypox Model with Singular and Non-Singular Kernels

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ARTICLE HISTORY
Received 13 April 2024
Revised 11 May 2024
Accepted 26 May 2024
Published 8 June 2024

KEYWORDS
Monkeypox model
Fractional-order model
Caputo derivative
Atangana-Baleanu-Caputo derivative
Parameter estimation

ABSTRACT. *The singularity of the kernel of the Caputo fractional derivative has become an issue, leading many researchers to consider the Atangana-Baleanu-Caputo (ABC) fractional derivative in epidemic models where the kernel is non-singular. In this context, the objective of this study is to compare the calibration and forecasting performance of fractional-order monkeypox models with singular and nonsingular kernels, represented by the model with respect to the Caputo operator and the ABC operator, respectively. We have proposed a monkeypox epidemic model with respect to the ABC operator (MPXABC), where the model with respect to the Caputo derivative (MPXC) has been proposed in previous research. We have analyzed the existence and uniqueness of the solution. Three equilibrium points of the model are endemic, human endemic, and monkeypox-free, and their global stability has been investigated. The global dynamics of the MPXABC are the same as those of the MPXC. In evaluating the performance, we collected secondary data on weekly monkeypox cases from June 1 to November 23, 2022, in the USA. Parameter estimation has been performed using the least squares method, while the solutions of the model have been determined numerically using a predictor-corrector scheme. The benchmark for performance has been determined based on the root mean square error. Data calibration and forecasting indicate that the MPXC generally has the best performance for each value of the derivative order. For certain values of derivative order, the MPXABC performs better than the corresponding first-order model. However, generally, the corresponding first-order model performs better than the MPXABC. Depending on the data trends and the specified orders, the MPXC outperforms the MPXABC. Thus, the singularity issue of the Caputo derivative does not always have a negative impact on model fitting to data.*

1. Introduction

The global attention on the monkeypox outbreak has persisted since early 2022. This infectious illness is caused by the infection of the monkeypox virus, initially detected in humans in the Democratic Republic of the Congo in 1970 [1]. An outbreak occurred in Africa in 2003, followed by dormancy [2]. In 2022, monkeypox spread rapidly and extensively across numerous countries worldwide. From May 1 to the end of 2022, a total of 84,045 monkeypox cases were recorded, resulting in 74 fatalities [3]. Despite its low mortality rate, the rapid spread led to the World Health Organization declaring monkeypox a global health emergency [4]. Understanding the dynamics of monkeypox disease is crucial to preventing worst-case scenarios.

Mathematical modeling can provide insights into the dynamics of disease transmission [5, 6]. Mathematical models have been widely used to observe transmission phenomena involving calibration and forecasting [7–13]. For instance, Trisilowati et al. [8] employed a fractional-order SEIQRD model to forecast the COVID-19 cases in Indonesia; Chowell et al. [9] used a multiple-wave logistic model for SARS; and Qureshi and Yusuf [10] utilized an MSEIR model for calibrating chickenpox cases. The first monkeypox epidemic model was proposed based on the SIR mechanism by Bhunu and Mushayabasa [14]. Subsequently, some researchers considered some interventions, i.e., quarantine, vaccination, and hospitalization [15–17]. In addition to intervention models, some researchers also consider fractional-order derivatives in monkeypox models.

Fractional-order models are interesting to study due to the presence of fractional-order derivatives, which involve mathematical terms that represent memory effects [8]. This means that future conditions are influenced by all previous information [18]. This also relates to the epidemiology of diseases, where all cases in the previous period exhibited increasing or decreasing behaviours, thus affecting the number of current cases. Implemented fractional-order derivatives in models including Riemann-Liouville, Caputo, and Atangana-Baleanu-Caputo derivatives [19–21]. The Riemann-Liouville derivative is rarely used in models due to its non-local initial value [22]. Meanwhile, the Caputo derivative is more applicable because it involves a local initial value similar to first-order models [8]. However, in 2016, Atangana and Baleanu raised concerns about the singular kernel of the Caputo derivative [23]. Therefore, they [23] proposed a new fractional derivative operator with local initial value and a Mittag-Leffler kernel, which is a nonsingular kernel. This is commonly referred to as the Atangana-Baleanu-Caputo (ABC) deriva-

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Homepage : http://ejurnal.ung.ac.id/index.php/JJBM/index / E-ISSN : 2723-0317

tive.

Recently, Musafir et al. [20] proposed a fractional-order monkeypox model with respect to the Caputo derivative incorporating quarantine and hospitalization. The effect of the Caputo fractional derivative on their model was explored through dynamic analysis, parameter estimation, sensitivity analysis, forecasting of monkeypox cases, and examination from an epidemiological perspective. Vaccination program is not considered in the model because vaccination is recommended as a control for monkeypox outbreak by the World Health Organization (WHO). [24]. For this reason, Musafir et al. [25] performed optimal control of vaccination and rodent culling in the fractional-order monkeypox epidemic model.

The singularity issue of the Caputo derivative leads some researchers to consider the ABC derivative operator, which has a non-singular kernel [21, 26–28]. For example, the first-order monkeypox model constructed by Peter et al. [29] was modified by El-Mesady et al. [30] considering the Caputo operator. Qurashi et al. [28] issued the clarity objective of the kernel singularity; thus, they modified the model constructed by Peter et al. [29] with respect to the ABC operator. The question is how the singularity kernel problem affects the calibration and forecasting performance of the data. Based on this question, we aim to investigate the performance of a fractional-order monkeypox epidemic model with singular and non-singular kernels in the calibration and forecasting of data. We modify the fractional-order monkeypox model proposed by Musafir et al. [20] by considering the ABC derivative. Models with respect to the Caputo and ABC derivatives are fitted to the weekly monkeypox data in the USA. The model fitting is followed by determining the RMSE values as benchmarks for comparing the models.

We begin the study by providing preliminary definitions and the fractional-order monkeypox epidemic model. The existence and uniqueness of the solution are analyzed in Section 3. We investigate the existence of equilibrium points and their global stability in Section 4. In Section 5, the considered models are calibrated against data, and existing phenomena are examined. Finally, we emphasize the conclusions in Section 6.

2. Model Formulation

There are several definitions as preliminaries for the study of fractional order models. We first introduce the Riemann-Liouville fractional integral. This is followed by the Caputo fractional derivative.

Definition 1. [22] Define L_p as the p–norm Lebesgue space. Let t be the independent time variable on $[0,T]$, and f be a time-dependent function. The definition of the Riemann-Liouville fractional integral of $f(t)$ with order $\alpha > 0$ is

$$ {}_{0}^{RL}\mathcal{I}_t^{\alpha} f(t) = \frac{1}{\Gamma(n)} \int_0^t (t-s)^{\alpha-1} f(s) ds, $$

with ${}_{0}^{RL}\mathcal{I}_t^{\alpha} f(t) \in L_1([0,T])$.

