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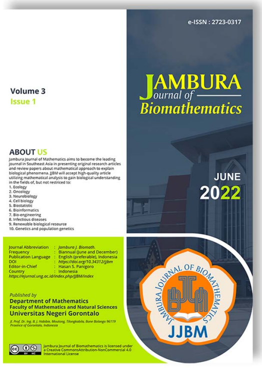

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Research Article

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A Fractional Mathematical Model for Controlling and Understanding Transmission Dynamics in Computer Virus Management Systems

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ABSTRACT. *The constant danger of computer viruses and malware makes it difficult to safely simulate the management of computer systems over time for both networks and individual users. The present study proposes a novel six-compartment fractional model that builds on existing classical frameworks and examines the existence and uniqueness of its solution, indicating that it is both mathematically and biologically well-posed. Additionally, we compute the fundamental reproduction number R_0 and use sensitivity analysis to investigate the impact of various factors on the model's behavior. The Laplace Adomian Decomposition Method is employed for numerical analysis, and its findings have the potential to transform computer security and network management by providing robust countermeasures and eradication tactics. The complex properties of the fractional-order model are further explored by examining the memory effect of fractional order on system dynamics. The research findings offer valuable insights for virus managers in developing and implementing effective management methods and can successfully prevent the spread of computer viruses by leveraging these discoveries. In conclusion, this study provides significant insights and solutions for protecting the integrity of digital domains and network infrastructure.*

1. Introduction

Malicious programs known as computer viruses can ruin files, harm systems, and steal data. They pose serious risks, including data loss and personal information theft, and proliferate undetected. Improved virus prevention and response tactics are made possible by mathematical models. Mathematical modeling needs in understanding of diseases that are transmissible. The dynamics of computer viruses, which pose a threat to networks and are essential for maintaining network security, are modeled using the Caputo fractional-order derivative. An examination of models such as FSSIP for virus mitigation in networks is conducted, and analysis using the Laplace-Adomian decomposition approach emphasizes the significance of preventive measures. [1–7], To understand these intricate dynamics, we delve into fractional calculus, a mathematical field exploring non-integer-order derivatives and integrals. Its origins date back to Leibniz's 1695 letter to L'Hospital, introducing semi-derivatives. In today's era of scientific and technological progress, information science gains increasing significance. Fractional calculus provides a valuable tool for analyzing and solving fractional differential equations, extending traditional calculus to non-integer orders [8–11].

Researchers emphasized the importance of human connections during epidemics and conducted stability analysis using the Laplace-Adomian decomposition method and the homotopy perturbation method [12–21]. A study established the validity of a nine-compartment model for coronavirus infection through robust epidemiological analysis [22], to describe coronavirus dynamics, the fractional-order Caputo's derivative model was approximated using Laplace-Adomian decomposition, and the basic reproduction threshold (R_0) was calculated using the next-generation matrix approach. The impact of smoking was addressed separately. Numerical simulations were meticulously conducted to gain insight into model parameters [23–26]. The World Health Organization's declaration of COVID-19 as a global public health emergency in March 2020 prompted heightened research efforts. This study comprehensively analyzes an SEIRV epidemic model, incorporating optimal control techniques and employing the Caputo fractional derivative of order $(0, 1]$. The stability of the SEIRV model is thoroughly examined, and an optimal control strategy is delineated. Real-time statistics from India ground the parameters of the fractional order SEIRV model in COVID-19 scenarios. The model's numerical solution was achieved using the Adam-Bash-Forward-Moulton predictor-corrector method. Findings indicate superior performance of the fractional-order SEIRV model compared to its integral-order counterpart in assessing COVID-19 transmission dynamics. The initial model undergoes stability analysis of equilibrium points, while the second model, incorporating a conformable fractional derivative, is discretized and scrutinized for stability and poten-

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tial Neimark-Sacker bifurcation, considering a variable parameter. Comparative analysis reveals chaotic tendencies in the discrete, conformable fractional order model, supported by numerical simulations [27–29]. In another study, a nonlinear fractional model is addressed via the reduced differential transform method, utilizing the Caputo fractional derivative for precise results. The model is visually represented through a signal flow diagram, and simulation is carried out using stimulation within MATLAB [30–32].

Various approaches exist for solving linear and nonlinear differential equations, often involving complex symbolic calculations or significant computational resources. Precise solutions for these models pose formidable challenges. In referenced research, the fractional optimal control problem concerning variational inequalities is explored, employing the Riemann-Liouville fractional time derivative and investigating the existence and uniqueness of fractional differential variation inequalities within a Sobolev space [7, 33–35]. It demonstrates solutions with both Dirichlet and Neumann boundary conditions, all within a confined region. The establishment of optimality conditions for the quadratic performance function of the fractional Cauchy problem is a focal point. These conditions encompass the adjoint problem as well as the first-order optimality test for the Euler-Lagrange equation [36].

