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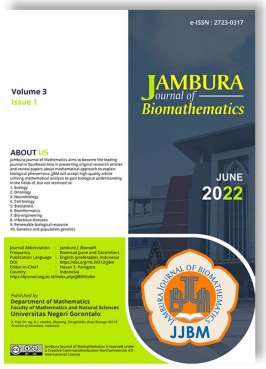

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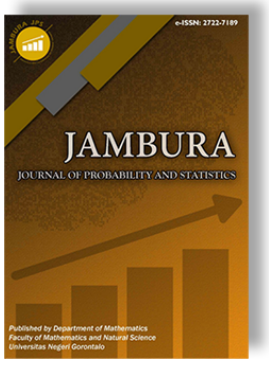

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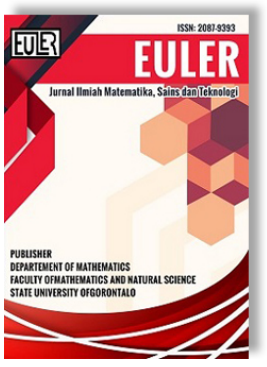

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Research Article

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A Qualitative Analysis of Leukemia Fractional Order SICW Model

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ABSTRACT. *Using a series of fundamental differential equations, including the Caputo derivative, which makes it easier to specify the initial conditions of the differential equations, we present a fractional order concept of leukemia in this study. The universality, positivity, and boundedness of solutions are first established. The local stability properties of the equilibrium are studied using the fractional Routh-Hurwitz stability criteria. The differential equation system has been solved using unconventional finite difference techniques. The Leukemia Fractional Order SICW model introduces several innovative elements compared to traditional epidemiological and disease models. This stands out due to its integration of fractional-order differential equations, inclusion of leukemic cells and immune cells compartments, simulation of treatment strategies, consideration of waning immunity, and its application to leukemia-specific scenarios. These elements collectively make it a valuable tool for studying leukemia dynamics, exploring treatment options, and improving our understanding of how the immune system interacts with cancer cells in leukemia patients. Numerical simulations of the model are shown at the conclusion to interpret our theoretical outcomes in support of various fractional orders of derivative ξ options. From there, we can observe how the evolution of the system components is impacted by the fractional derivative ξ.*

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1. Introduction

Over time, mathematical models have evolved into one of the most crucial instruments for studying the dynamics of non-communicable illnesses like leukemia. Although "blood cancer" is a more prevalent term, leukemia ("white blood") is the most accurate description. Cancer that develops in the tissues responsible for producing blood. The spongy, soft inside of a bone called bone marrow is responsible for making platelets, white blood cells, and red blood cells. Symptoms like anemia and shortness of breath manifest when the body does not have enough red blood cells, which transport oxygen to all of the body's cells. The immune system's white blood cells battle infections, and the blood-clotting platelets stop bleeding. A specific kind of white blood cell known as a lymphocyte is produced by the lymph nodes and the spleen. Antibodies are made by lymphocytes, which also help the body's immune system fight infections. Each of the two circulatory systems in the body-the lymphatic and blood vessel systems-receives millions of cells per day from all blood-forming tissues. Leukocytes, or aberrant, immature white blood cells, are discharged into these circulatory systems in millions when leukemia develops. These cells are immature, thus they are unable to perform their essential role of preventing infection. The production of normal white blood cells to combat infections, platelets to control bleeding, and red blood cells to avoid anemia is crowded out in advanced leukemia due to the unchecked expansion of aberrant cells (Indian Cancer Society). Approximately 256,000 adults and children worldwide were diagnosed with leukemia in 2000, and 209,000 of them died away. This is equivalent to around 0.35% of all fatalities from all causes and 3% of the almost seven million cancer-related deaths that year. Leukemia was the 12th most prevalent kind of neoplastic illness and the 11th most common cause of cancer-related mortality out of the sixteen distinct locations the body studied [1].

The study of an infectious agent spreading from one patient's cell to another has produced a large body of research. A thorough overview of the key concepts generated by these models has been put out by [2]. Numerous illnesses that involve the immune system have been simulated in written works, such as hepatitis B sickness by [3] and TB by [4, 5]. A comprehensive mathematical analysis of cancer immunotherapy is presented by [6]. They demonstrated an immunological model of cancer treatment in which cancer and healthy cells are seen as competitors for the same resources. It was believed that the anti-cancer cells preyed on the cancer cells. Additionally, a mathematical model illustrating the rivalry between tumours and the immune system while taking antibodies into account was published by [7]. Leukemia was first modelled by [8–10]. In order to better understand the mechanisms governing the various blood cell popula-