Definition 2. [22] Let $\alpha \in (0,1)$, t be the independent time variable on $[0,T]$, and f be a time-dependent function with a continuous first derivative. The definition of the Caputo fractional derivative of $f(t)$ with order α is

$$ {}_{0}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds. $$

Notice that the kernel $(t-s)^{-\alpha}$ is singular, which is undefined as $s = t$. Hence, we consider the Atangana-Baleanu-Caputo (ABC) derivative, which has a nonsingular kernel in the form of the Mittag-Leffler kernel. The ABC fractional derivative and its integral are defined as follows.

Definition 3. [23] Let $\alpha \in (0,1)$, t be the independent time variable on $[0,T]$, and f be a time-dependent function with a continuous first derivative. The definition of the Atangana-Baleanu fractional derivative in Caputo sense (ABC) of $f(t)$ with order α is

$$ {}_{0}^{ABC}D_t^{\alpha} f(t) = \frac{N(\alpha)}{(1-\alpha)} \int_0^t \mathcal{E}_{\alpha} \left[-\frac{\alpha}{1-\alpha}(t-s)^{\alpha} \right] f'(s) ds, $$

where $N(\alpha)$ is a normalization function satisfied $N(0) = N(1) = 1$ and $\mathcal{E}_{\alpha}(\cdot)$ is the Mittag-Leffler function with parameter α. Then, its fractional integral is

$$ {}_{0}^{ABC}\mathcal{I}_t^{\alpha} f(t) = \frac{1-\alpha}{N(\alpha)} f(t) + \frac{\alpha}{\Gamma(\alpha)N(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, $$

with ${}_{0}^{ABC}\mathcal{I}_t^{\alpha} f(t) \in L_1([0,T])$.

The fractional-order monkeypox epidemic model with respect to the Caputo derivative proposed by Musafir et al. [20] involves compartments S_h (susceptible human subpopulation), E_h (exposed human subpopulation), I_h (infected human subpopulation), Q_h (quarantined human subpopulation), H_h (hospitalized human subpopulation), S_n (susceptible nonhuman subpopulation), E_n (exposed nonhuman subpopulation), and I_n (infected nonhuman subpopulation). The model with respect to the Caputo operator (MPXC) proposed by Musafir et al. [20] is expressed as

$$ \begin{aligned} {}_{0}^{C}D_t^{\alpha} S_h &= \Pi_h - (\beta_1 I_h + \beta_2 I_n) S_h - \mu_h S_h, \\ {}_{0}^{C}D_t^{\alpha} E_h &= (\beta_1 I_h + \beta_2 I_n) S_h - (\nu_h + \mu_h) E_h, \\ {}_{0}^{C}D_t^{\alpha} I_h &= \nu_h E_h - (\sigma_{h1} + \rho_h + \gamma_{h1} + \delta_{h1} + \mu_h) I_h, \\ {}_{0}^{C}D_t^{\alpha} Q_h &= \rho_h I_h - (\sigma_{h2} + \gamma_{h2} + \delta_{h2} + \mu_h) Q_h, \\ {}_{0}^{C}D_t^{\alpha} H_h &= \sigma_{h1} I_h + \sigma_{h2} Q_h - (\gamma_{h3} + \delta_{h3} + \mu_h) H_h, \\ {}_{0}^{C}D_t^{\alpha} S_n &= \Pi_n - \beta_3 I_n S_n - \mu_n S_n, \\ {}_{0}^{C}D_t^{\alpha} E_n &= \beta_3 I_n S_n - (\nu_n + \mu_n) E_n, \\ {}_{0}^{C}D_t^{\alpha} I_n &= \nu_n E_n - (\delta_n + \mu_n) I_n, \end{aligned} \tag{1} $$

where the description of variables and parameters is listed in Table 1.

Table 1. Definition of the parameters of the model.

Parameter	Definition	Unit
Π_h	Recruitment rate of human	$\frac{\text{human}}{(\text{week})^\alpha}$
Π_n	Recruitment rate of nonhuman	$\frac{\text{nonhuman}}{(\text{week})^\alpha}$
μ_h, μ_n	Natural death rates of human and nonhuman, respectively	$\frac{1}{(\text{week})^\alpha}$
β_1	Infection rate from human-to-human contact	$\frac{1}{(\text{human})\times(\text{week})^\alpha}$
β_2	Infection rate from nonhuman-to-human contact	$\frac{1}{(\text{nonhuman})\times(\text{week})^\alpha}$
β_3	Infection rate from nonhuman-to-nonhuman contact	$\frac{1}{(\text{nonhuman})\times(\text{week})^\alpha}$
ν_h, ν_n	Incubation rates for E_h and E_n, respectively	$\frac{1}{(\text{week})^\alpha}$
ρ_h	Quarantine rate for I_h	$\frac{1}{(\text{week})^\alpha}$
σ_{h1}, σ_{h2}	Hospitalization rate for I_h and Q_h, respectively	$\frac{1}{(\text{week})^\alpha}$
$\gamma_{h1}, \gamma_{h2}, \gamma_{h3}$	Recovery rates for I_h, Q_h, and H_h, respectively	$\frac{1}{(\text{week})^\alpha}$
$\delta_{h1}, \delta_{h2}, \delta_{h3}, \delta_n$	Mortality rates for I_h, Q_h, H_h, and I_n, respectively	$\frac{1}{(\text{week})^\alpha}$

Model (1) represents a fractional-order monkeypox model with a singular kernel. Since the singularity of the kernel is an issue, we consider implementing the ABC operator into model (1) to propose a model with a nonsingular kernel. The transmission mechanisms involved in the model are similar to model (1). Here, we investigate the basic properties and analyze the global stability of the equilibrium points of the model with respect to the ABC derivative. Then, we implement model (1) and our proposed model into the monkeypox data. Our proposed model with respect to the ABC operator (MPXABC) is expressed as

$$\begin{aligned}
{}_0^{ABC}D_t^\alpha S_h &= \Pi_h - (\beta_1 I_h + \beta_2 I_n) S_h - \mu_h S_h,\\
{}_0^{ABC}D_t^\alpha E_h &= (\beta_1 I_h + \beta_2 I_n) S_h - k_1 E_h,\\
{}_0^{ABC}D_t^\alpha I_h &= \nu_h E_h - k_2 I_h,\\
{}_0^{ABC}D_t^\alpha Q_h &= \rho_h I_h - k_3 Q_h,\\
{}_0^{ABC}D_t^\alpha H_h &= \sigma_{h1} I_h + \sigma_{h2} Q_h - k_4 H_h,\\
{}_0^{ABC}D_t^\alpha S_n &= \Pi_n - \beta_3 I_n S_n - \mu_n S_n,\\
{}_0^{ABC}D_t^\alpha E_n &= \beta_3 I_n S_n - k_5 E_n,\\
{}_0^{ABC}D_t^\alpha I_n &= \nu_n E_n - k_6 I_n,
\end{aligned} \tag{2}$$

where $k_1 = (\nu_h + \mu_h)$, $k_2 = (\sigma_{h1} + \rho_h + \gamma_{h1} + \delta_{h1} + \mu_h)$, $k_3 = (\sigma_{h2} + \gamma_{h2} + \delta_{h2} + \mu_h)$, $k_4 = (\gamma_{h3} + \delta_{h3} + \mu_h)$, $k_5 = (\nu_n + \mu_n)$, and $k_6 = (\delta_n + \mu_n)$.