Furthermore, another study delves into the ramifications of the Caputo fractional order concerning the KDV equation, accentuating its utility within the framework of wave modeling, as demonstrated through the homotopy perturbation approach [34, 37].

In the specific context of an epidemic model denoted as SAEIQRS (Susceptible-Antidotal-Exposed-Infected-Quarantined-Recovered-Susceptible), the differential transformation technique (DTM) is employed. A comparison of solutions between the Differential Transform Method (DTM) and the fourth-order Runge-Kutta technique (RK4) illustrates their concurrence [7]. The study investigates the estimation of glucose supply in the human bloodstream using mathematical modeling that incorporates the incomplete I-function (IIF) [38]. The results reveal significant scenarios across parameters, with important applications in biology and medicine. In a fractional blood alcohol model, [39] investigates the Caputo and modified Atangana-Baleanu derivatives (MABC), paying particular attention to liver metabolism, absorption rates, and stomach alcohol content. The link between blood and stomach alcohol concentrations is revealed by analytical data obtained using the Laplace transform and Mittag-Leffler methods. The influence of fractional factors is emphasized by graphic analyses, which provide fresh perspectives with possible medical uses [40]. One of the main causes of death worldwide, particularly for those over 65 and children under five, is pneumonia. Stability, reproduction numbers, and equilibrium points are the main topics of this study's analysis of disease dynamics utilizing an SVEIR model. MATLAB21 simulations demonstrate that the disease may be eradicated by raising vaccination rates over a certain threshold. Uses the Caputo fractional derivative to investigate a novel fractional TB model [41]. Two treatment strategies are considered while computing solutions using the generalized Euler's method (GEM): primary therapy for infected individuals and protective treatment for latent populations. The six-dimensional compartmental model includes the susceptible, latent, infected, recovered, and treatment classes. The stability of the equilibrium point is examined, and MATLAB 22 is used for graphical simulations, providing precise and straightforward insights into the model's dynamics [42]. Pneumonia causes over 2,000,000 deaths annually, mainly in children under five and the elderly in developing nations. The study uses a generalized SVEIR model with Caputo fractional derivatives to analyze pneumonia dynamics. Solutions are computed using the generalized Euler's method, and stability analysis reveals that increased vaccination coverage can eliminate the disease. Sensitivity analysis highlights transmission rates and progression to contagiousness as key factors, stressing the need to improve treatment effectiveness to prevent disease spread [43]. The study explores the application of fractional operators and special functions to understand physical processes. It introduces a fractional integral operator with an I-function in its kernel, used to solve several fractional differential equations (FDEs). These equations model various physical phenomena across fields such as physics, biology, engineering, and chemistry. The study establishes key relations involving the new fractional operator with incomplete I-function, classical Riemann Liouville operators, Hilfer fractional derivatives, and the generalized composite fractional derivative (GCFD) operator, followed by the identification and analysis of several exceptional cases [44]. The study explores the use of non-integer order derivatives in modeling contagious diseases, specifically applying a fractional model to dengue fever. The Hilfer fractional model is used to analyze epidemic dynamics. The study employs the Laplace homotopy analysis transform method (LHATM) for numerical analysis, incorporating homotopy analysis and Laplace transforms. The uniqueness and convergence of the solution are also considered. MATLAB21a is used for numerical simulations, comparing results for both integer and non-integer orders within the interval $(0, 1)$.

Computer viruses pose a common hazard to information security, and examining the fractional-order model of a computer virus outbreak through various approaches can yield valuable information. The Laplace-Adomian decomposition technique (LADM) is utilized to derive an analytical solution, facilitating an investigation into the influence of Caputo fractional order derivatives on various factors. Analyzing virus transmission dynamics is crucial for countermeasures, and this study uses fractional-order derivatives to enhance understanding. Fractional calculus accurately captures virus-spreading properties, offering insights into propagation rates and the role of network structure. Future research can explore evolving viruses, real-world data, and the interplay with other factors, ultimately improving security and resilience against cyber threats.

In order to better understand the dynamics of computer viruses, this study proposes a new six-compartment fractional model that takes memory effects into account. Through thorough validation, it guarantees mathematical and biological robustness and offers useful information through sensitivity analysis and reproduction number computation. It provides a revolutionary method for protecting digital systems and network infrastructure by utilizing the Laplace Adomian Decomposition Method to facilitate the creation of effective countermeasures and eradication strategies.

