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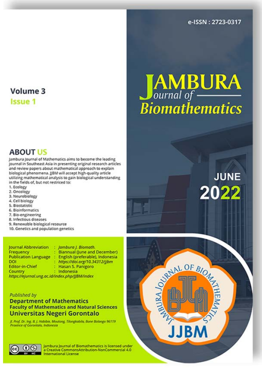
Volume 5, Issue 2, Pages 63–70, December 2024

Received 2 June 2024, Revised 14 July 2024, Accepted 18 October 2024, Published Online 1 December 2024

To Cite this Article : A.N. Salsabila and D. Savitri,"Dynamical Analysis of Holling Tanner Prey Predators Model with Add Food in Second Level Predators", *Jambura J. Biomath*, vol. 5, no. 2, pp. 63–70, 2024, *https://doi.org/10.37905/jjbm.v5i2.25753*

JOURNAL INFO • JAMBURA JOURNAL OF BIOMATHEMATICS



Homepage	:	http://ejurnal.ung.ac.id/index.php/JJBM/index
Journal Abbreviation	:	Jambura J. Biomath.
Frequency	:	Biannual (June and December)
Publication Language	:	English
DOI	:	https://doi.org/10.37905/jjbm
Online ISSN	:	2723-0317
Editor-in-Chief	:	Hasan S. Panigoro
Publisher	:	Department of Mathematics, Universitas Negeri Gorontalo
Country	:	Indonesia
OAI Address	:	http://ejurnal.ung.ac.id/index.php/jjbm/oai
Google Scholar ID	:	XzYgeKQAAAAJ
Email	:	editorial.jjbm@ung.ac.id

Jambura Journal of Biomathematics, Volume 5, Issue 2, Pages 63–70, December 2024 https://doi.org/10.37905/jjbm.v5i2.25753

Research Article

Dynamical Analysis of Holling Tanner Prey Predators Model with Add Food in Second Level Predators

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ARTICLE HISTORY
Received 2 June 2024
Revised 14 July 2024
Accepted 18 October 2024
Published 1 December 2024

KEYWORDS
Holling Tanner
bifurcation
additional food
dynamic analysis

ABSTRACT. *This article discusses the Holling Tanner prey predator model and Holling type II response function with additional food in the second level predator. The dynamic analysis of the system begins with determining the equilibrium point, analyzing the stability of the equilibrium point, and numerical simulation with python. The results of the dynamic analysis obtain seven equilibrium points, namely E_1 extinction in three populations, point E_2 extinction in the population of prey and first level predator, point E_3 extinction in the first and second level predator populations, point E_4 extinction in the second level predator population, point E_5 extinction in the first level predator population, and point E_6 the three populations are not extinction. The results of the stability analysis around the equilibrium point E_1, E_2, E_3 are shown to be saddle unstable, then E_4, E_5, E_6 are asymptotically stable with certain conditions. Numerical simulation is applied to determine the validity of the analytical results. The simulation results illustrate changes in the system solution in the form of phase portraits. The bifurcation diagram of the numerical continuation of the maximum predation rate parameter of the second-level predator (β) shows the existence of Hopf bifurcation when maximum predation rate parameter of the second-level predator with $\beta = 2.1014232$ and Transcritical bifurcation when maximum predation rate parameter of the second-level predator with $\beta = 3.197$.*

1. Introduction

Organisms in the world living side by side and interdependent, in the life of organisms there are several species that create a population, in the association of each population with other populations forming a community, and than community will relate to its environment which forms a complete structure called an ecosystem or ecological system. [1]. In ecosystems there is an interaction or relation between prey and predator which becomes a prey-predators system, where prey is an animal that is eaten, and predators are animals that eat other animals to survive. Prey-predators interactions have a positive effect on predator density and a negative effect on prey density [2].

In ecological studies, the process of interaction between prey and predator is known as the response function. It is a food density function for the predator that describes the amount of food eaten. This response function has been the basis of many ecological models and population dynamics studies. Holling (1965) introduced a response function form divided into 3, namely, Holling type I, type II, and type III response functions. Several researchers have used Holling's response function in prey-predator interactions by taking into factor that affect the population growth. [3–6]. In the development of the prey-predator model, there are various modifications in the predation process. The simple ecological model became the forerunner of other predator-prey models, one of which is the Leslie-Gower model [7, 8], which focuses on a system with two populations, one prey and one predator. Next is the Holling Tanner model, this model was proposed by James T. Tanner, This model considers factors such as logistic growth and Holling type II response function. [9]. Holling Tanner model was studied by [10] and [11] explains that prey predator dynamics follow the Holling Tanner model in which a generalist predator eats its favorite food as long as it is available and grows logistically with an intrinsic growth rate and carrying capacity proportional to the size of its prey. Furthermore, in order to provide a better understanding of the dynamics of complex ecological systems, the study of three-population models is very interesting. The model with three populations has been studied by Savitri [12] which discusses prey-predator interactions with Holling type II response functions and includes the addition of food and anti-predator defenses.

Predators in prey-predator interactions perform an important role in balancing the overall structure of the ecosystem. One way to maintain and control balance is by additional food for the predators [13]. Supplementary food is food that is found in the digestive tract in small amounts [14]. In the process of model development, there are many modifications of the prey predator model and the predation process of the predator on the prey. Several developments of prey-prey models with additional food for predators have been studied using different response functions [15–17].