3. Existence and Uniqueness of Solution

In order to show the existence and uniqueness of the solution of model (2), we first observe that each kernel satisfies the Lipschitz condition. The existence and uniqueness of solutions for model (2) are investigated as follows. Consider the following kernel functions:

$$\begin{aligned}
F_1(t, S_h) &= \Pi_h - (\beta_1 I_h + \beta_2 I_n) S_h - \mu_h S_h,\\
F_2(t, E_h) &= (\beta_1 I_h + \beta_2 I_n) S_h - k_1 E_h,\\
F_3(t, I_h) &= \nu_h E_h - k_2 I_h,\\
F_4(t, Q_h) &= \rho_h I_h - k_3 Q_h,\\
F_5(t, H_h) &= \sigma_{h1} I_h + \sigma_{h2} Q_h - k_4 H_h,\\
F_6(t, S_n) &= \Pi_n - \beta_3 I_n S_n - \mu_n S_n,\\
F_7(t, E_n) &= \beta_3 I_n S_n - k_5 E_n,\\
F_8(t, I_n) &= \nu_n E_n - k_6 I_n.
\end{aligned}$$

Suppose that $\vec{X} = (S_h, E_h, I_h, Q_h, H_h, S_n, E_n, I_n)$ and $\vec{\mathcal{X}} = (\bar{S}_h, \bar{E}_h, \bar{I}_h, \bar{Q}_h, \bar{H}_h, \bar{S}_n, \bar{E}_n, \bar{I}_n)$ are bounded solutions of model (2). Hence, $\|I_h\| \le K$ and $\|I_n\| \le K^*$ for some positive constants K and K^*. For kernel F_1 with respect to variables S_h and $\bar{S}_h$, we obtain

$$\begin{aligned}
\|F_1(t, S_h) - F_1(t, \bar{S}_h)\| &= \|\Pi_h - (\beta_1 I_h + \beta_2 I_n) S_h - \mu_h S_h\\
&\quad -\Pi_h + (\beta_1 I_h + \beta_2 I_n)\bar{S}_h + \mu_h \bar{S}_h\|\\
&= \|(\beta_1 I_h + \beta_2 I_n + \mu)(S_h - \bar{S}_h)\|\\
&= \|\beta_1 I_h + \beta_2 I_n + \mu\| \|S_h - \bar{S}_h\|\\
&\le (\beta_1 \|I_h\| + \beta_2 \|I_n\| + \mu) \|S_h - \bar{S}_h\|\\
&\le c_1 \|S_h - \bar{S}_h\|,
\end{aligned}$$

where $c_1 = \beta_1 K + \beta_2 K^* + \mu_h$. Hence, F_1 satisfies the Lipschitz condition. Similarly, we can show that

$$\begin{aligned}
\|F_2(t, E_h) - F_2(t, \bar{E}_h)\| &\le c_2 \|E_h - \bar{E}_h\|,\\
\|F_3(t, I_h) - F_3(t, \bar{I}_h)\| &\le c_3 \|I_h - \bar{I}_h\|,\\
\|F_4(t, Q_h) - F_4(t, \bar{Q}_h)\| &\le c_4 \|Q_h - \bar{Q}_h\|,\\
\|F_5(t, H_h) - F_5(t, \bar{H}_h)\| &\le c_5 \|H_h - \bar{H}_h\|,\\
\|F_6(t, S_n) - F_6(t, \bar{S}_n)\| &\le c_6 \|S_n - \bar{S}_n\|,\\
\|F_7(t, E_n) - F_7(t, \bar{E}_n)\| &\le c_7 \|E_n - \bar{E}_n\|,\\
\|F_8(t, I_n) - F_8(t, \bar{I}_n)\| &\le c_8 \|I_n - \bar{I}_n\|,
\end{aligned}$$

where $c_2 = k_1$, $c_3 = k_2$, $c_4 = k_3$, $c_5 = k_4$, $c_6 = \beta_3 K^* + \mu_n$, $c_7 = k_5$, and $c_8 = k_6$. Hence, F_k satisfy the Lipschitz condition for all $k = 2, 3, \ldots, 8$. In addition, if $0 < c_i < 1$ for all $i = 1, 2, \ldots, 8$, then all kernels F_k are contracted.

Let $\vec{X} = (X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8) = (S_h, E_h, I_h, Q_h, H_h, S_n, E_n, I_n)$. According to the fractional integral in Definition 3, the solution of model (2) is expressed by the Volterra function:

$$\begin{aligned}
X_i(t) &= X_i(0) + (1-\alpha)\frac{1}{N(\alpha)} F_i(t, X_i)\\
&\quad + \alpha \frac{1}{\Gamma(\alpha) N(\alpha)} \int_0^t (t-s)^{\alpha-1} F_i(s, X_i) ds,
\end{aligned} \tag{3}$$

for $i = 1, 2, \ldots, 8$. From which, we have the iterative scheme:

$$\begin{aligned}
X_{i,n}(t) &= X_i(0) + (1-\alpha)\tfrac{1}{N(\alpha)} F_i(t, X_{i,n-1})\\
&\quad + \alpha \tfrac{1}{\Gamma(\alpha) N(\alpha)} \int_0^t (t-s)^{\alpha-1} F_i(s, X_{i,n-1}) ds,
\end{aligned} \tag{4}$$

for $i = 1, 2, \ldots, 8$. Notice that the solution of model (2) can be approximated by scheme (4) as $n \to \infty$ with the initial guess