2. Preliminaries

Definition 1. The fractional integration of order α is defined as

$$(D_{t_0}^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int (t-s)^{\alpha-1} f(s)ds, \ \alpha \geq 0, \ t \geq t_0,$$
$$(D_{t_0}^{\alpha} f)(t) = f(t).$$

Definition 2. Gamma function $\Gamma(p)$ is defined as

$$\Gamma(p) = \int_0^{\infty} \ell^{-x} x^{p-1} dx.$$

Definition 3. The fractional derivative of order α and $n = [\alpha]$ the Riemann-Liouville fractional time derivative of order α can be defined as:

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^t (t-u)^{n-\alpha-1} f(u)du.$$

Also non-integer time Fractional derivative in the origin is defined as:

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-u)^{n-\alpha-1} f(u)du,$$

for a function $f(t)$, $f(t) = 0$, if $t \geq 0$. Where $[\alpha] = n$ and c is constant, then $D_t^{\alpha} c = 0$.

Theorem 1. *If $n-1 < \alpha < n$ where $n \in N$ and $\alpha \in R$ then*

$$\lim_{\alpha \to n} D_t^{\alpha} f(t) = f^{(n)}(t),$$
$$\lim_{\alpha \to n-1} D_t^{\alpha} f(t) = f^{(n-1)}(t) - f^{(n-1)}(0).$$

Proof. By the formula

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(y)dy}{(t-y)^{\alpha+1-n}}.$$

We will use the integration by part, obtain

$$\int_0^t u(y)v^{'}(y)dy = u(y)v(y)_0^t - \int_0^t u^{'}(y)v(y)dy,$$
$$u(y) = f^{(n)}(y),$$
$$v(y) = -(t-y)^{n-\alpha},$$
$$u^{'}(y) = f^{(n+1)},$$
$$v^{'}(y) = (t-u)^{n-\alpha-1}.$$

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left[\frac{f^{(n)}(y)(t-y)^{n-\alpha}}{n-\alpha} \Bigg|_0^t + \frac{1}{n-\alpha} \int_o^t (t-y)^{n-\alpha} f^{(n+1)}(y)dy \right].$$

Using the property of Γ Function:

$$\Gamma(n-\alpha+1) = (n-\alpha)\Gamma(n-\alpha),$$
$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left[f^{(n)}(0) + \int_0^t f^{(n+1)}(y)(t-y)^{n-\alpha} dy \right],$$
$$\lim_{\alpha \to n} D_t^{\alpha} f(t) = \left[f^{(n)}(0) + \int_0^t f^{(n+1)}(y)(t-y)^{n-\alpha} dy \right],$$
$$= f^{(n)}(0) + f^{(n)}(y)_0^t,$$
$$= f^{(n)}(t).$$
$$\lim_{\alpha \to 1} D_t^{\alpha} f(t) = \left[f^{(n)}(0) + \int_0^t f^{(n+1)}(y)(t-y)^{n-\alpha} dy \right],$$
$$= f^{(n)}(0) + (t-y) f^{(n)}(y)_0^t,$$
$$= f^{(n-1)}(t) - f^{(n-1)}(0).$$

□

Definition 4. The Laplace Transform of a function is defined as

$$F(s) = \int_0^{\infty} \ell^{-st} f(t)dt.$$

The corresponding inverse Laplace transform is defined as

$$f(t) = \frac{1}{2\pi} \lim_{t \to \infty} \int_{-\infty}^{\infty} \ell^{-st} F(s)dt = L^{-1}F(s).$$

Theorem 2. $f(t) = t\lambda^{\infty}$.

Proof. By definition of Laplace transform

$$F(s) = \int_0^{\infty} \ell^{-st} t^{\lambda} dt.$$

We introduce the change of variable $x = ts$, we have also $dx = sdt$.

$$F(s) = \int_0^{\infty} \ell^{-x} \frac{x^{\lambda}}{s^{\lambda}} \frac{dx}{s},$$
$$\Rightarrow F(s) = \frac{1}{S^{\lambda+1}} \int_0^{\infty} \ell^{-x} x^{\lambda} dx,$$
$$= \frac{\Gamma(\lambda+1)}{s^{\lambda+1}}.$$

The direct and inverse Laplace transform are

$$L(t^{\lambda}) = \frac{\Gamma(\lambda+1)}{s^{\lambda+1}},$$
$$L^{-1}\left(\frac{1}{s^{\lambda+1}}\right) = \frac{t^{\lambda}}{\Gamma(\lambda+1)}.$$

□

Definition 5. If$\alpha > 0$,

$$f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-y)^{\alpha-1} f(y) dy.$$

Laplace transform of Riemann -Liouville is

$$L[D_t^{\alpha} f(t)] = L\left[\frac{1}{\Gamma(n-\alpha)}\left(\frac{dt^n}{d^n t}\right)\int_0^t (t-u)^{n-\alpha-1} f(u) du\right],$$
$$L\left(\frac{dt^n}{d^n t}\right) t^{n-a} f(t),$$
$$L[f^n(t)] = s^n F(s) - s^{n-1} f^{'}(0) \cdots f^{(n-1)}(0).$$