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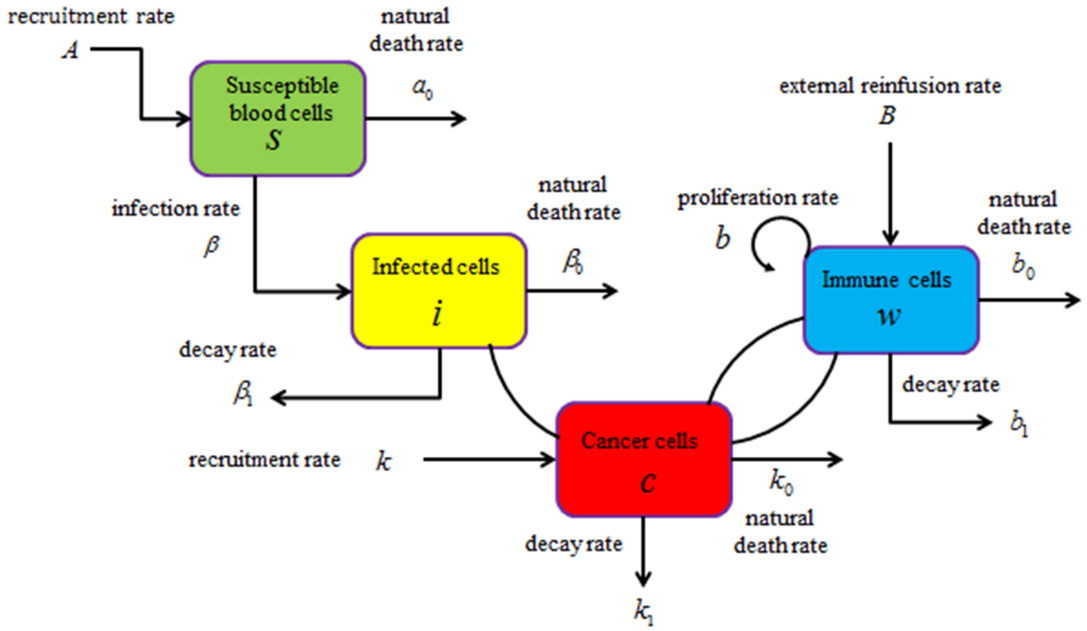

Figure 1. Transfer diagram showing dynamics of cancer cells.

tions and to evaluate potential strategies for the prevention and management of noncommunicable illnesses in people, dynamical models of noncommunicable diseases in humans were being developed. In order to represent leukemia and its therapy, we must first take into consideration how leukemia is spread through the blood circulation system. Our goal is to outline a mathematical model of leukemia and examine the effects of adoptive cell transfer treatment on the blood cells.

From the work of Aggarwal and Bhadauria [11], we provide a four-segment paradigm on leukemia in this study. With the help of Moore and Li's research, we improved this model by adding a decay rate parameter β_1 for infected cells. We take into account this parameter since certain infected cells also experience losses as a result of their interactions with cancer cells. In the numerical simulation part, we also evaluate our suggested model in comparison to the prior model. Let s represents the population of blood cells that are vulnerable, i represents the population of blood cells that are infected, c represents the population of leukemic cells (abnormal cells), and w represents the population of white blood cells or immune cells. The population of blood cells that are vulnerable to infection begins with a source term A entering the bloodstream from organs such the bone marrow, lymph nodes, and thymus. The natural mortality rates of vulnerable blood cells, infected cells, cancer cells, and immune cells are represented by the parameters a_0, k_0, β_0 and b_0 , respectively. The parameter represents the continual loss rate of blood cells that are vulnerable to infection by cancer cells.

The parameter β represents the continual loss rate of blood cells that are vulnerable to infection by cancer cells. The loss rate constant of cancer cells as a result of interactions with immune cells is represented by the parameter k_1. Due to the presence of leukemia or cancer cells in the blood, which is indicated by the letter b_1, certain immune cells will also degenerate. B is the frequency at which T cells or other immune cells are infused externally into cancer patients. Immune cells will continue to multiply if cancer recurs at a steady rate b.The transfer diagram of the model is shown in Figure 1 in the following.

1.1. *Framework of SICW Epidemic model*

Our improved paradigm is administered by the subsequent scheme of ordinary differential equations

$$\begin{aligned}
\frac{ds}{dt} &= A - a_0 s - \beta sc, \\
\frac{di}{dt} &= \beta sc - \beta_0 i - \beta_1 ci, \\
\frac{dc}{dt} &= k - k_0 c - k_1 cw, \\
\frac{dw}{dt} &= B + bc - b_0 w - b_1 wc.
\end{aligned} \tag{1}$$

All of the model's parameters are either positive, zero, or both; they cannot be negative or have no biological significance.

Numerous research scholars have noted that fractional mathematical models can give additional accurate data on real-life physical processes and that fractional calculus modelling is quite ideal and efficient for providing a precise overview of remembrance as well as some physical attributes of diverse materials and processes, and that are completely absent from traditional or integer-order equations [12–16]. The hydrology and groundwater flow [14, 17] , diffusion-like waves, pattern formation in chemical and biological processes [18–20], non-linear movement of earthquakes [19], viscoelastic materials [21], and muscular blood vessel model [22] are just a few examples of the physical schemes confronted through different disciplines that have been discussed by fractional differential equations.Additionally, models with memory easily relate to fractional-order issues, which occur in several biological contexts [23–25]. As a result, the dynamics of leukaemia with fractional derivative are what motivate this work. As far as we are aware, the mentioned model has not yet been examined in the literature. The resulting mathematical findings are therefore novel and intriguing.