$X_{1,0} = S_h(0)$, $X_{2,0} = E_h(0)$, $X_{3,0} = I_h(0)$, $X_{4,0} = Q_h(0)$, $X_{5,0} = H_h(0)$, $X_{6,0} = S_n(0)$, $X_{7,0} = E_n(0)$, and $X_{8,0} = I_n(0)$. We next define the difference between successive terms

$$\begin{aligned}\Phi_{i,n} &= X_{i,n} - X_{i,n-1}\\ &= (1-\alpha)\frac{1}{N(\alpha)}\left[F_i(t, X_{i,n-1}) - F_i(t, X_{i,n-2})\right]\\ &\quad +\alpha\frac{1}{\Gamma(\alpha)N(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[F_i(s, X_{i,n-1}) - F_i(s, X_{i,n-2})\right] ds,\end{aligned}$$

for $i = 1, 2, \ldots, 8$. By using the fact that F_k satisfies the Lipschitz condition for $k = 1, 2, \ldots, 8$, we have

$$\begin{aligned}\|\Phi_{i,n}\| &\leq (1-\alpha)\tfrac{1}{N(\alpha)}c_i\|\Phi_{i,n-1}\|\\ &\quad +\alpha\tfrac{1}{\Gamma(\alpha)N(\alpha)}c_i\int_0^t \|\Phi_{i,n-1}\|(t-s)^{\alpha-1}ds,\end{aligned} \tag{5}$$

for $i = 1, 2, \ldots, 8$. If the difference of successive terms in (5) is approximated by an iterative process, then we get

$$\|\Phi_{i,n}\| \leq X_{i,0}\left((1-\alpha)\frac{1}{N(\alpha)}c_i + \frac{t^\alpha}{\Gamma(\alpha)N(\alpha)}c_i\right)^n,$$

for all $i = 1, 2, \ldots, 8$.

If we assume that there is $t = t_* > 0$ such that $\left((1-\alpha)\frac{1}{N(\alpha)}c_i + \frac{t^\alpha}{\Gamma(\alpha)N(\alpha)}c_i\right) < 1$ for all $i = 1, 2, \ldots, 8$, then we have $\|\Phi_{k,n}\| = 0$ as $n \to \infty$ for all $k = 1, 2, \ldots, 8$. Hence, if the assumption is satisfied, then model (2) has a solution, which is expressed by eq. (3).

We next show that the solution of model (2) is unique. Suppose the contrary, so that there is another solution of model (2), that is, $\hat{Z} = (\hat{Z}_1, \hat{Z}_2, \hat{Z}_3, \hat{Z}_4, \hat{Z}_5, \hat{Z}_6, \hat{Z}_7, \hat{Z}_8) = (\hat{S}_h, \hat{E}_h, \hat{I}_h, \hat{Q}_h, \hat{H}_h, \hat{S}_n, \hat{E}_n, \hat{I}_n)$. Then, for $i = 1, 2, \ldots, 8$, we get

$$\begin{aligned}\|X_i - \hat{Z}_i\| &= (1-\alpha)\frac{1}{N(\alpha)}\left\|F_i(t, X_i) - F_i(t, \hat{Z}_i)\right\|\\ &\quad +\alpha\frac{1}{\Gamma(\alpha)N(\alpha)}\left\|\int_0^t (t-s)^{\alpha-1}\left[F_i(s, X_i) - F_i(s, \hat{Z}_i)\right] ds\right\|\\ &\leq (1-\alpha)\frac{1}{N(\alpha)}c_i\|X_i - \hat{Z}_i\|\\ &\quad +\alpha\frac{1}{\Gamma(\alpha)N(\alpha)}c_i\int_0^t \|X_i - \hat{Z}_i\|(t-s)^{\alpha-1}ds\\ &\leq \|X_i - \hat{Z}_i\|\left((1-\alpha)\frac{1}{N(\alpha)}c_i + \frac{t^\alpha}{\Gamma(\alpha)N(\alpha)}c_i\right)^n.\end{aligned}$$

If $t = t_*$, we have $\|X_i - \hat{Z}_i\| \leq 0$ for $i = 1, 2, \ldots, 8$. Hence, we strictly obtain $\|X_i - \hat{Z}_i\| = 0$, which implies $X_i = \hat{Z}_i$ for $i = 1, 2, \ldots, 8$. This leads to a contradiction. Therefore, model (2) has a unique solution if the assumption is satisfied. The following theorem has been obtained:

Theorem 1. *Model (2) has a unique solution if there is $t_* > 0$ such that*

$$\frac{(1-\alpha)c_i}{N(\alpha)} + \frac{c_i t_*^\alpha}{\Gamma(\alpha)N(\alpha)} < 1,$$

for all $i = 1, 2, \ldots, 8$, where $c_1 = \beta_1 K + \beta_2 K^* + \mu_h$, $c_2 = k_1$, $c_3 = k_2$, $c_4 = k_3$, $c_5 = k_4$, $c_6 = \beta_3 K^* + \mu_n$, $c_7 = k_5$, and $c_8 = k_6$.

4. Existence and Global Stability of Equilibrium Points

In this section, we provide the equilibrium points and their stability of model (2). The equilibrium points of model (2) are obtained by solving system (2) where the left-hand sides of the equations are equal to zero [31]. Thus, the system obtained is the same as that in Musafir et al. [20]. The equilibrium points of model (2), as determined by Musafir et al. [20], are as follows:

1. Monkeypox-free equilibrium $\mathbf{X}_1 = \left(\frac{\Pi_h}{\mu_h}, 0, 0, 0, 0, \frac{\Pi_n}{\mu_n}, 0, 0\right)$, which always exists.
2. Human-endemic equilibrium $\mathbf{X}_2 = \left(\tilde{S}_h, \tilde{E}_h, \tilde{I}_h, \tilde{Q}_h, \tilde{H}_h, \tilde{S}_n, 0, 0\right)$ where

$$\begin{aligned}\tilde{S}_h &= \frac{k_1 k_2}{\beta_1 \nu_h},\\ \tilde{E}_h &= \frac{\mu_h k_2\left(\mathcal{R}_{0h} - 1\right)}{\beta_1 \nu_h},\\ \tilde{I}_h &= \frac{\mu_h\left(\mathcal{R}_{0h} - 1\right)}{\beta_1},\\ \tilde{Q}_h &= \frac{\rho_h \mu_h\left(\mathcal{R}_{0h} - 1\right)}{\beta_1 k_3},\\ \tilde{H}_h &= \frac{\mu_h\left(\sigma_{h1} k_3 + \sigma_{h2}\rho_h\right)\left(\mathcal{R}_{0h} - 1\right)}{\beta_1 k_3 k_4},\\ \tilde{S}_n &= \frac{\Pi_n}{\mu_n},\end{aligned}$$

   and $\mathcal{R}_{0h} = \frac{\Pi_h \beta_1 \nu_h}{\mu_h k_1 k_2}$ is the basic reproduction number for human-to-human transmission. $\mathbf{X}_2$ exists if $\mathcal{R}_{0h} > 1$.
3. Endemic equilibrium $\mathbf{X}_3 = \left(S_h^*, E_h^*, I_h^*, Q_h^*, H_h^*, S_n^*, E_n^*, I_n^*\right)$ where