Definition 6. Adomian polynomials, we will denote these polynomial by A_0, $A_1, \ldots, A_n$, the Adomian method consists in the decomposition the unknown function $y(t)$in a series of the form $y(t) = y_0 + y_1 + y_2 + \cdots + y_n$, where y_n can be expressed in term of Adomian polynomials A_n. The Adomian polynomial are defined

$$A_n = \frac{1}{n} \frac{d^n}{d\lambda^n} \left[G(t) \sum_{j=0}^{n} y_j \lambda^j\right]_{\lambda=0}.$$

Definition 7. Let g(y) be a function defined on the interval $(0, t)$ and let $n - 1 < \alpha < n$ be a real number, where n is the smallest integer greater than α. The Caputo fractional derivative of order α is given by:

$$D^m g(y) = \frac{1}{\Gamma(n-1)} \int_0^t (y-t)^{m-\alpha-1} g^{(m)}(t) dt.$$

The Caputo fractional derivative has several important properties, including:

1. Linearity: for functions
$$D^{\alpha}(af(y) + bg(y) = aD^{\alpha} f(y) + bD^{\alpha} g(y) f(x),$$
where a and b are contains in Chain rule: If $D^{\alpha} f(g(x))$ is differentiable and $D^{\alpha} f(x)$ is Caputo differentiable, then
$$D^{\alpha}[f(g(x))] = D^{\alpha}[f(x)] \cdot g'(x).$$
2. Initial conditions: for $n - 1 < \alpha < n$ the Caputo fractional derivative satisfies $\Gamma(n-\alpha+1)\Gamma(n+1) \times n - \alpha$.
3. Caputo fractional integration: the Caputo fractional derivative is related to fractional integration through the relationship
$$D^{\alpha} \int_p^{\alpha} (y-t)^{n-\alpha-1} \rho^{(n)}(t) dt = f(y).$$

3. Methods

3.1. Model formulations

A nonlinear classical-order mathematical model was developed by [7], and it has now been modified to a fractional-order version. This reformulation aims to capture the memory effect and offer a deeper understanding of the concept of computer viruses.

$$\begin{aligned}
\frac{d^{\alpha} S(t)}{dt^{\alpha}} &= \pi - \beta S(t)I(t) - (\psi + \mu)S(t) + \eta R(t),\\
\frac{d^{\alpha} A(t)}{dt^{\alpha}} &= \psi S(t) - (\mu + \phi)A(t) - \rho A(t)I(t),\\
\frac{d^{\alpha} E(t)}{dt^{\alpha}} &= \beta S(t)I(t) - (\mu + \gamma)E(t) + \rho A(t)I(t),\\
\frac{d^{\alpha} I(t)}{dt^{\alpha}} &= \gamma E(t) - (\mu + \theta + \sigma + \omega)I(t),\\
\frac{d^{\alpha} Q(t)}{dt^{\alpha}} &= \omega I(t) - (\mu + \theta + \delta)Q(t),\\
\frac{d^{\alpha} R(t)}{dt^{\alpha}} &= \sigma I(t) + \delta Q(t) + \omega A(t) - (\mu + \eta)R(t).
\end{aligned} \tag{1}$$

Given the initial condition: $S_0 = n_1$, $A_0 = n_2$, $E_0 = n_3$, $I_0 = n_4$, $Q_0 = n_5$, $R_0 = n_6$.

The fractional order derivative model of eq. (1) can be transformed into a Caputo derivative, as shown in eq. (2). The variables and parameters of the model are described in Table 1.

$$\begin{aligned}
{}^c D^{\alpha_1} S(t) &= \pi - \beta S(t)I(t) - (\mu + \psi)S(t) + \eta R(t),\\
{}^c D^{\alpha_2} A(t) &= \psi S(t) - (\phi + \mu)A(t) - \rho A(t)I(t),\\
{}^c D^{\alpha_3} E(t) &= \beta S(t)I(t) - (\gamma + \mu)E(t) + \rho A(t)I(t),\\
{}^c D^{\alpha_4} I(t) &= \gamma E(t) - (\mu + \omega + \sigma + \theta)I(t),\\
{}^c D^{\alpha_5} Q(t) &= \omega I(t) - (\mu + \theta + \delta)Q(t),\\
{}^c D^{\alpha_6} R(t) &= \sigma I(t) + \delta Q(t) - (\eta + \mu)R(t) + \omega A(t),\\
{}^c D^{\alpha} 0 &\le \alpha \le 1.
\end{aligned} \tag{2}$$

Signifies Caputo's fractional-order derivative, while α indicates the fractional time derivative.