An interesting problem in the interaction of prey and predator populations happens in the aquatic ecosystem in Indonesia in the Flores Sea. One of the species found in the Flores Sea is skipjack (Katsuwonus pelamis). The interaction process in skip-

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jack follows Holling's type II response function by requiring time to search, eat, and digest food. The interaction starts with the zooplankton species Euterpina acutifrons (Copepoda) which occupies the position of prey. The zooplankton species are eaten by anchovies which are positioned as first-level predators. Anchovy becomes the main food for skipjack which is positioned as a second-level predator, according to [18] additional food types of skipjack in the Flores Sea are squid (Loligo sp.) and other fish. Skipjack is a Carnivore because it can be seen that the most consumed food and the main food of skipjack are types of fish (anchovies, shrimp, squid) [19].

Based on the research reviewed by [12, 20] and [21], there are additional food parameters for predators. Sahoo modeled a three population prey predator system using Holling Type II response function with additional food. The results showed that additional food is an effective parameter to control the dynamics of the three species prey predator system. Basheer's research shows the Holling Tanner model with additional food for the predator and the goals of the additional food are for the predator to survive when the prey population is low, increase the number of carrying capacity of the predator, and increase in predator births. The authors will analyze the model using the Holling Tanner model and the Holling type II response function in the existence of additional food for predators.

The main objective of this research is to study the one prey and two predators model with the dynamic analysis which includes equilibrium point analysis, local stability, and numerical simulation used to support the stability analysis and interpretation. To make it easier to understand the determination of stability and types is to use the equilibrium point stability criteria table [22].

The rest of the paper is structured as follows. The research concept in Section 2. Results and discussion in Section 3 which includes the model is presented in Section 3.1, the existence of equilibrium points are discussed in Section 3.2, local stability analysis is carried out in Section 3.3, numerical simulation are given in Section 3.4. A Conclusion follows Section 4.

2. Methods

The prey predators interaction model studied is a Holling-Tanner mathematical model and uses a Holling type II response function with additional food in the second-level predator. The following are the stages in the research concept:

1. Literature study
   Literature study related to the interaction model in one prey two predator Holling Tanner system and Holling type II response function with the additional food in predators.
2. Model construction
   Model construction is made by some literature on the prey predator system which is then developed into a model.
3. Equilibrium point
   Equilibrium point analysis was conducted to determine the equilibrium point by zeroing the right-hand segment of each equation of the interaction model in one prey two predator Holling Tanner system and Holling type II response function with the additional food in predators.
4. Local Stability
   Local Stability is the use of Jacobian matrix to obtain eigenvalues used in determining the Local Stability properties of the system around the equilibrium point.
5. Numerical simulation
   Numerical simulation is carried out to show changes in the dynamic system around the equilibrium point. Numerical simulation is also used to determine the suitability of the results of system analysis and model interpretation. The results of numerical simulations are graphically represented using python and matlab programs.

3. Results and Discussion

3.1. Mathematical models

The prey-predator interaction model in this study combine and modify assumptions based on previous research, namely the Holling Tanner prey-predator model using a ratio-dependent response function in the presence of additional food on predators [21], and one-prey-two-predators model using Holling type II response function in the presence of additional food for the predators. [20]. The model is categorized into three populations, namely prey (x), first-level predator (y), and second-level predator (z). From the ecosystem found in the Flores Sea, it can be stated as follows, zooplankton are prey, zooplankton are eaten by anchovies as first-level predators, and anchovies are eaten by skipjack as second-level predators. The growth of zooplankton follows logistic growth and is reduced due to the maximum predation rate by anchovy by following Holling type II response function. Furthermore, anchovy growth will increase with the conversion of energy from zooplankton and decrease due to the maximum predation rate by skipjack and natural mortality, and the growth of skipjack follows the Holling Tanner model with a carrying capacity proportional to the size of the anchovy, next the growth of skipjack increases with the conversion of energy from anchovies and with the additional food. Based on the above assumptions, the construction of the Holling Tanner prey-predator interaction model and Holling type II response function in the presence of additional food for predators is as follows

$$\begin{aligned}
\frac{dx}{dt} &= rx\left(1-\frac{x}{K}\right)-\frac{\alpha xy}{m+x},\\
\frac{dy}{dt} &= \frac{\alpha_1 xy}{m+x}-\mu y-\frac{\beta yz}{nA+m+y},\\
\frac{dz}{dt} &= \beta_1 z\left(\rho+\frac{A-z}{nA+y}\right).
\end{aligned} \tag{1}$$

3.2. Equilibrium point

To determine the equilibrium point of the system eq. (1) can be determined by solving the equation with $\frac{dx}{dt}=\frac{dy}{dt}=\frac{dz}{dt}=0$. From solving eq. (1), seven equilibrium points are obtained as below.

1. Equilibrium point $E_1=(0,0,0)$
   Represents the extinction of the prey, first level predator, and second level predator populations.
2. Equilibrium Point $E_2=(0,0,A+\rho nA)$
   Represents the extinction of the prey population and the first-level predator, while the second-level predator population exist.
3. Equilibrium point $E_3=(K,0,0)$
   represents the prey population existing, while the first and

Table 1. Description of parameters in the model (1)

Parameters	Interpretasion
r	Prey intrinsic growth rate
K	Environmental carrying capacity
α	Maximum predation rate of a first level predator
m	time taken by the predator to find, eat and digest the food
α_1	Predation conversion of prey by a first level predator
μ	Natural death
β	Maximal predation rate of second level predators
nA	Quality and quantity of additional food in second level predators
β_1	Predation conversion of first level predators by second-level predators
ρ	Intrinsic growth rate of second-level predator

second level predator population are extinction.

4. Equilibrium point $E_4 = (\frac{\mu m}{\alpha_1-\mu}, \frac{r m \alpha_1(k\alpha_1 - K\mu - \mu m)}{(\alpha_1-\mu)^2\alpha K}, 0)$
Expresses the condition that the population of prey and first-level predators does not extinction or exist, while the population of second-level predators experiences extinction.
5. Equilibrium point $E_5 = (K, 0, A(1-\rho n))$
Represents the situation where the prey population and the second-level predator exist, while the first-level predator population experiences extinction.
6. Equilibrium point $E_6 = (x^*, y^*, z^*)$
Expressed the condition that the populations of prey, first-level predators, and second-level predators do not experience extinction or exist.