$$\begin{aligned}S_h^* &= \frac{\Pi_h \nu_h - k_1 k_2 I_h^*}{\mu_h \nu_h},\\ E_h^* &= \frac{k_2 I_h^*}{\nu_h},\\ I_h^* &= \frac{1}{2}\left[\frac{\beta_1 \Pi_h \nu_h - k_1 k_2\left(\beta_2 I_n^* + \mu_h\right)}{\beta_1 k_1 k_2}\right.\\ &\quad \left. + \sqrt{\left(\frac{\beta_1 \Pi_h \nu_h - k_1 k_2\left(\beta_2 I_n^* + \mu_h\right)}{\beta_1 k_1 k_2}\right)^2 + \frac{4\beta_2 \Pi_h \nu_h I_n^*}{\beta_1 k_1 k_2}}\right],\\ Q_h^* &= \frac{\rho_h I_h^*}{k_3},\\ H_h^* &= \frac{\left(\sigma_{h1} k_3 + \sigma_{h2}\rho_h\right) I_h^*}{k_3 k_4},\\ S_n^* &= \frac{k_5 k_6}{\beta_3 \nu_n},\\ E_n^* &= \frac{\mu_n k_6\left(\mathcal{R}_{0n} - 1\right)}{\beta_3 \nu_n},\\ I_n^* &= \frac{\nu_n\left(\mathcal{R}_{0n} - 1\right)}{\beta_3},\end{aligned}$$

   and $\mathcal{R}_{0n} = \frac{\Pi_n \beta_3 \nu_n}{\mu_n k_5 k_6}$ is the basic reproduction number of nonhuman-to-nonhuman transmission. $\mathbf{X}_3$ exists if $\mathcal{R}_{0n} > 1$.

The stability of the equilibrium points of model (1) with respect to the Caputo operator has been analyzed by Musafir et al. [20], both locally and globally. If $\mathcal{R}_{0h} < 1$ and $\mathcal{R}_{0n} < 1$, then equilibrium $\mathbf{X}_1$ is asymptotically stable. If $\mathcal{R}_{0h} > 1$ and $\mathcal{R}_{0n} < 1$, then equilibrium $\mathbf{X}_2$ exists and is asymptotically stable. If $\mathcal{R}_{0n} > 1$, then equilibrium $\mathbf{X}_3$ exists and is always asymptotically stable.

In 2020, Taneco-Hernández and Vargas-De-León [31] proposed the inequality properties of the fractional-order derivative with respect to the ABC operator. Let $X(t) \in \mathbb{R}^+$ be a continu-

Table 2. The estimated parameter values of model (2) for $\alpha = 0.84, 0.87, 0.9, 0.94$, and 0.97.

Parameter	Derivative order				
	$\alpha = 0.84$	$\alpha = 0.87$	$\alpha = 0.9$	$\alpha = 0.94$	$\alpha = 0.97$
Π_h	4.7916×10^5	4.9526×10^5	4.9366×10^5	5.7152×10^5	1.1994×10^6
Π_n	1.5770×10^5	1.6292×10^5	1.6681×10^5	1.7921×10^5	4.4915×10^5
μ_h	0.2517	0.2361	0.2112	0.2014	0.1770
μ_n	0.4574	0.4429	0.4281	0.4076	0.3876
β_1	5.2436×10^{-14}	4.9630×10^{-14}	5.1062×10^{-14}	5.1230×10^{-10}	5.5844×10^{-9}
β_2	1.4414×10^{-10}	1.4650×10^{-10}	1.5619×10^{-10}	1.5580×10^{-10}	1.4895×10^{-10}
β_3	4.1440×10^{-7}	4.3040×10^{-7}	4.4046×10^{-7}	4.7967×10^{-7}	4.8430×10^{-7}
ν_h	0.2046	0.2068	0.2112	0.2051	0.1951
ν_n	0.3191	0.3288	0.3362	0.3580	0.3642
ρ_h	0.3613	0.3457	0.3120	0.3326	0.3296
σ_{h1}	0.3613	0.3457	0.3120	0.3326	0.3296
σ_{h2}	0.0300	0.0300	0.0300	0.0300	0.0300
γ_{h1}	0.3989	0.3842	0.3525	0.3718	0.3695
γ_{h2}	0.0300	0.0300	0.0300	0.0300	0.0300
γ_{h3}	0.0300	0.0300	0.0300	0.0300	0.0300
δ_{h1}	0.2489	0.2306	0.1909	0.2151	0.2103
δ_{h2}	0.0004	0.0004	0.0004	0.0004	0.0004
δ_{h3}	0.0004	0.0004	0.0004	0.0004	0.0004
δ_n	0.6536	0.6428	0.6333	0.6152	0.6127

ous and derivable time-dependent function. For any $t \geq 0$,

$$ {}_0^{ABC}D_t^\alpha \left[X(t) - X^* - X^* \ln \left(\frac{X(t)}{X^*} \right) \right] \leq \left(1 - \frac{X^*}{X(t)} \right) {}_0^{ABC}D_t^\alpha X(t), \tag{6} $$

for some positive constant X^*. Ine quality (6) is exactly used to show the global stability of the equilibrium points of model (1) with respect to the Caputo operator. Furthermore, Taneco-Hernández and Vargas-De-León [31] proposed LaSalle's invariance principle for the global stability of equilibrium points in the model with respect to the ABC operator, which also resembles that of the model with respect to the Caputo operator. Similarly, the following theorems are obtained. Readers interested in studying the proofs of these theorems can refer to Musafir et al. [20].

Theorem 2. *Monkeypox-free equilibrium* $\mathbf{X}_1$ *of model (2) is globally asymptotically stable if both* $\mathcal{R}_{0h} < 1$ *and* $\mathcal{R}_{0n} < 1$.

Theorem 3. *Let human-endemic equilibrium* $\mathbf{X}_2$ *of model (2) exists. That equilibrium is globally asymptotically stable if* $\mathcal{R}_{0n} < 1$.