3.2. Determine the solution's existence and uniqueness

Theorem 3. *Let*

$$\begin{array}{lcll}
v_1^{'} & = & w_1(v_1, v_2, v_3, \ldots, v_n, t), v_1(t_0) & = & v_1 0\\
v_2^{'} & = & w_2(v_1, v_2, v_3, \ldots, v_n, t), v_2(t_0) & = & v_2 0\\
v_3^{'} & = & w_3(v_1, v_2, v_3, \ldots, v_n, t), v_3(t_0) & = & v_3 0\\
\vdots & & & & \\
v_n^{'} & = & w_n(v_1, v_2, v_3, \ldots, v_n, t), v_n(t_0) & = & v_n 0
\end{array}$$

Suppose U represents a region in $(n+1)$ dimensional space (with one dimension) for t and n dimensions for vector x. When the partial derivative $\frac{\partial w}{\partial v_i}$, where i are continuous within U,

$$U = \{(v, t) \ : \ |t - t_0| \le a, \ |v - v_0| \le b\},$$

then there constant $\delta \ge 0$ Such that there exist a unique continuous vector solution $v = [v_1(t), v_2(t), v_3(t), \ldots, v_n(t)]$ in the

Table 1. Parameters description use in model (1)

Symbol	Description
$N(t)$	The total number of population.
$S(t)$	The quantity of computers susceptible at a specified moment, exhibiting neither immunity nor infection.
$A(t)$	The aggregate count of computers with antidotal capabilities at a specific point in time, whether they have been updated or not.
$E(t)$	The count of vulnerable computers that are susceptible to infection at any given point in time.
$I(t)$	The count of computers that require simultaneous cleanup due to being infected.
$Q(t)$	The number of quarantined infected machines at any given time.
$R(t)$	Transient immunity for uninfected computers at a given time.
π	The frequency with which new computers are added to the network (birth rate).
μ	The natural death rate (crashing of the computers due to other reason than attack of virus).
θ	The crashing rate of computers due to the attack of virus.
B	Transmission rate of virus attack when susceptible computer contact.
B	The rate at which susceptible computers begin the antidotal process.
ϕ	The speed at which a virus infiltrates when antidotal computers make contact with infected computers prior to receiving the latest update (A to E).
ρ	The pace at which antidotal computers achieve recovery (A to R).
σ, ω	Are the coefficient rate of transition from the infectious class to the recovered class (I to R) the same as the coefficient rate of transition from the infectious class to the quarantined class (I to Q).
δ, η	Do the rate coefficients of the quarantine class and the recovery class (q to r) differ from the coefficient rate of transition from the recovery class to the susceptible class (R to S).

interval $|t - t_0| \leq \delta$.

Proof.

$$\begin{aligned}
w_1 &= {}^cD^{\alpha_1}S(t) = \pi - \beta S(t)I(t) - (\mu + \psi)S(t) + \eta R(t),\\
w_2 &= {}^cD^{\alpha_1}A(t) = \psi S(t) - (\phi + \mu)A(t) - \rho A(t)I(t),\\
w_3 &= {}^cD^{\alpha_1}E(t) = \beta S(t)I(t) - (\mu + \gamma)E(t) + \rho A(t)I(t),\\
w_4 &= {}^cD^{\alpha_1}I(t) = \gamma E(t) - (\mu + \theta + \sigma + \omega)I(t),\\
w_5 &= {}^cD^{\alpha_1}Q(t) = \omega I(t) - (\mu + \omega + \delta)Q(t),\\
w_6 &= {}^cD^{\alpha_1}R(t) = \sigma I(t) + \delta Q(t) - (\eta + \mu)R(t) + \omega A(t),\\
U &= \{(S, A, E, I, Q, R) : |S - S_0| \leq a, |A - A_0| \leq b,\\
&\qquad |E - E_0| \leq c, |I - I_0| \leq d, |Q - Q_0| \leq e, |R - R_0| \leq f\}.
\end{aligned}$$

Applying partial differentiation yields the subsequent result, establishing the model's distinct solution:

$$w_1 = {}^cD^{\alpha_1}S(t) = \pi - \beta S(t)I(t) - (\mu + \psi)S(t) + \eta R(t),$$

$$\begin{aligned}
\frac{\partial w_1}{\partial S} &= |-(\mu + \psi)\beta|, & \frac{\partial w_1}{\partial A} &= 0,\\
\frac{\partial w_1}{\partial E} &= 0, & \frac{\partial w_1}{\partial I} &= |-\beta|,\\
\frac{\partial w_1}{\partial Q} &= 0, & \frac{\partial w_1}{\partial R} &= \eta.
\end{aligned}$$

Furthermore, by computing the partial derivative of the second function i.e.

$$w_2 = {}^cD^{\alpha_1}A(t) = \psi S(t) - (\mu + \phi)A(t) - \rho A(t)I(t),$$

to obtain the following:

$$\begin{aligned}
\frac{\partial w_2}{\partial S} &= \psi, & \frac{\partial w_2}{\partial A} &= |-(\mu + \phi)|,\\
\frac{\partial w_2}{\partial E} &= 0, & \frac{\partial w_2}{\partial I} &= |-\rho|,\\
\frac{\partial w_2}{\partial Q} &= 0, & \frac{\partial w_2}{\partial R} &= 0.
\end{aligned}$$