3.3. Local stability

The Jacobian matrix of the prey predator system can be obtained by taking partial derivatives of eq. (1), resulting in the following Jacobian matrix:

$$J(x,y,z) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix}, \quad (2)$$

with the following elements

$$\begin{aligned}
\frac{\partial f_1}{\partial x} &= r\left(1-\frac{x}{K}\right) - \frac{rx}{K} - \frac{\alpha y}{m+x} + \frac{\alpha x y}{(m+x)^2}, \\
\frac{\partial f_1}{\partial y} &= -\frac{\alpha x}{m+x}, \qquad \frac{\partial f_1}{\partial z} = 0, \\
\frac{\partial f_2}{\partial x} &= \frac{\alpha_1 y}{m+x} - \frac{\alpha_1 x y}{(m+x)^2}, \\
\frac{\partial f_2}{\partial y} &= \frac{\alpha_1 x}{m+x} - \mu - \frac{\beta z}{nA+m+y} + \frac{\beta y z}{(nA+m+y)^2}, \\
\frac{\partial f_2}{\partial z} &= -\frac{\beta y}{nA+m+y}, \qquad \frac{\partial f_3}{\partial x} = 0, \\
\frac{\partial f_3}{\partial y} &= -\frac{\beta_1 z(A-z)}{(nA+y)^2}, \\
\frac{\partial f_3}{\partial z} &= \beta_1\left(\rho + \frac{A-z}{nA+y}\right) - \frac{\beta_1 z}{nA+y}.
\end{aligned}$$

Theorem 1. *The equilibrium point E_1 $(0,0,0)$ is always unstable*

Proof. By substituting E_1 $(0,0,0)$ to the eq. (2), we have

$$J_{E_1} = \begin{bmatrix} r & 0 & 0 \\ 0 & -\mu & 0 \\ 0 & 0 & \beta_1\left(\rho+\frac{1}{n}\right) \end{bmatrix},$$

than we get eigen values $\lambda_1 = r$, $\lambda_2 = -\mu$, and $\lambda_3 = \beta_1\left(\rho+\frac{1}{n}\right)$. Since λ_1, λ_2, and λ_3 positive, equilibrium point E_1 $(0,0,0)$ is always unstable. □

Theorem 2. *The equilibrium point E_2 $(0,0,A+\rho nA)$ is always unstable*

Proof. By substituting E_2 $(0,0,A+\rho nA)$ to the eq. (2), we have

$$J_{E_2} = \begin{bmatrix} r & 0 & 0 \\ 0 & -\mu - \frac{\beta(An\rho+A)}{nA+m} & 0 \\ 0 & \frac{\beta_1(An\rho+A)\rho}{An} & -\frac{\beta_1(An\rho+A)}{nA} \end{bmatrix},$$

than eigen values for E_2 are $\lambda_1 = r$, $\lambda_2 = \frac{-\mu nA - \mu m - \beta An\rho - \beta A}{nA+m}$, and $\lambda_3 = -\frac{\beta_1(n\rho+1)}{n}$, Since λ_1 positive, equilibrium point E_2 $(0,0,A+\rho nA)$ is always unstable. □

Theorem 3. *The equilibrium point $E_3(K,0,0)$ is always ustable*

Proof. By substituting $E_3(K,0,0)$ to the eq. (2), we have

$$J_{E_3} = \begin{bmatrix} -r & \frac{\alpha K}{m+K} & 0 \\ 0 & \frac{\alpha_1 K}{m+K} - \mu & 0 \\ 0 & 0 & -\beta_1\left(\rho+\frac{1}{n}\right) \end{bmatrix},$$

than eigen values for E_3 are $\lambda_1 = -r$, $\lambda_2 = \frac{\alpha_1 K - \mu m - \mu K}{m+K}$, and $\lambda_3 = \frac{\beta_1(\rho n+1)}{n}$. Since λ_3 positive, equilibrium point $E_3(K,0,0)$ is always unstable. □

Theorem 4. *The equilibrium point E_4 $(\frac{\mu m}{\alpha_1-\mu}, \frac{r m\alpha_1(k\alpha_1-K\mu-\mu m)}{(\alpha_1-\mu)^2\alpha K}, 0)$ is asymptotically stable if $B_{33} < 0$, $T_1 < 0$, dan $D_1 > 0$.*

Proof. By substituting $E_4\ (\frac{\mu m}{\alpha_1-\mu}, \frac{rm\alpha_1(k\alpha_1-K\mu-\mu m)}{(\alpha_1-\mu)^2\alpha K}, 0)$ eq. (2), we have

$$J_{E_4} = \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{21} & B_{22} & B_{23} \\ 0 & 0 & B_{33} \end{bmatrix},$$
$$b_{11} = r\left(1 - \frac{m\mu}{(\alpha_1-\mu)K}\right) + \frac{m^2\mu r\alpha_1(\alpha_1 K - \mu K - m\mu)}{(\alpha r\mu)^3\left(m+\frac{m\mu}{\alpha_1-\mu}\right)^2 K} - \frac{rm\mu}{(\alpha_1-\mu)K} - \frac{rm\alpha_1(\alpha_1 K - \mu K - m\mu)}{(\alpha_1-\mu)^2)\left(m+\frac{m\mu}{\alpha_1-\mu}\right)K},$$