The rest of this work is divided into the following sections:

Section 2 presents formulation of the fractional-order leukemia model, as well as other findings and concepts related to fractional calculus, are. Additionally demonstrated is the mathematical and biological soundness of our paradigm. Utilizing Routh-Hurwitz criterion, Section 3 focuses on the issue of local stability of positive equilibrium point. Numerical simulations are undertaken in Section 4 to demonstrate the theoretical findings. Finally, Section 5 brings this essay to a close.

2. Formulation of Fractional order of SICW Epidemic Model

The modeling of the dynamics of epidemic illnesses has been extensively studied, but it has only ever used integer-order (delay) differential equations. During the past few years, it has been discovered that models utilizing fractional order differential equations (FODEs) may successfully represent a wide variety of events in several domains [26–28]. Now, we introduce fractional form of Caputo arbitrary order derivative of order ξ, $0 < \xi < 1$ into model (1) is stated as

$$\begin{aligned} {}^{c}D^{\xi}s &= A - a_0 s - \beta sc, \\ {}^{c}D^{\xi}i &= \beta sc - \beta_0 i - \beta_1 ci, \\ {}^{c}D^{\xi}c &= k - a_0 s - k_0 c - k_1 cw, \\ {}^{c}D^{\xi}w &= B + bc - b_0 w - b_1 wc. \end{aligned} \tag{2}$$

Thus, we get a set of fractional differential equations eq. (2). System eq. (2), with $\xi = 1$ provides the classical SICW model explored in [11]. Initial states $s \geq 0, i \geq 0, c \geq 0$ and $w \geq 0$ are provided. We will review certain fundamental ideas associated to the ξ fractional integral together with the fractional differential operator in order to explore the consequences of a decay rate parameter β_1 of infected cells.

Definition 1. The Riemann-Liouville fractional integral of order $\xi > 0$ a function $f(t) \in C[a, b]$ is defined as

$$I_a^{\xi} f(t) = \frac{1}{\Gamma(\xi)} \int_a^t (t-s)^{\xi-1} f(s) ds,$$

where $\Gamma(\xi)$ is the Euler Gamma function.

Definition 2. For every ξ and $n = [\xi]$ the Riemann-Liouville derivate of order ξ can be defined as

$${}_aD_t^{\xi} f(t) = \frac{1}{\Gamma(n-\xi)} \left(\frac{d}{dx}\right)^n \int_a^t (t-s)^{n-\xi-1} f(s) ds.$$

The Riemann-Liouville fractional derivative is theoretically the most established and was historically the first fractional derivative notion. However, in the case of the Riemann-Liouville fractional differential equation, the initial value is often given in the form of a fractional derivative, which is not feasible. Consequently, the Caputo fractional derivative is employed, which is defined as follows.

Definition 3. Let $\xi > 0, n = [\xi]$. The Caputo derivative of order ξ is defined as

$${}_a^{c}D_t^{\xi} f(t) = \frac{1}{\Gamma(n-\xi)} \int_a^t (t-s)^{n-\xi-1} \left(\frac{d}{ds}\right)^n f(s) ds.$$

2.1. Non-negative solutions

We now demonstrate that the suggested model is properly posed (2). The evidence requires the results listed below.

Result 1. ([29])(Generalized mean value theorem) Let the function $u(t) \in C[a, b]$ and its fractional derivative $D^{\xi}u(t) \in C(a, b]$ for $0 < \xi \leq 1$, $a, b \in \mathcal{R}$ then we have

$$u(t) = u(a) + \frac{1}{\Gamma(n-\xi)} D^{\xi}u(\eta)(t-a)^{\xi}, 0 \leq \eta \leq t, \forall t \in (a, b]. \tag{3}$$

Result 2. ([30])The results as follows if we assume that the function $u(t)$ is ξ differentiable on (a, b):

1. If $D^{\xi}u(t) < 0$ for all $t \in (a, b)$ then $u(t)$ is decreasing on (a, b).
2. If $D^{\xi}u(t) > 0$ for all $t \in (a, b)$ then $u(t)$ is increasing on (a, b).
3. If $D^{\xi}u(t) = 0$ for all $t \in (a, b)$ then $u(t)$ is constant on (a, b).