In a similar manner, deriving the partial derivative of the third function i.e.

$$w_3 = {}^cD^{\alpha_1}E(t) = \beta S(t)I(t) - (\gamma + \mu)E(t) + \rho A(t)I(t),$$

to obtain the following:

$$\begin{aligned}
\frac{\partial w_3}{\partial S} &= \beta, & \frac{\partial w_3}{\partial A} &= \phi_1,\\
\frac{\partial w_3}{\partial E} &= |-(\mu + \gamma)|, & \frac{\partial w_3}{\partial I} &= \beta + \phi_1,\\
\frac{\partial w_3}{\partial Q} &= 0, & \frac{\partial w_3}{\partial R} &= 0.
\end{aligned}$$

Calculating the partial derivative of the fourth function is similar i.e.

$$w_4 = {}^cD^{\alpha_1}I(t) = \gamma E(t) - (\mu + k_1)I(t) - \sigma_1 I(t) - \sigma_2 I(t),$$

to obtain the following;

$$\begin{aligned}
\frac{\partial w_4}{\partial S} &= 0, & \frac{\partial w_4}{\partial A} &= 0,\\
\frac{\partial w_4}{\partial E} &= \gamma, & \frac{\partial w_4}{\partial I} &= |-(\mu + \kappa_1) - \sigma_1 - \sigma_2|,\\
\frac{\partial w_4}{\partial Q} &= 0, & \frac{\partial w_4}{\partial R} &= 0.
\end{aligned}$$

Computing the partial derivative of the fifth function i.e.

$$w_5 = {}^cD^{\alpha_1}Q(t) = \sigma_2 I(t) - (\mu + k_1)Q(t) - \delta Q(t),$$

to obtain the following;

$$\begin{aligned}
\frac{\partial w_5}{\partial S} &= 0, & \frac{\partial w_5}{\partial A} &= 0,\\
\frac{\partial w_5}{\partial E} &= 0, & \frac{\partial w_5}{\partial I} &= \sigma_2,\\
\frac{\partial w_5}{\partial Q} &= |-(\mu + \kappa_1) - \delta|, & \frac{\partial w_5}{\partial R} &= 0.
\end{aligned}$$

The sixth function's partial derivative, or, ultimately, yields the following result;

$$w_6 = {}^cD^{\alpha_1}R(t) = \sigma_1 I(t) + \delta Q(t) - \mu R(t) + \phi_2 A(t) - \eta R(t),$$

$$\begin{aligned}\frac{\partial w_6}{\partial S} &= 0, & \frac{\partial w_6}{\partial A} &= 0,\\ \frac{\partial w_6}{\partial E} &= 0, & \frac{\partial w_6}{\partial I} &= \sigma_1,\\ \frac{\partial w_6}{\partial Q} &= \delta, & \frac{\partial w_6}{\partial R} &= \left|-\mu-\eta\right|.\end{aligned}$$

Theorem 1 implies that the function is continuous and limited due to its existence of the partial derivative. The model's uniqueness of the solution means that it is both mathematically and biologically well posed. □

3.3. *Basic reproduction number*

Basic reproduction number (R_0) in epidemiology measures the average number of secondary infections caused by one infected person. It predicts the potential for epidemics or pandemics: $R_0 < 1$ leads to declining infections, while $R_0 > 1$ leads to increasing infections. Factors affecting R_0 include transmission mode, infectious period, population susceptibility, contact rate, virulence, and environmental conditions. Mathematical models like the SIR model estimate R_o by considering these factors. R_o assumes a fully susceptible population but can be reduced by immunity from past infections or vaccinations (R_e). Public health interventions like social distancing, mask-wearing, and vaccinations aim to decrease R_o. Monitoring R_o helps guide disease control, resource allocation, and outbreak management to minimize the impact of infectious diseases. Top of Form

$$F = \begin{pmatrix} \mu & \beta & \psi \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$V = \begin{pmatrix} (\mu+\theta)+\sigma+\delta & 0 & 0 \\ \gamma & (\mu+\theta)+\phi & 0 \\ \sigma & \delta & (\mu+\eta) \end{pmatrix} R_0 = FV^{-1},$$

$$R_0 = \frac{\beta\gamma(\mu+\theta+\delta)\delta + ((\mu+\theta)+\sigma+\rho+\mu\gamma(\mu+\eta)+\psi\sigma(\mu+\eta)}{(\mu+\theta+\delta)(\mu+\theta)(\mu+\eta)}.$$

3.4. *Sensitivity analysis of R_0*

This analysis is performed to gain further insight into the transmission dynamics of the virus while considering the impact of each parameters of the model on R_0. The normalized sensitivity index given by $p_g^{R_0} = \frac{\partial R_0}{\partial g}\cdot\frac{g}{R_0}$ and the computed results are presented on Table 2.