Theorem 4. *Let endemic equilibrium* $\mathbf{X}_3$ *of model (2) exists. That equilibrium is always globally asymptotically stable.*

5. Implementation to Data

In this section, we compare fractional-order monkeypox models with singular and non-singular kernel. The model with a singular kernel is represented by model (1), which is a model with respect to the Caputo operator (MPXC). Meanwhile, the model with a nonsingular kernel is represented by model (2), which is a model with respect to the ABC operator (MPXABC). For this purpose, we collect weekly monkeypox data taken from the website ourworldindata.org/monkeypox for the period from June 1 to November 23, 2022, in the United States [3]. The data is depicted in Figure 1, which spans 25 weeks. It can be observed that the trend of the data collection tends to follow a normal distribution. We perform parameter estimation of the model on the data with an 18-week calibration period and a 7-week forecast period. To evaluate the performance of each operator in the model, we determine the value of the root mean square error (RMSE). Let $(\mathbf{t}, I_h(\mathbf{t}))$ be the solution of the variable I_h and $(\mathbf{t}^{data}, I^{data}(\mathbf{t}^{data}))$ be the N-sized time-dependent data collection. The formulation of RMSE is

$$ RMSE = \sqrt{\frac{\sum_{i=1}^{N} \left(I_h(t_i^{data}) - I^{data}(t_i^{data}) \right)}{N}}. $$

Model calibration to the data is performed via parameter estimation using the least squares method. We employ the built-in *lsqcurvefit* MATLAB program for parameter estimation. At the same time, we employ the predictor-corrector schemes to determine the numerical solution of models (1) and (2) [32, 33]. The initial values of the model for parameter estimation are (333 133 413,30,25,10,5,100 000 000,2,10). We consider the population and monkeypox cases in the United States as of June 1, 2022, to be 333 133 413 and 25 humans, respectively [20]. The initial guesses for the parameters were as follows:

$$ \begin{gathered} 706.36 \leq \Pi_h \leq 706\ 358\ 490.56, \\ 211.53 \leq \Pi_n \leq 211\ 538\ 461.54, \\ 0 \leq \beta_1, \beta_2, \beta_3 \leq 0.01, \\ 0 \leq \mu_h, \nu_h, \sigma_{h1}, \rho_h, \gamma_{h1}, \delta_{h1}, \delta_{h2}, \delta_{h3}, \sigma_{h2}, \gamma_{h2}, \gamma_{h3}, \mu_n, \nu_n, \delta_n \leq 1. \end{gathered} $$

The values of α used in Musafir et al. [20] for parameter estimation are 0.8, 0.84, 0.87, 0.9, 0.94, and 0.97. In their study, the calibration of the MPXC has the best performance when $\alpha = 0.97$. Meanwhile, the forecasting of the MPXC has the best performance when $\alpha = 0.94$. In our study, the considered values of α for both derivatives are 0.84, 0.87, 0.9, 0.94, 0.97, and 1. The

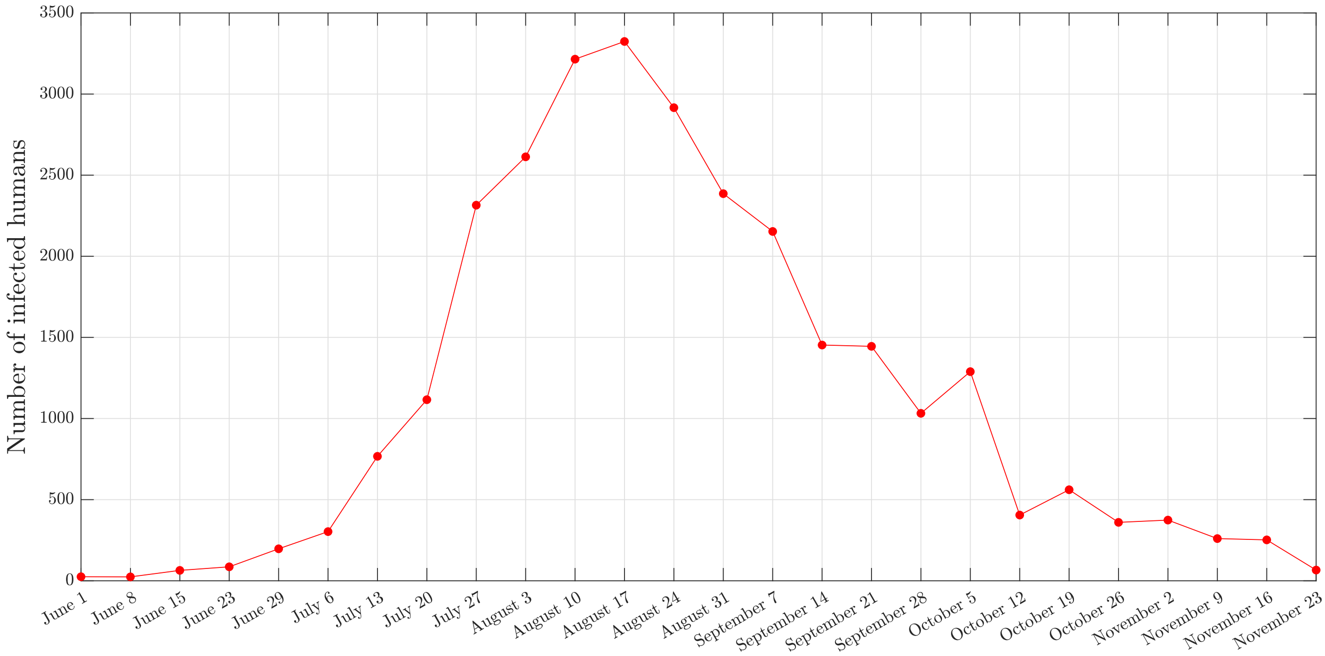

Figure 1. The profile of weekly monkeypox case data from June 1 to November 23, 2022, in the USA.