with the following elements

$$B_{12} = -\frac{\alpha m\mu}{(\alpha_1-\mu)\left(m+\frac{m\mu}{\alpha_1-\mu}\right)},$$
$$B_{21} = \frac{\alpha_1(rm\alpha_1(K\alpha_1 - K\mu - m\mu))}{((\alpha_1-\mu)^2\alpha K)m + \frac{m\mu}{\alpha_1-\mu}} - \frac{{\alpha_1}^2 m^2 r\mu(K\alpha_1 - K\mu - m\mu)}{(\alpha_1-\mu)^3\alpha K\left(m+\frac{m\mu}{\alpha_1-\mu}\right)^2},$$
$$B_{22} = \frac{\alpha_1 m\mu}{(\alpha_1-\mu)\left(\frac{m+m\mu}{\alpha_1-\mu}\right)} - \mu,$$
$$B_{23} = -\frac{\beta rm\alpha_1(K\alpha_1 - K\mu - \mu m)}{(\alpha_1-\mu)^2\alpha K\left(nA + m + \frac{rm\alpha_1(K\alpha_1-K\mu-\mu m)}{(\alpha_1-\mu)^2\alpha K}\right)},$$
$$B_{33} = \beta_1\left(\rho + \frac{A}{nA + \frac{rm\alpha_1(K\alpha_1-K\mu-\mu m)}{(\alpha_1-\mu)^2\alpha K}}\right),$$

than eigen values for E_4 $\lambda_1 = B_{33}$, λ_2 and λ_3 obtained from $\lambda^2 - T_1\lambda + D_1 = 0$, where $T_1 = B_{11} + B_{22}$ and $D_1 = B_{11}B_{22} - B_{12}B_{21}$, so equilibrium point E_4 is asymptotically stable if $B_{33} < 0$, $T_1 < 0$, and $D_1 > 0$. □

Theorem 5. *The equilibrium point $E_5\ (K, 0, A(1-\rho n))$ is stable if $\lambda_2 < 0$ when $\beta > \frac{(\alpha_1 K-\mu m-\mu K)(nA+m)}{(m+K)(A(1+\rho n)}$, and $\lambda_3 < 0$ when $\beta_1\rho < \frac{\beta_1(1-\rho n)}{n} - \frac{\beta_1 A-\beta_1 A(1-\rho n)}{nA}$.*

Proof. By substituting $E_5\ (K, 0, A(1-\rho n))$ to the eq. (2), we have

$$J_{E_5} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ 0 & C_{22} & 0 \\ 0 & C_{32} & C_{33} \end{bmatrix}$$

with the following elements

$$C_{11} = -r,$$
$$C_{12} = -\frac{\alpha K}{m+K},$$
$$C_{22} = \frac{\alpha_1 K}{m+K} - \mu - \frac{\beta A(1+\rho n}{nA+m},$$
$$C_{32} = -\frac{\beta_1 A(1+\rho n)(A-(1+\rho n))}{(nA)^2},$$
$$C_{33} = \beta_1\left(\rho + \frac{A - A(1+\rho n)}{nA}\right) - \frac{\beta_1(1+\rho n)}{n}.$$

than eigen values for E_5 are

$$\lambda_1 = C_{11} = -r,$$
$$\lambda_2 = C_{22} = \frac{\alpha_1 K}{m+K} - \mu - \frac{\beta A(1+\rho n)}{nA+m},$$
$$\lambda_3 = C_{33} = \beta_1\left(\rho + \frac{A - A(1+\rho n)}{nA}\right) - \frac{\beta_1(1+\rho n)}{n}.$$

So stability of equilibrium point E_5 are

a. Stable if $\lambda_2 < 0$ when $\frac{\alpha_1 K}{m+K} - \mu - \frac{\beta A(1+\rho n)}{nA+m} < 0$ or $\beta > \frac{(\alpha_1 K-\mu m-\mu K)(nA+m)}{(m+K)(A(1+\rho n)}$, and $\lambda_3 < 0$ when $\beta_1\left(\rho + \frac{A-A(1+\rho n)}{nA}\right) - \frac{\beta_1(1+\rho n)}{n} < 0$ or $\beta_1\rho < \frac{\beta_1(1-\rho n)}{n} - \frac{\beta_1 A-\beta_1 A(1-\rho n)}{nA}$

b. Unstable if $\lambda_2 > 0$ when $\beta < \frac{(\alpha_1 K-\mu m-\mu K)(nA+m)}{(m+K)(A(1+\rho n)}$, and $\lambda_3 > 0$ when $\beta_1\rho > \frac{\beta_1(1-\rho n)}{n} - \frac{\beta_1 A-\beta_1 A(1-\rho n)}{nA}$

□

Theorem 6. *The equilibrium point $E_6\ (x^*, y^*, z^*)$ is asymtotically stable if $h_1 > 0$, $h_3 > 0$, dan $h_1h_2 > h_3$.*

Proof. By substituting $E_6\ (x^*, y^*, z^*)$ to the eq. (2), we have

$$J_{E_6} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{21} & D_{22} & D_{23} \\ 0 & D_{32} & D_{33} \end{bmatrix} \tag{3}$$

with each of its components are

$$D_{11} = r\left(1 - \frac{x^*}{K}\right) - \frac{rx^*}{K} - \frac{\alpha y^*}{m+x^*} + \frac{\alpha x^* y^*}{(m+x^*)^2},$$
$$D_{12} = -\frac{\alpha x^*}{m+x^*},$$
$$D_{21} = \frac{\alpha_1 y^*}{m+x^*} - \frac{\alpha_1 x^* y^*}{(m+x^*)^2},$$
$$D_{22} = \frac{\alpha_1 x^*}{m+x^*} - \mu - \frac{\beta z^*}{nA+m+y^*} + \frac{\beta y^* z^*}{(nA+m+y^*)^2},$$
$$D_{23} = -\frac{\beta y^*}{nA+m+y^*},$$
$$D_{32} = -\frac{\beta_1 z^*(A-z^*)}{(nA+y^*)^2},$$
$$D_{33} = \beta_1\left(\rho + \frac{A-z^*}{nA+y^*}\right) - \frac{\beta_1 z^*}{nA+y^*}.$$