Theorem 1. *The fractional-order SICW model (2) has only one solution. Additionally, the solution persists in $\mathcal{R}_+^4$ and is non-negative for every $t > 0$.*

Proof. We draw the conclusion that there is a solution for model (2) in $(0, \infty)$ that is both unique and consistent with the findings of Lin's Theorem 3.1 and Remark 3.2 [31]. We now establish the positively invariant nature of the domain

$$\mathcal{R}_+^4 = \{(s, i, c, w) \in \mathcal{R}_+^4 : s \geq 0, i \geq 0, c \geq 0, w \geq 0.\}$$

In order to determine if a vector field $(A - a_0 s - \beta sc,\ \beta sc - \beta j - \beta ci,\ k - k_0 c - k_1 cw,\ B + bc - b_0 w - b_1 wc)$ is tangent to or points toward the interior $\mathcal{R}_+^4$ of a coordinate space, we look at its direction on each coordinate space. Since,

$$\begin{aligned} {}^{c}D^{\xi}s(t)|_{s=0} &= A \geq 0, \\ {}^{c}D^{\xi}i(t)|_{i=0} &= \beta sc \geq 0, \\ {}^{c}D^{\xi}c(t)|_{c=0} &= k \geq 0, \\ {}^{c}D^{\xi}w(t)|_{w=0} &= B + bc \geq 0. \end{aligned} \tag{4}$$

From Result 1, 2 and eq. (4), the vector field $(A - a_0 s - \beta sc,\ \beta sc - \beta j - \beta ci,\ k - k_0 c - k_1 cw,\ B + bc - b_0 w - b_1 wc)$ on each coordinate plane is either tangent to the coordinate plane or and points to the interior of $\mathcal{R}_+^4$. As a result, the region of the domain $\mathcal{R}_+^4$ is positively invariant. The solution to the model (2) is bounded, as shown by the ensuing result, which we assert without offering any evidence. □

Result 3. The solutions of the system eq. (2) are bounded within a region Γ, where

$$\begin{aligned}\Gamma &= \left\{(s,i,c,w): 0<s(t)\le\frac{A}{a_0},\ 0<s(t)+i(t)\le\frac{A}{\delta},\right.\\ &\quad \left. 0<c\le\frac{k}{k_0},\ \text{and } 0<w(t)\le\frac{Bk_0+bk}{k_0^2}\right\},\\ \delta &= \min\{a_0,\beta_0,\beta_1 c\}.\end{aligned}$$

By positive, we mean that the population is sustained, and by boundedness, we indicate that there is a natural limit on population expansion as a result of scarce resources.

2.2. *Existence and uniqueness of global solution*

We will use the subsequent result 4 to demonstrate that the system eq. (2) has a unique global solution. Define $f : \mathbb{R}^n \to \mathbb{R}^n$ with $n \ge 1$. consider the fractional-order scheme below:

$$\begin{cases} D^{\xi} &= f(x),\\ x(t) &= x_0,\end{cases} \tag{5}$$

with $\xi \in (0,1]$, $t \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$. We will apply the following result 4, which is a direct consequence from [16], to globalize the solution of scheme eq. (5).

Result 4. Suppose that the function meets the criteria listed below:

1. f and $\dfrac{\partial f}{\partial x}$ are continuous on $\mathcal{R}^n$.
2. $||f(x)|| \le \gamma + \eta + ||x||$ for all $x \in \mathcal{R}^n$ where γ and η are two positive constants. Then, system eq. (6) has a unique solution on $[t_0, +\infty)$.

Theorem 2. *The fractional-order initial value problem eq. (2) has unique solution.*

Proof. Let $X = (s,i,c,w)^T$, then model (2) can be rewritten as follows:

$$D^{\xi}X = F(X), \tag{6}$$

where

$$F(X) = \begin{pmatrix} A - a_0 s - \beta sc\\ \beta sc - \beta_0 i - \beta_1 ci\\ k - k_0 c - k_1 cw\\ B + bc - b_0 w - b_1 wc\end{pmatrix}.$$

First, it is clear that F satisfies the first criterion of Result 4. Next, let's rewrite F as a vector function to satisfy the second condition.

$$F(X) = M_0 + (cM_1 + M_2)X,$$

where

$$M_0 = \begin{pmatrix} A\\ 0\\ k\\ B\end{pmatrix},$$

$$M_1 = \begin{pmatrix} -\beta & 0 & 0 & 0\\ \beta & \beta_1 & 0 & 0\\ 0 & 0 & 0 & -k_1\\ 0 & 0 & 0 & -b_1\end{pmatrix},$$

$$M_2 = \begin{pmatrix} -a_0 & 0 & 0 & 0\\ 0 & -\beta_0 & 0 & 0\\ 0 & 0 & -k_0 & 0\\ 0 & 0 & b & -b_0\end{pmatrix}.$$

It follows that there exist $\gamma = ||M_0||$ and $\eta = |c|||M_1|| + ||M_2||$ such that

$$F(X) \le \gamma + \eta||X||.$$

Then, model (2) has a unique solution on $[0, +\infty)$. □

3. Stability criteria for the Fractional order of SICW model

In this part, we go through whether equilibria exist and their local stability of model (2). To find the equilibria, we should equate to zero the right-hand side in model (2). It has only one equilibrium point, namely $E^*(s^*, i^*, c^*, w^*)$ whose components s^*, i^*, c^* and w^* are positive solutions of the subsequent algebraic equations.