Table 2. Sensitivity index of each parameter on R_0

Parameters	Sensitivity Indices
ρ	0.5789005232
β	0.07619778136
ε	0.3417607759
ψ	0.001013075916
μ	−0.1234818938
θ	−0.1320257419
ϕ	0
ω	0
σ	0.1013075916

Table 2 outlines the sensitivity of each parameter's impact on the basic reproduction number $\Re_0$. In Figure 1 below this highlights the need to address their roles by implementing factors potentially capable of lowering their value because positive sensitivity on R_0 on virus.

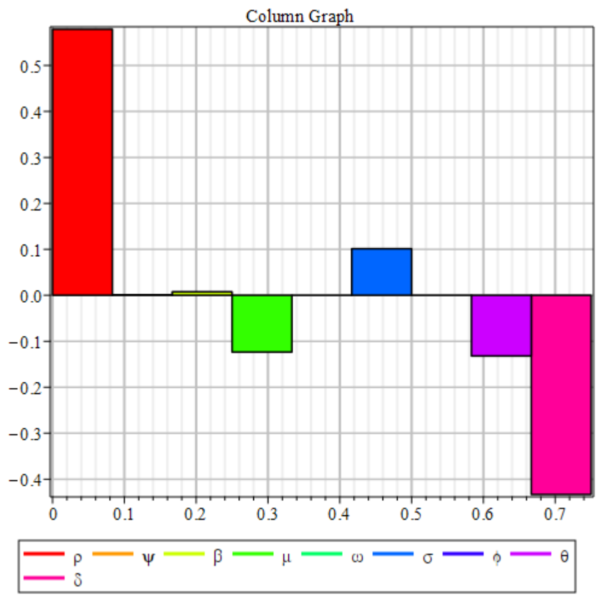


Figure 1. Sensitivity chart of essential parameters contained in $\Re_0$.

4. The Laplace Adomian Decomposition Method

Examine the fractional order epidemic model (2) under the initial condition, where the independence of each component is preserved, and they adhere to the specified relationship.

$$M(t) = S(t)+A(t)+E(t)+I(t)+Q(t)+R(t),$$

where N is the total population, the nonlinear term in the model is $S(t)I(t)$ and $A(t)I(t)$.

Applying Laplace transform to both side of the model (2), we have eq. (3)

$$\begin{aligned}
L\{{}^cD^{\alpha_1}S(t)\} &= L\{\pi-\mu s(t)-\beta S(t)I(t)-\mu S(t)+\eta R(t)\},\\
L\{{}^cD^{\alpha_2}A(t)\} &= L\{\psi S(t)-\mu A(t)-\phi_2 A(t)-\phi_1 A(t)I(t)\},\\
L\{{}^cD^{\alpha_3}E(t)\} &= L\{\beta S(t)I(t)-\mu I(t)-\gamma E(t)+\phi_1 A(t)I(t)\},\\
L\{{}^cD^{\alpha_4}I(t)\} &= L\{\gamma E(t)-(\mu+k_1)E(t)-\sigma_1 I(t)-\sigma_2 I(t)\},\\
L\{{}^cD^{\alpha_5}Q(t)\} &= L\{\sigma_2 I(t)-(\mu+k_1)Q(t)-\delta Q(t)\},\\
L\{{}^cD^{\alpha_6}R(t)\} &= L\{\sigma_1 I(t)+\delta Q(t)-\mu R(t)+\phi_2 A(t)-\eta R(t)\}.
\end{aligned} \tag{3}$$

From eq. (3), we obtain

$$\begin{aligned}
S^{\alpha_1}S(t) &= S^{\alpha_1-1}S(0)+L\{\pi-\beta S(t)I(t)-(\psi+\mu)S(t)+\eta R(t)\},\\
S^{\alpha_2}A(t) &= S^{\alpha_2-1}A(0)+L\{\psi S(t)-(\phi+\mu)A(t)-\rho A(t)A(t)\},\\
S^{\alpha_3}E(t) &= S^{\alpha_3-1}E(0)+L\{\beta S(t)I(t)-(\mu+\gamma)E(t)+\rho A(t)I(t)\},\\
S^{\alpha_4}I(t) &= S^{\alpha_4-1}I(0)+L\{\gamma E(t)-(\mu+\theta+\sigma+\omega)I(t)\},\\
S^{\alpha_5}Q(t) &= S^{\alpha_5-1}Q(0)+L\{\omega I(t)-(\mu+\theta+\delta)Q(t)\},\\
S^{\alpha_6}R(t) &= S^{\alpha_6-1}R(0)+L\{\sigma I(t)+\delta Q(t)+\delta A(t)-(\eta+\mu)R(t)\},
\end{aligned} \tag{4}$$