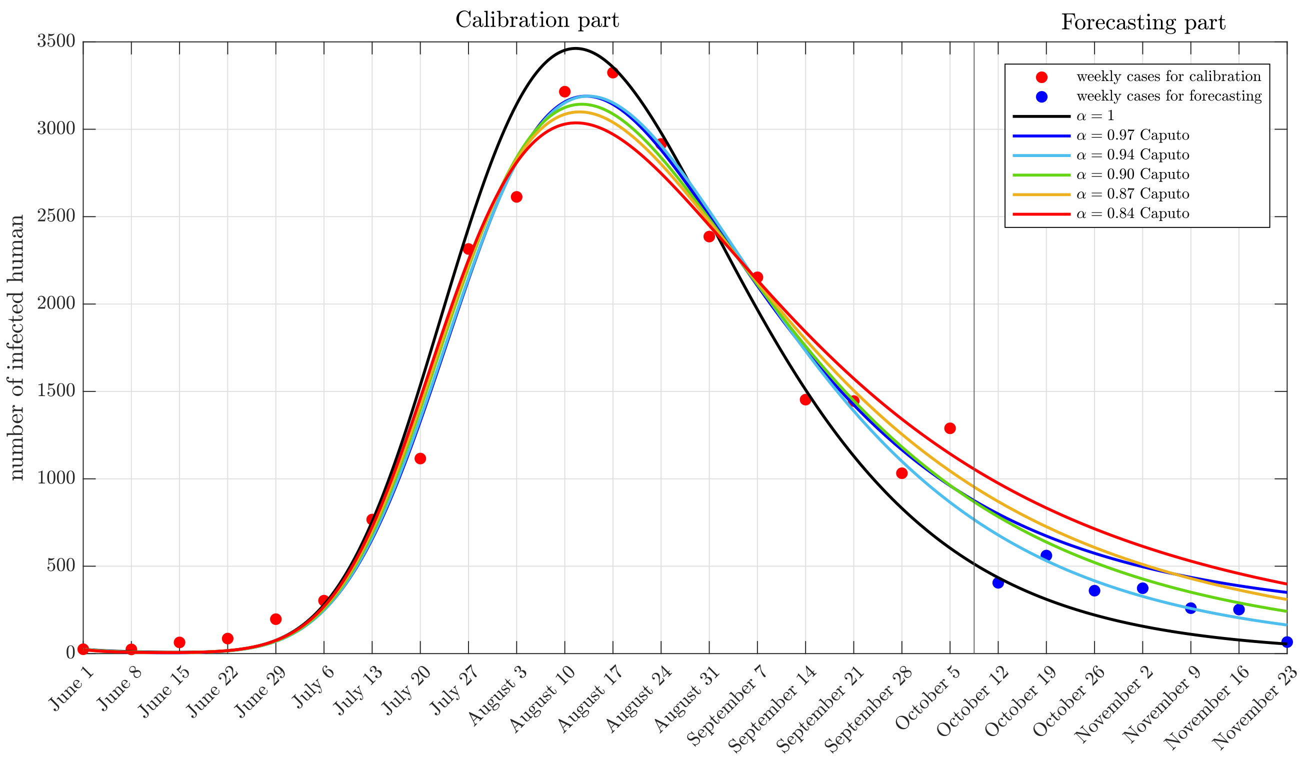

Figure 2. The fitted solution curve of model (1) (MPXC) and different values of α, namely $1, 0.97, 0.94, 0.9, 0.87, 0.84$. The red dots represent the time-dependent monkeypox data for calibration. The blue dots represent the time-dependent monkeypox data for forecasting.

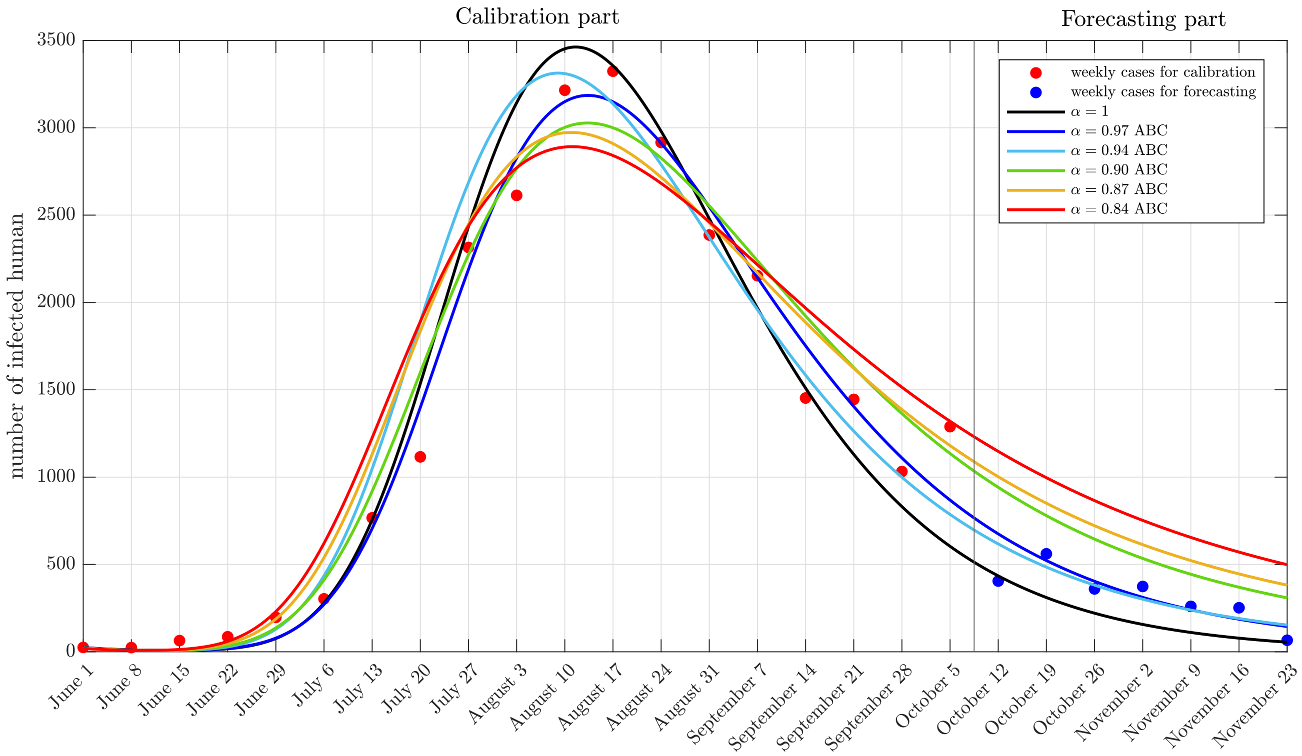

Figure 3. The fitted solution curve of model (2) (MPXABC) and different values of α, namely $1, 0.97, 0.94, 0.9, 0.87, 0.84$. The red dots represent the time-dependent monkeypox data for calibration. The blue dots represent the time-dependent monkeypox data for forecasting.

estimated values of each parameter in model (1) for each α are provided in the study by Musafir et al. [20]. Meanwhile, the estimated values of each parameter in model (2) for each α, except $\alpha = 1$, are listed in Table 2.

The solution curves of the MPXC and the estimated parameter values are depicted in Figure 2. It is observed that the solution curves provide good calibration to the data trend for each value of α. Changes in the value of the derivative order result in systematic calibration, with higher orders leading to higher epidemic peaks, except for $\alpha = 0.97$. The first-order model calibration is further from the data compared to the MPXC. Additionally, it can be seen that the solution curves for $\alpha = 0.97, 0.94, 0.90$ provide better forecasting results compared to other curves.