This results in the characteristic equation, namely:

$$\lambda^3 + h_1\lambda^2 + h_2\lambda + h_3 = 0 \tag{4}$$

where

$$h_1 = -(D_{11} + D_{22} + D_{33}),$$
$$h_2 = D_{11}D_{22} + D_{11}D_{33} + D_{22}D_{33} - D_{23}D_{32} - D_{12}D_{21},$$
$$h_3 = -D_{11}D_{22}D_{33} + D_{11}D_{23}D_{32} + D_{12}D_{21}D_{33}.$$

Based on the Routh-Hurwitz criterion, the equilibrium point E_6 (x^*, y^*, z^*) is asymptotically stable if $h_1 > 0$, $h_3 > 0$, and $h_1h_2 > h_3$. □

Table 2. Parameter Values

Parameter	Values	Source
r	0.6	[23]
K	10	[23]
α	5	[20]
m	0.5	[23]
α_1	4.6	[20]
μ	0.4	[20]
β	2.1014232 3.197 5	Assumption
n	0.28	[21]
A	0.5	[21]
β_1	0.5322	[21]
ρ	2.12	[21]

3.4. Numerical Simulation

Numerical simulations are used to show changes in dynamic behavior around the equilibrium point. Population growth among prey, first level predators, and second level predators can be shown by the following phase portrait that describes the population dynamics with the support of *Pyton software*. The following are the values of the parameters to calculate the simulation.

For the numerical simulation process, there are three stages, namely the first simulation using the parameter value of the maximum predation rate of the second level predator with an assumption of $\beta = 2.1014232$, the second simulation assumed $\beta = 3.179$, and the third simulation assume $\beta = 5$. The numerical simulation results of each point are presented using a phase portrait with Python *software*.

The next simulation to determine changes in the system solution by performing numerical continuation is shown in the bifurcation diagram using MatCont. Numerical continuation is applied to the maximum predation rate parameter of the second-level predator (β).

a. Numerical simulation through phase portraits
Numerical simulation based on the parameters in Table 2 with $\beta = 2.1014232$, illustrated through a phase portrait using Python *software* is below.

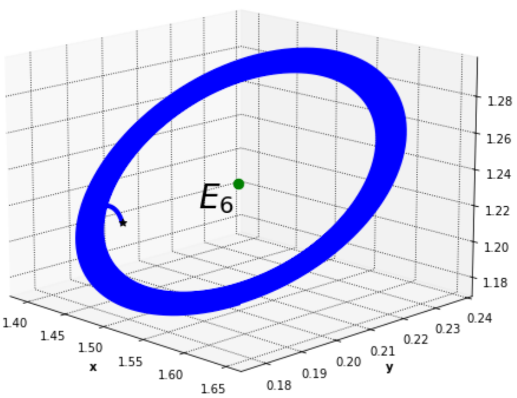
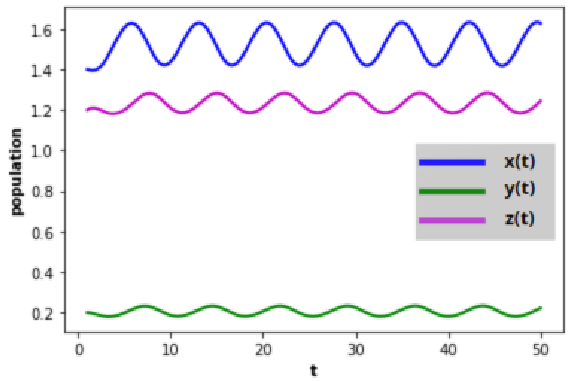
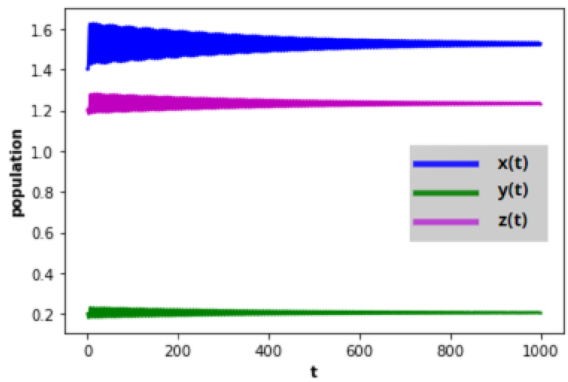


Figure 1. Phase Portrait $\beta = 2.1014232$

Figure 1 with $\beta = 2.1014232$ there are six points of exist, namely E_1, E_2, E_3, E_4, E_5, and E_6 with the equilibrium point E_6 is stable. Shows the phase portrait with initial value [1.4,0.2,1.2] to the point $E_6(1.52433, 0.20589, 1.23328)$ so it can be concluded that the stability of the equilibrium point $E_6(1.52433, 0.20589, 1.23328)$ in the form of a spiral, asymptotically stable. The stability at the point $E_6(1.52433, 0.20589, 1.23328)$ shows that the population of zooplankton, anchovy, and skipjack exist or can survive together. The following graph will show time series with initial value [1.4,0.2,1.2] with the corresponding parameters in Table 2 and at the time of maximum predation rate of second-level predators $\beta = 2.1014232$.