$$A - a_0 s - \beta sc = 0, \tag{7}$$
$$\beta sc - \beta_0 i - \beta_1 ci = 0, \tag{8}$$
$$k - k_0 c - k_1 cw = 0, \tag{9}$$
$$B + bc - b_0 w - b_1 wc = 0. \tag{10}$$

From eq. (9) we have,

$$c = \frac{k}{1 + k_1 w}. \tag{11}$$

Using eq. (11) in eq. (10) we have the following quadratic equation in w:

$$k_1 b_0 w^2 + (k_0 b_0 + k b_1 - B k_1)w - (kb + Bk_0) = 0 \tag{12}$$

The positive endemic equilibrium satisfies eq. (2) and it can be easily observed that eq. (12) has a unique positive root by escartes' rule of sign, w^* (say). Using value of w^* in eq. (11) we get

$$\begin{aligned} c^* &= \frac{k}{1 + k_1 w^*},\\ s^* &= \frac{A(k_0 + k_1 w^*)}{a_0 k_0 + \beta k + a_0 k_1 w^*},\\ i^* &= \frac{\beta A k_0 (k_0 + k_1 w^*)}{(a_0 k_0 + \beta k + a_0 k_1 w^*)(\beta_0 (k_0 + k_1 w^*) + \beta_1 k)}.\end{aligned}$$

The Jacobian matrix of model (2) is

$$J = \begin{pmatrix} -a_0 - \beta c & 0 & -\beta s & 0\\ \beta c & \beta_0 - \beta_1 c & \beta s - \beta_1 i & 0\\ 0 & 0 & -k_0 - k_1 w & -k_1 c\\ 0 & 0 & b - b_1 w & -b_0 - b_1 c\end{pmatrix}. \tag{13}$$

To examine the stability of equilibrium points, we employ the following stability theorem.

Theorem 3. *Consider the following autonomous* $\frac{d^{\xi}u(t)}{dt^{\xi}} = f(u(t)); u(0) = 0; 0 < \xi < 1$ *nonlinear fractional order system:*
The following system's equilibrium points are solutions to the equation $f(u(t)) = 0$. *If all eigenvalues* (λ_j) *of the Jacobian*

matrix $J = \frac{\partial f}{\partial u}$ evaluated at equilibrium u^* meet the condition

$$|arg(\lambda_j)| > \frac{\xi\pi}{2}, \tag{14}$$

then an equilibrium point u^* is locally asymptotically stable.

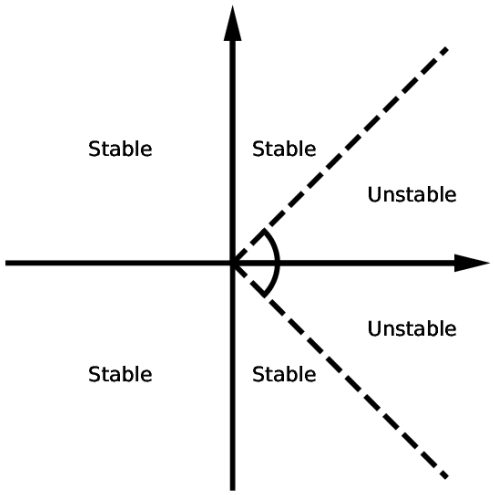

Figure 2. The stability region of the fractional order system with $0 < \xi \leq 1$ is shown as per criterion eq. (14) .

Now, we'll concentrate about the model (2) endemic (positive) equilibrium's asymptotic stability. The Jacobian matrix evaluated at the endemic equilibrium is given as:

$$J(E_+) = \begin{pmatrix} -a_0 - \beta c & 0 & -\beta s^* & 0 \\ \beta c^* & -\beta_0 - \beta_1 c^* & \beta s^* - \beta_1 i^* & 0 \\ 0 & 0 & -k_0 - k_1 w^* & -k_1 c^* \\ 0 & 0 & b - b_1 w^* & -b_0 - b_1 c^* \end{pmatrix} \tag{15}$$

From eq. (15) the latent equation of $J(E_+)$ is

$$\lambda^4 + h_0\lambda^3 + h_1\lambda^2 + h_2\lambda + h_3 = 0 \tag{16}$$

Where

$$\begin{aligned} h_0 =& D_1 + D_2 + D_3 + D_4 + D_5 + D_6, \\ h_1 =& D_1D_2 + D_4D_6 + D_1D_4 + D_1D_6 + D_2D_4 + D_2D_6 \\ &+ D_1k_1c^*, \\ h_2 =& D_1D_4D_6 + D_2D_4D_6 + D_1D_2D_4 + D_1D_2D_6 + D_1^2k_1c^* \\ &+ D_1D_2k_1c^*, \\ h_3 =& D_1D_2D_4D_6 + D_1^2D_2k_1c^*, \\ D_1 =& a_0 + \beta c^*, D_2 = \beta_0 + \beta_1 c^*, \\ D_3 =& \beta s^* - \beta_1 i^*, D_4 = k_0 + k_1 w^*, \\ D_5 =& b - b_1 w^*, D_6 = b_0 + b_1 c^*. \end{aligned}$$