The solution curves of the MPXABC and the estimated parameter values are depicted in Figure 3. It is observed that the influence of the ABC operator compared to the Caputo operator shows that the estimated curves are irregular for each α. The solution curves for $\alpha = 0.87, 0.84$ provide the furthest calibration compared to the others, although the solution curves are still relevant to the data trend. Meanwhile, the calibrations for $\alpha = 0.97, 0.94$ provide curves that are relatively close to the data trend. For other values of α, the model calibration produces good curves, but not as well as the solution curves of calibrations for $\alpha = 0.97, 0.94$. The solution curves for $\alpha = 0.97, 0.94$ provide forecasts that are quite close to the data trend. Meanwhile, the curves for $\alpha = 0.84$ provide the furthest forecasts from the data trend.

To provide precise results, we have calculated the RMSE for each parameter estimation in the calibration and forecasting. The RMSE values for each parameter estimation in the calibration are listed in Table 3, while the RMSE values for parameter estimation in the forecasting are listed in Table 4. Specifically, the MPXC calibration for $\alpha = 0.97$ has the minimum RMSE, followed by $\alpha = 0.90, 0.94, 0.87$. This confirms the solution curves depicted in Figure 2, which show that the curves for $\alpha = 0.97, 0.94, 0.90$ of the MPXC have relatively good calibration to the data. Generally, the MPXC calibration is better than MPXABC calibration, except for $\alpha = 0.96$. On the other hand, the best MPXABC calibration is when $\alpha = 0.97$. This also confirms the solution curve for $\alpha = 0.97$ depicted in Figure 3, which shows that its calibration provides relatively good results compared to the data trend. Furthermore, the RMSE of the MPXABC is relatively greater than that of the corresponding first-order model, especially for $\alpha = 0.94, 0.87, 0.84$.

Table 3. Root mean square error of 18-week calibration

Derivative order	Caputo operator	ABC operator
$\alpha = 1$	259.38	
$\alpha = 0.97$	**152.46**	167.20
$\alpha = 0.94$	163.13	295.67
$\alpha = 0.90$	163.04	221.66
$\alpha = 0.87$	171.04	279.87
$\alpha = 0.84$	191.95	328.38

In terms of RMSE values in the forecasting, the MPXC for $\alpha = 0.94$ provides the best forecasts, followed by $\alpha = 0.90$ and $\alpha = 0.97$. The MPXC for $\alpha = 0.84$ has a greater RMSE compared to the first-order model. Nevertheless, the MPXC outperforms the first-order model. There are varying values of α that indicate

Table 4. Root mean square error of 7-week forecast

Derivative order	Caputo operator	ABC operator
$\alpha = 1$	236.13	
$\alpha = 0.97$	176.35	154.18
$\alpha = 0.94$	**151.23**	256.29
$\alpha = 0.90$	166.97	239.49
$\alpha = 0.87$	194.99	299.65
$\alpha = 0.84$	242.49	375.53

the best performance in the calibration and forecasting. This has been studied by Musafir et al. [20], indicating that there is an influence of vaccination on the data period of calibration. On the other hand, the MPXC is not always better than the MPXABC. For $\alpha = 0.97$, the MPXABC provides better forecasts than the MPXC. For other values of α, the MPXABC has a greater RMSE than the MPXC. For values of α other than 0.97, the MPXABC has a greater RMSE than the corresponding first-order model in forecast. Thus, the MPXC generally performs better than the MPXABC in forecasting the data.

We consider the integrability issue of the Caputo derivative, leading many researchers to employ the ABC derivative instead of the Caputo derivative [21, 26–28]. Based on our study, the integrability issue of the Caputo derivative does not always have a negative impact on its performance. In fact, our study generally indicates that the Caputo derivative outperforms the ABC derivative in model (2). Hence, the performance of the model with respect to either the Caputo or ABC derivative in calibration and forecasting is relative to the data trend.

We previously noticed that the calibration and forecasting of model (2) are limited to values of α, namely 1, 0.97, 0.94, 0.9, 0.87, and 0.84. However, the results are sufficient to show that the implementation of the ABC operator does not always yield better results than the implementation of the Caputo operator. For future work, the performance of each operator can be compared by involving the estimation of the value of α.

6. Conclusion

We have considered a monkeypox epidemic model with respect to the Caputo operator (MPXC) as a model with a singular kernel. Then, we have proposed a monkeypox epidemic model with respect to the ABC operator (MPXABC) as a model with a non-singular kernel. The conditions for the existence and uniqueness of solutions of the MPXABC have been determined. The existence and global stability of equilibrium points have also been investigated. The conditions for the basic properties and stability of equilibrium points of the MPXABC are the same as those of the MPXC, which have been analyzed in the previous study. To evaluate the performance of both operators in the model, we have fitted the model to the data using the least squares method, followed by the predictor-corrector method to obtain numerical solutions. In addition, the values of the root mean square error (RMSE) are also determined as benchmarks for the performance of both operators in the model. Based on the fitted model and the values of the derivative order, the MPXC generally performs better than the first-order model in calibration and forecasting. Meanwhile, the MPXABC has relatively lower performance than the first-order model in both calibration and forecasting, except for certain derivative orders. For each derivative order, the calibration of the MPXC performs better than that of the MPXABC, except for certain derivative orders in forecasting. The MPXC generally has the best performance in calibrating and forecasting monkeypox cases. Thus, the singularity issue of the Caputo derivative does not always have a negative impact on the performance of fractional-order models.

Our study is relative to the data and the derivative order. Hence, future work should be performed by estimating the derivative orders and parameters of the model. Moreover, trends in data collection should also be considered to observe the performance of operators in the monkeypox models. Thus, the performance of the model with respect to the Caputo and ABC derivatives can be further studied and confirmed.

Author Contributions. **Musafir, R.R.**: Conceptualization, Methodology, Formal analysis, Software, Writing – original draft, Writing – reviewing & editing, Data Curation, Visualization. **Suryanto, A.**: Supervision, Conceptualization, Formal analysis, Methodology, Writing – original draft, Writing – reviewing & editing, Validation. **Darti, I.**: Supervision, Conceptualization, Formal analysis, Writing – reviewing & editing, Project Administration. **Trisilowati**: Supervision, Writing – reviewing & editing, Data Curation, Visualization, Validation. All authors read and approved the final manuscript.

Acknowledgement. The authors are thankful to both the editor and the anonymous reviewers for their comments and suggestions to improve the quality of the paper.

Funding. This work is funded by Faculty of Mathematics and Natural Science (FMIPA) through Public Funds DPA (Dokumen Pelaksanaan Anggaran) PTNBH (Perguruan Tinggi Negeri Berbadan Hukum) University of Brawijaya and based on FMIPA Professor Grant, with contract number: 4158.15/UN10.F09/PN/2023, dated June 05, 2023.

Conflict of interest. All authors declare no conflict of interest.

Data availability. The data utilized in this study was accessed on July 21, 2023, from the website ourworldindata.org/monkeypox, which provides free access to explore monkeypox data produced by the World Health Organization.

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