(a) Timeseries with $t = 50$



(b) Timeseries with $t = 1000$

Figure 2. Timeseries with Parameter Value $\beta = 2.1014232$ and Initial Value [1.4,0.2,1.2]

Figure 2 shows when the predation rate carried out by Skipjack Fish is 2.1014232 species with an initial value of population [1.4,0.2,1.2] will be stable towards the equilibrium point $E_6(1.52433, 0.20589, 1.23328)$ which is shown in the Figure 2b. With an initial value of 1.4 zooplankton towards the equilibrium point $x = 1.52433$. Anchovy population with an initial value of 0.2 species to the equilibrium point $y = 0.20589$. With an initial value of 1.2 species, skipjack also goes to the equilibrium point $z = 1.23328$. Based on time series pada Figure 2 the maximum predation rate by skipjack is 2.1014232 with an initial population value of [1.4,0.2,1.2], indicating that zooplankton, anchovy, and skipjack populations can coexist.

Numerical simulations based on the parameters in Table 2 with $\beta = 3.197$, illustrated through phase portraits using Python *software* is below. Figure 3 with

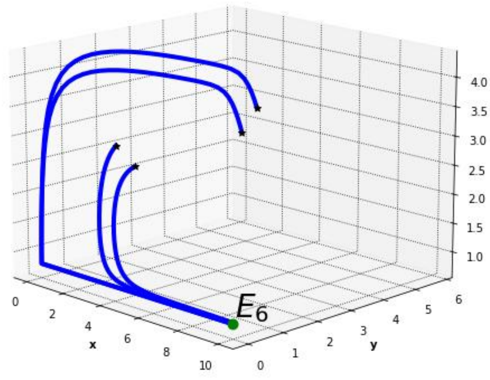


Figure 3. Phase Portrait $\beta = 3.197$

$\beta = 3.197$ there are six points of exist, namely E_1, E_2, E_3, E_4, E_5, and E_6 with the equilibrium point E_6 is stable. Shows the phase portrait with initial value [2.6,5,3] to the point $E_6(9.99879, 0.00015, 0.79712)$ so it can be concluded that the stability of the equilibrium point $E_6(9.99879, 0.00015, 0.79712)$ in the form of a node, asymptotically stable. The stability at the point $E_6(9.998797031, 0.0001515567043, 0.7971213002)$ shows that the population of zooplankton, anchovy, and skipjack exist or can survive together. The following graph will show time series with initial value [2.6,5,3] with the corresponding parameters in Table 2 and at the time of maximum predation rate of second-level predators $\beta = 3.197$.

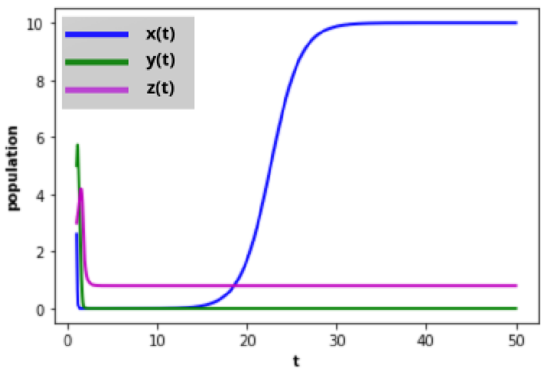


Figure 4. Timeseries with Parameter $\beta = 3.197$ and Initial Vale [2.6,5,3]

Figure 4 shows when the predation rate carried out by Skipjack Fish is 3,197 species with an initial value of population [2.6,5,3] will be stable towards the equilibrium point $E_6(9.99879, 0.00015, 0.79712)$. With an initial value of 2.6 zooplankton towards the equilibrium point $x = 9.99879$. anchovy population with an initial value of 5 species to the equilibrium point $y = 0.00015$. With an initial value of 3 species, skipjack also goes to the equilibrium point $z = 0.79712$. Based on time series in Figure 4 the maximum predation rate by skipjack is 3,197 with an initial population value of [2.6,5,3], indicating that zooplankton, anchovy and skipjack populations can coexist.

Numerical simulations based on the parameters in Table 2 with $\beta = 5$, illustrated through phase portraits using Python *software* is below.

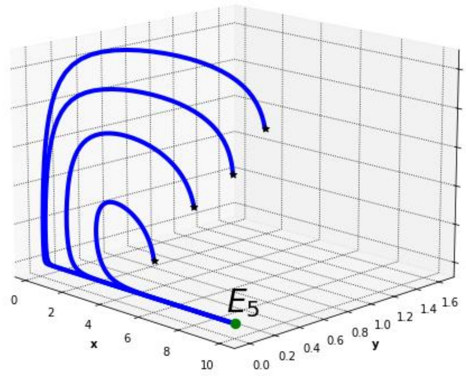


Figure 5. Phase Portrait $\beta = 5$

Figure 5 with $\beta = 5$ there are five points of exist, namely E_1, E_2, E_3, E_4, E_5 with the equilibrium point E_5 is stable. Shows the phase portrait with initial value [2.8, 0.5, 0.8] to the point $E_5(10, 0, 0.7968)$ so it can be concluded that the stability of the equilibrium point $E_5(10, 0, 0.7968)$ in the form of a node, asymptotically stable. The stability at the point $E_5(10, 0, 0.7968)$shown in the phase portrait means that the population of anchovy is heading towards extinction, while zooplankton and skipjack exist or can coexist.The following graph will show time series with initial value [2.8, 0.5, 0.8] with the corresponding parameters in Table 2 and at the time of maximum predation rate of second-level predators $\beta = 5$