If $D(\phi)$ denotes the discriminant of the polynomial $\phi(\lambda) = \lambda^4 + h_0\lambda^3 + h_1\lambda^2 + h_2\lambda + h_3$ where all the coefficients are real. Then let denote

$$D(\phi) = \begin{vmatrix} 1 & h_0 & h_1 & h_2 & h_3 & 0 & 0 \\ 0 & 1 & h_0 & h_1 & h_2 & h_3 & 0 \\ 0 & 0 & 1 & h_0 & h_1 & h_2 & h_3 \\ 4 & 3h_0 & 2h_1 & h_2 & 0 & 0 & 0 \\ 0 & 4 & 3h_0 & 2h_1 & h_2 & 0 & 0 \\ 0 & 0 & 4 & 3h_0 & 2h_1 & h_2 & 0 \\ 0 & 0 & 0 & 4 & 3h_0 & 2h_1 & h_2 \end{vmatrix} \tag{17}$$

From [32], we have the proposition.

Theorem 4. *Suppose that E_+ exists in $\mathcal{R}_+^4$*

1. *Let Ω_1, Ω_2, Ω_3 be Routh-Hourwitz determinants:*

$$\begin{aligned} \Omega_1 &= h_0, \\ \Omega_2 &= \begin{vmatrix} h_0 & 1 \\ h_2 & h_1 \end{vmatrix}, \\ \Omega_3 &= \begin{vmatrix} h_0 & 1 & 0 \\ h_2 & h_1 & h_0 \\ h_4 & h_3 & h_2 \end{vmatrix} = \begin{vmatrix} h_0 & 1 & 0 \\ h_2 & h_1 & h_0 \\ 0 & h_3 & h_2 \end{vmatrix}. \end{aligned}$$

   Hence, as $\xi = 1$, the equilibrium point E_+ is locally asymptotically stable if

$$\Omega_1 > 0,\ \Omega_2 > 0,\ \Omega_3 = 0,\ h_3 > 0. \tag{18}$$

   For all $\xi \in [0, 1)$, for E_+ to be locally asymptotically stable, these constraints eq. (18) are sufficient but not necessary. Since most biologically intriguing systems are one, two and three dimensions we will investigate the problem eq. (14) from $n = 1$ to 4.
2. *If $D(\phi) > 0$, $h_0 > 0$, $h_1 < 0$ and $\xi > \frac{2}{3}$, then the equilibrium point E_+ is unstable.*
3. *If $D(\phi) < 0$, $h_0 > 0$, $h_1 > 0$, $h_2 > 0$, $h_3 > 0$ and $\xi < \frac{1}{3}$, then the equilibrium point E_+ is stable. Also if $D(\phi) < 0$, $h_0 < 0$, $h_1 > 0$, $h_2 < 0$, $h_3 > 0$, then the equilibrium point E_+ is unstable.*
4. *If $D(\phi) < 0$, $h_0 > 0$, $h_1 > 0$, $h_2 > 0$, $h_3 > 0$, and $h_1 = \frac{h_0h_3}{h_2} + \frac{h_2}{h_0}$ then the equilibrium point E_+ is locally asymptotically stable for all $\xi \in (0, 1)$.*
5. *If $h_3 > 0$ is the necessary condition for the equilibrium point E_+ to be locally asymptotically stable.*

4. Numerical Methods and Simulations

Approximation and numerical approaches must be utilized since the majority of fractional-order differential equations lack accurate analytic solutions. The fractional-order differential equations have been solved using a variety of analytical and numerical techniques. One can utilise an unconventional finite difference approach to solve the model (2) numerically (NFDM). The notion of the nonstandard finite difference technique is covered in [33]. Mickens first developed the nonstandard finite difference methods in the 1980s as a potent numerical approach that retains important features of precise solutions of the differential equation concerned [34]. The model (2) can be discretized as follows using this approach [35].