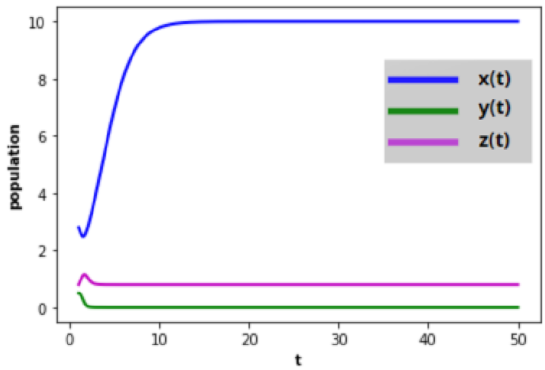


Figure 6. Timeseries with Parameter $\beta = 5$ and Initial Value [2.8, 0.5, 0.8]

Figure 6 shows that when the predation rate carried out by Skipjack Fish is 5 species with an initial population value of [2.8, 0.5, 0.8], it will stabilize towards the equilibrium point $E_5(10, 0, 0.7968)$. With an initial zooplankton value of 2.8 species to the equilibrium point $x = 10$. Anchovy population with an initial value of 0.5 species to the equilibrium point $y = 0$ or extinction. With an initial value of 0.8 species, skipjack also goes to the equilibrium point $z = 0.7968$. Based on time series in Figure 6 shows that the zooplankton and skipjack populations have a larger population than the anchovy population which results in the anchovy population experiencing extinction. The maximum predation rate by skipjack is 5 with an initial population value of [2.8, 0.5, 0.8] indicating that the zooplankton and skipjack populations can coexist, while the anchovy population cannot coexist or become extinct.

b. Numerical Continuation of the Parameter β
Numerical continuation is applied to the eq. (1) system by operating on the value of β, which is the parameter of the maximum predation rate of the second-level predator. The result of numerical continuation on the parameter β causes a change in the stability of the equilibrium point E_5 and E_6 which is illustrated with Python *software* using MatCont in the form of a bifurcation diagram below

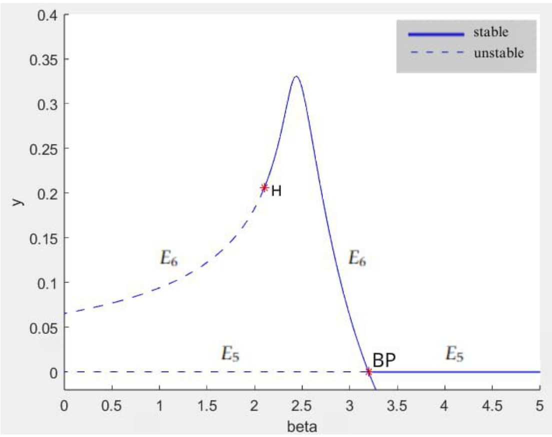


Figure 7. Bifurcation Diagram

The numerical continuation results on Figure 7 show the bifurcation diagram of the equilibrium point of the system due to a change in the value of β. Continuation begins when $\beta < 2.1014232$ shows that around the equilibrium points $E_1, E_2, E_3, E_4, E_5, E_6$ are unstable. Then when $\beta = 2.1014232$ Hopf occurs. The type of stability of E_6 changes from unstable to stable, after crossing a bifurcation point called Hopf bifurcation at $\beta = 2.1014232$. The numerical continuation results also match the simulation results shown in the phase portrait. After passing the bifurcation point, the β is moved forward to meet BP *(Branch Point)* at $\beta = 3.197$. The BP phenomena is called Transcritical bifurcation which is characterized by the crossing of two branches of equilibrium points in the bifurcation diagram, namely equilibrium points E_5 and E_6. This means that there is a change in the stability of the two equilibrium points after passing BP, namely the E_5 equilibrium point which was unstable becomes stable and the E_6 point which was stable becomes unstable when passing the maximum predation rate parameter value of the second-level predator $\beta = 3.197$.

4. Conclusion

The results of the dynamic analysis resulted in six equilibrium points, namely $E_1 = (0, 0, 0)$ represents the extinction of the three populations, $E_2 = (0, 0, A + \rho n A)$ represents the extinction of the first level prey and predator populations, $E_3 = (K, 0, 0)$ represent extinction in the first and second level predator populations, $E_4 = (\frac{\mu m}{\alpha_1 - \mu}, \frac{r m \alpha_1 (k\alpha_1 - K\mu - \mu m)}{(\alpha_1 - \mu)^2 \alpha K}, 0)$ represent extinction in the second level predator population, $E_5 = (K, 0, A(1 - \rho n))$ represents the extinction of the first level predator population, and $E_6 = (x^*, y^*, z^*)$ represents that the three populations did not go extinct. The results of the stability analysis around the equilibrium point E_1, E_2, E_3 are shown to be saddle unstable, then E_4, E_5, E_6 are asymptotically stable with certain conditions. The most interesting of this article is the numerical simulation carried out by taking the value of the maximum predation rate parameter in the second level predator (β). Numerical simulations were carried out with several values of the parameter β where the equilibrium point that was originally unstable became stable and otherwise. The change in the type of stability of the equilibrium point E_6 to stable is characterized by the Hopf bifurcation when $\beta = 2.1014232$, and the Transcritical bifurcation occurs when $\beta = 3.197$ it makes E_6 unstable but changes the point E_5 which was unstable to stable.

Author Contributions. **Salsabila, A.N.**: Conceptualization, methodology, formal analysis, investigation, resources, data curation, writing original draft preparation, writing review and editing. **Savitri, D.**: Conceptualization, methodology, software, validation, writing review and editing, visualization, supervision.

Acknowledgement. The authors are thankful the editors and reviewers who have supported us in improving this manuscript.

Funding. This research received no external funding.

Conflict of interest. The authors declare no conflict of interest.

Data availability. The data supporting this research are available in the main references and several parameter values are cited from some references. See Table 2.