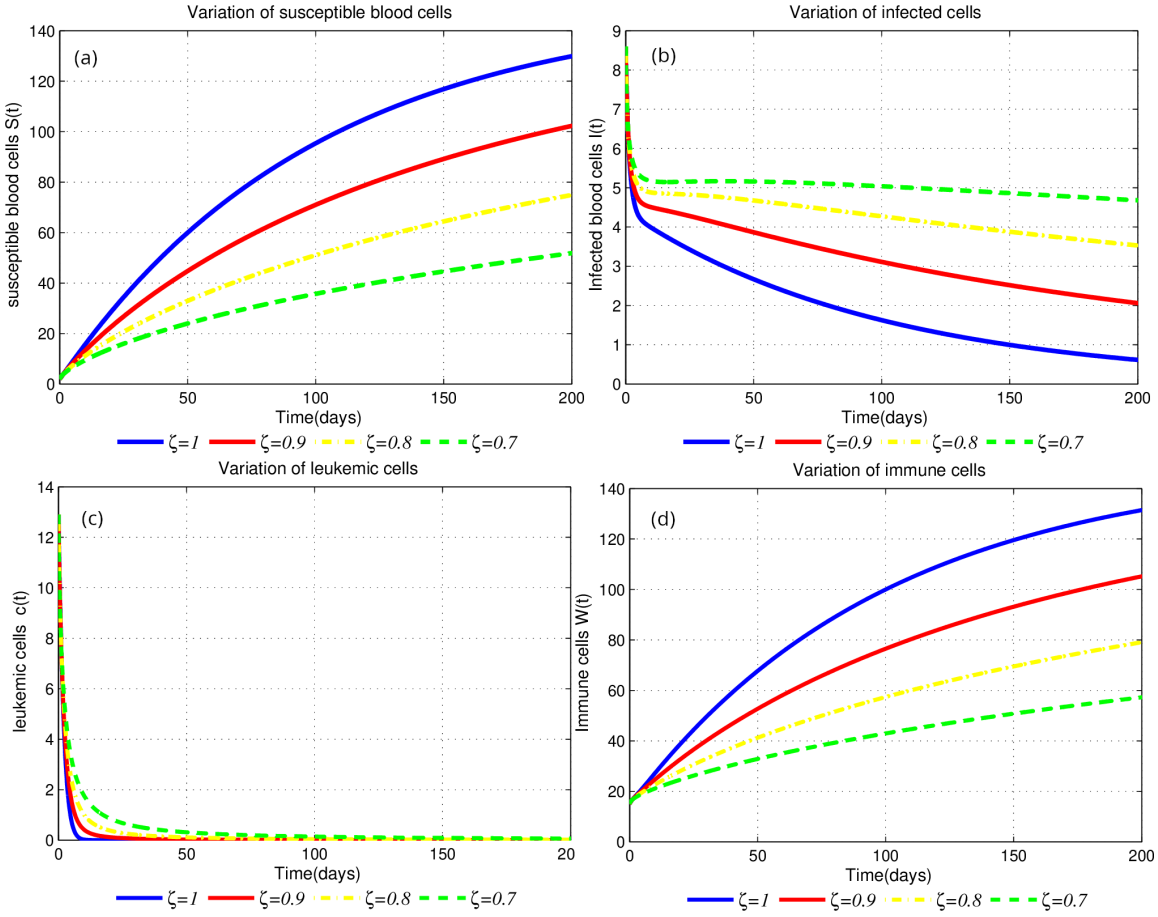

Figure 3. This figure illustrates the effect of stability of the endemic equilibrium E_+ for various values of ξ. (a)-(d) illustrate the variations of susceptible, infected, Leukemic and immune cells.

$$
\begin{aligned}
\sum_{r=0}^{\nu+1} \delta_r^{\xi} s_{\nu+1-r} &= A - a_0 s_{\nu+1} - \beta s_{\nu+1} c_{\nu} \\
\sum_{r=0}^{\nu+1} \delta_r^{\xi} i_{\nu+1-r} &= \beta s_{\nu+1} c_{\nu} - \beta_0 i_{\nu+1} - \beta_1 c_{\nu+1} i_{\nu} \\
\sum_{r=0}^{\nu+1} \delta_r^{\xi} c_{\nu+1-r} &= k - k_0 c_{\nu+1} - k_1 c_{\nu+1} w_{\nu} \\
\sum_{r=0}^{\nu+1} \delta_r^{\xi} w_{\nu+1-r} &= B + b c_{\nu} - b_0 w_{\nu+1} - b_1 c_{\nu+1} w_{\nu}
\end{aligned}
\tag{19}
$$

Performing some algebraic operations to eq. (19) produces the subsequent relations.

$$
\begin{aligned}
s_{\nu+1} &= \frac{A - \sum_{r=1}^{\nu+1} \delta_r^{\xi} s_{\nu+1-r}}{\delta_0^{\xi} + a_0 + \beta c_{\nu}} \\
i_{\nu+1} &= \frac{\beta s_{\nu+1} c_{\nu} - \beta_1 c_{\nu+1} i_{\nu} - \sum_{r=1}^{\nu+1} \delta_r^{\xi} i_{\nu+1-r}}{\delta_0^{\xi} + \beta} \\
c_{\nu+1} &= \frac{k - \sum_{r=1}^{\nu+1} \delta_r^{\xi} c_{\nu+1-r}}{\delta_0^{\xi} + k_0 + k_1 w_{\nu}} \\
w_{\nu+1} &= \frac{B + b c_{\nu} - b_1 w_{\nu} c_{\nu+1} - \sum_{r=1}^{\nu+1} \delta_r^{\xi} w_{\nu+1-r}}{\delta_0^{\xi} + b_0}
\end{aligned}
$$

In order to demonstrate our theoretical findings mentioned in the preceding sections, we run few numerical simulations of the luekemia transmission model (2). Figures 3 and 4 depict the approximate results $s(t)$, $i(t)$, $c(t)$ and $w(t)$. By choosing $A = 1.5$, $A_0 = 0.01$, $\beta = 0.003$, $\beta_0 0.03$, $\beta_1 = 0.02$, $k = 0.003$, $k_0 = 0.01$, $k_1 = 0.03$, $B = 1.5$, $b = 0.5$ and various values of ξ, we get the endemic equilibrium point E_+. Then, according to the Theorem 4, the endemic equilibrium point is locally asymptotically stable (Figure 3).