References

[1] S. W. Utomo, I. Sutriyono, and R. Rizal, "*Pengertian, ruang lingkup ekologi dan ekosistem*," Jakarta: Universitas Terbuka, 2012.
[2] R. L. Smith *et al.*, "Elements of ecology," 1998.
[3] D. Savitri, N. W. Hidajati, and H. S. Panigoro, "Implementasi algoritma genetika dalam mengestimasi kepadatan populasi jackrabbit dan coyote," *Jambura Journal of Biomathematics (JJBM)*, vol. 3, no. 1, pp. 23–28, 2022. DOI:10.34312/jjbm.v3i1.11935
[4] A. L. Firdiansyah and N. Nurhidayati, "Dynamics in two competing predators-one prey system with two types of holling and fear effect," *Jambura Journal of Biomathematics (JJBM)*, vol. 2, no. 2, pp. 58–67, 2021. DOI:10.34312/jjbm.v2i2.11264
[5] K. Das *et al.*, "Bifurcation analysis of phytoplankton-fish model through parametric control by fish mortality rate and food transfer efficiency," *Jambura Journal of Biomathematics (JJBM)*, vol. 4, no. 2, pp. 146–154, 2023. DOI:10.37905/jjbm.v4i2.21480
[6] D. Mukherjee, "Stability and bifurcation of a two competing prey-one predator system with anti-predator behavior," *Jambura Journal of Biomathematics (JJBM)*, vol. 3, no. 1, pp. 1–11, 2022. DOI:10.34312/jjbm.v3i1.13820

[7] H. M. Ulfa, A. Suryanto, and I. Darti, "Dynamics of leslie-gower predator-prey model with additional food for predators," *International Journal of Pure and Applied Mathematics*, vol. 115, no. 2, pp. 751–765, 2017. DOI:10.12732/ijpam.v115i2.1

[8] H. S. Panigoro *et al.*, "The influence of additive allee effect and periodic harvesting to the dynamics of leslie-gower predator-prey model," *Jambura Journal of Mathematics*, vol. 2, no. 2, pp. 87–96, 2020. DOI:10.34312/jjom.v2i2.4566

[9] J. T. Tanner, "The stability and the intrinsic growth rates of prey and predator populations," *Ecology*, vol. 56, no. 4, pp. 855–867, 1975. DOI:10.2307/1936296

[10] R. Gupta, "Dynamics of a Holling–Tanner model," *American Journal of Engineering Research (AJER)*, vol. 6, no. 4, pp. 132–140, 2017.

[11] M. Banerjee and S. Banerjee, "Turing instabilities and spatio-temporal chaos in ratio-dependent Holling–Tanner model," *Mathematical biosciences*, vol. 236, no. 1, pp. 64–76, 2012. DOI:10.1016/j.mbs.2011.12.005

[12] D. Savitri, "Numerical study of one prey-two predator model considering food addition and anti-predator defense," in *E3S Web of Conferences*, vol. 328, no. 1, p. 06003, 2021. DOI:10.1051/e3sconf/202132806003

[13] N. A. Campbell, J. B. Reece, and L. G. Mitchell, *BIOLOGI: Edisi Kelima Jilid 3*. Jakarta: Erlangga, 2004.

[14] G. Nikolsky, "*The ecology of fishes*." London: Academic Press, 1963.

[15] P. Srinivasu, B. Prasad, and M. Venkatesulu, "Biological control through provision of additional food to predators: a theoretical study," *Theoretical Population Biology*, vol. 72, no. 1, pp. 111–120, 2007. DOI:10.1016/j.tpb.2007.03.011

[16] W. Mahmudah and M. Rifai, "Analisis kestabilan model predator-prey dengan penambahan makanan alternatif dan fungsi respon Holling tipe III," *Jurnal Ilmiah Matematika dan Pendidikan Matematika*, vol. 10, no. 2, pp. 133–146, 2020. DOI:10.36456/buanamatematika.v10i2.2728

[17] A. Mufidah and D. Savitri, "Analisis kestabilan model mangsa pemangsa dengan makanan tambahan pada pemangsa menggunakan fungsi respon holling tipe iv," *Jurnal Riset dan Aplikasi Matematika (JRAM)*, vol. 7, no. 1, pp. 80–94, 2023. DOI:10.26740/jram.v7n1.p80-94

[18] A. Mallawa, F. Amir, and M. Zainuddin, "Keragaan biologi populasi ikan cakalang (Katsuwonus pelamis) yang tertangkap dengan purse seine pada musim timur di perairan Laut Flores," *Jurnal IPTEKS Pemanfaatan Sumberdaya Perikanan*, vol. 1, no. 2, pp. 129–145, 2014.

[19] W. Susaniati, A. Mallawa, and F. Amir, "Kebiasaan makan ikan cakalang di Perairan laut Flores Sulawesi Selatan," *Agrokompleks*, vol. 19, no. 1, pp. 51–57, 2019.

[20] B. Sahoo, "Predator-prey system with seasonally varying additional food to predators," *International Journal of Basic and Applied Sciences*, vol. 1, no. 4, pp. 363–373, 2012. DOI:10.14419/ijbas.v1i4.205

[21] A. Basheer, E. Quansah, and R. D. Parshad, "The effect of additional food in Holling Tanner type models," *International Journal of Dynamics and Control*, vol. 7, no. 4, pp. 1195–1212, 2019. DOI:10.1007/s40435-019-00580-3

[22] W. E. Boyce, R. C. DiPrima, and D. B. Meade, *Elementary differential equations*. John Wiley & Sons, 2017.

[23] D. Savitri and H. S. Panigoro, "Bifurkasi hopf pada model prey-predator-super predator dengan fungsi respon yang berbeda," *Jambura Journal of Biomathematics (JJBM)*, vol. 1, no. 2, pp. 65–70, 2020. DOI:10.34312/jjbm.v1i2.8399