Figure 4 further shows that the solution of model (1) converges to the endemic equilibrium point E^*. for fixed value of $\xi = 0.9$ and for various initial conditions. Therefore, we may notice that fractional derivative ξ affects the evolution of the states from all of these Figures.

5. Discussion and Concluding Remarks

In the current study, in order to explain how leukemia spreads, we have suggested as well as investigated an unique fractional order model that makes use of the Caputo fractional derivative. First, we have proven that non-negative solutions exist and are bounded. The Routh-Hurwitz criteria are used to determine the endemic equilibrium's local stability. We ran the numerical simulation for several values of the parameter ξ, and found that this parameter affects how the model states change over time. Integer order system provides fewer benefits than fractional order system. We can represent a higher order system using a lower order model when describing a system by frac-

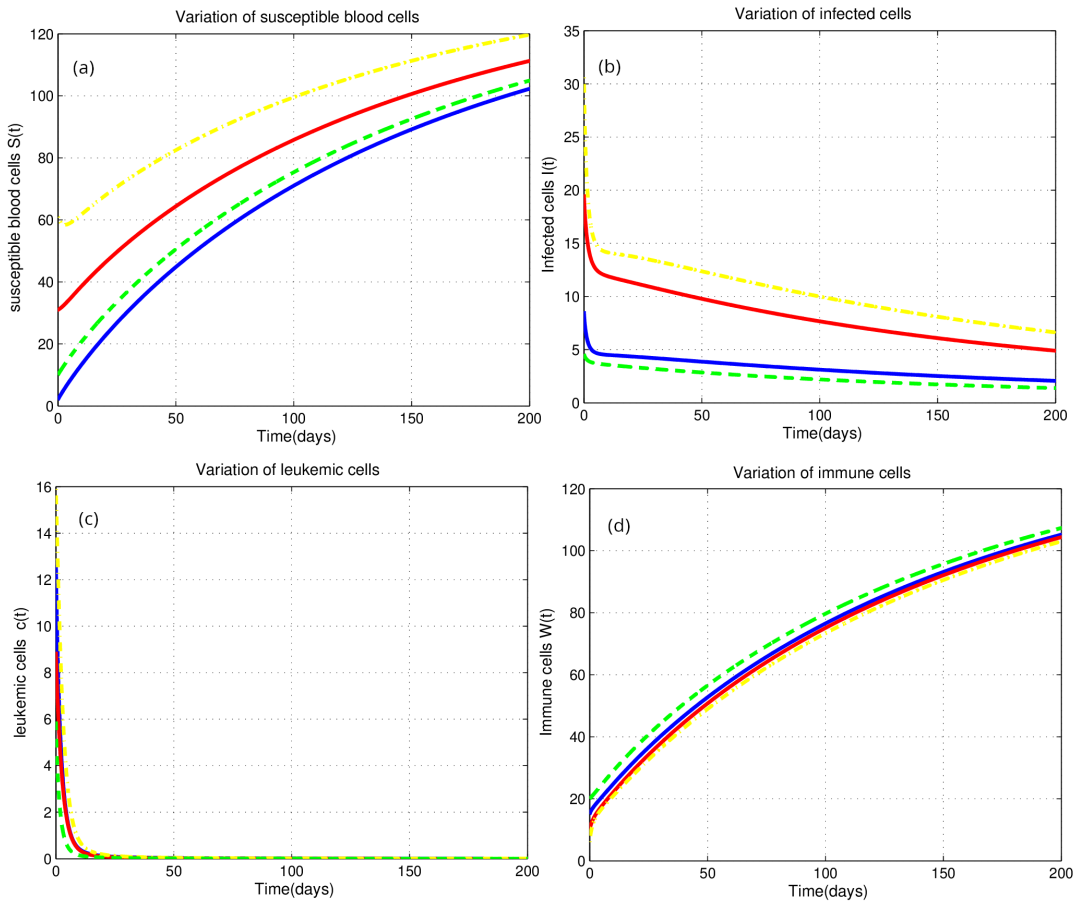

Figure 4. This depicts the effect of stability of the endemic equilibrium E_+ with fixed value of $\xi = 0.9$ and various initial values of each state variable.(a)-(d) illustrate the variations of susceptible, infected, Leukemic and immune cells.

tional order. Fractional order systems frequently have a greater influence on controlling theories than integer order systems. We have used Predictor-Corrector method to simulate model (2) with the help of MATLAB. The numerical analysis of the fractional order system is quite advanced. Numerous techniques are utilized in numerical studies, including the Homotopy perturbation method, the Taylor basis approximations approach, the Adomian decomposition method, and others. In the future, we'll strive to assess this work using a variety of numerical techniques to provide better outcomes.

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