where eq. (4) implies

$$\begin{aligned}
S(t) &= S^{-1}S(0)+\frac{1}{s^{\alpha_1}}L\{\pi-\beta S(t)I(t)-(\psi+\mu)S(t)+\eta R(t)\},\\
A(t) &= S^{-1}A(0)+\frac{1}{s^{\alpha_2}}L\{\psi S(t)-(\phi+\mu)A(t)-\rho A(t)A(t)\},\\
E(t) &= S^{-1}E(0)+\frac{1}{s^{\alpha_3}}L\{\beta S(t)I(t)-(\mu+\gamma)E(t)+\rho A(t)I(t)\},
\end{aligned}$$

$$I(t) = S^{-1}I(0) + \frac{1}{s^{\alpha_4}}L\{\gamma E(t) - (\mu+\theta+\sigma+\omega)I(t)\},$$
$$Q(t) = S^{-1}Q(0) + \frac{1}{s^{\alpha_5}}L\{\omega I(t) - (\mu+\theta+\delta)Q(t)\},$$
$$R(t) = S^{-1}R(0) + \frac{1}{s^{\alpha_6}}L\{\sigma I(t) + \delta Q(t) + \delta A(t) - (\eta+\mu)R(t)\}. \quad (5)$$

Assuming that the solution $S(t)$, $A(t)$, $E(t)$, $I(t)$, $Q(t)$, $R(t)$ are in form of infinite series given by

$$\begin{array}{rclrcl} S(t) &=& \sum_{n=0}^{\infty} S_n, & A(t) &=& \sum_{n=0}^{\infty} A_n, \\ E(t) &=& \sum_{n=0}^{\infty} E_n, & I(t) &=& \sum_{n=0}^{\infty} I_n, \\ Q(t) &=& \sum_{n=0}^{\infty} Q_n, & R(t) &=& \sum_{n=0}^{\infty} R_n, \end{array} \quad (6)$$

and non linear term involved in the model $S(t)I(t)$, $A(t)I(t)$ are decomposed by Adomain.

$$S(t)I(t) = \sum_{n=0}^{\infty} X_n,$$
$$A(t)I(t) = \sum_{n=0}^{\infty} Y_n. \quad (7)$$

Equation (6) is Adomain polynomial are gives by

$$X_n = \frac{1}{\Gamma(n+1)}\frac{d^n}{dt}\left[\sum_{k=o}^{n}\lambda^k s_K \sum_{k=0}^{n}\lambda^n i_k\right]\lambda,$$
$$Y_n = \frac{1}{\Gamma(n+1)}\frac{d^n}{dt}\left[\sum_{k=o}^{n}\lambda^k a_K \sum_{k=0}^{n}\lambda^n i_k\right]\lambda. \quad (8)$$

Putting eqs. (6) to (8) into eq. (5), using initial value condition

$$S(t) = \frac{n_1}{s} + \frac{1}{s^{\alpha_1}}L\{\pi - \beta X_n - (\mu+\psi)S_n + \eta R_n\},$$
$$A(t) = \frac{n_2}{s} + \frac{1}{s^{\alpha_2}}L\{\psi S_n - (\phi+\mu)A_n - \rho Y_n\},$$
$$E(t) = \frac{n_3}{s} + \frac{1}{s^{\alpha_3}}L\{\beta X_n - (\mu+\gamma)E_n + \rho Y_n\},$$
$$I(t) = \frac{n_4}{s} + \frac{1}{s^{\alpha_4}}L\{\gamma E_n - (\mu+\theta+\sigma+\omega)I_n\}, \quad (9)$$
$$Q(t) = \frac{n_5}{s} + \frac{1}{s^{\alpha_5}}L\{\omega I_n - (\mu+\theta+\delta)Q\},$$
$$R(t) = \frac{n_6}{s} + \frac{1}{s^{\alpha_6}}L\{\sigma I_n + \delta I_n + \omega A_n - (\mu+\eta)R_n\}.$$

In order to obtain the solution for each compartment, a process of iterating through the terms in eq. (9) and subsequently performing a Laplace inverse leads to the derivation of a comprehensive formula for the model.

$$\sum_{n=0}^{k} S_{n+1}(t) = L^{-1}\left[\frac{1}{s^{\alpha_1}}L\{\pi - \beta X_n - (\mu+\psi)S_n + \eta R_n\}\right],$$
$$\sum_{n=0}^{k} A_{n+1}(t) = L^{-1}\left[\frac{1}{s^{\alpha_2}}L\{\psi S_n - (\phi+\mu)A_n - \rho Y_n\}\right],$$
$$\sum_{n=0}^{k} E_{n+1}(t) = L^{-1}\left[\frac{1}{s^{\alpha_3}}L\{\beta X_n - (\mu-\gamma)E_n + \rho Y_n\}\right],$$
$$\sum_{n=0}^{k} I_{n+1}(t) = L^{-1}\left[\frac{1}{s^{\alpha_4}}L\{\gamma E_n - (\mu+\theta+\sigma+\omega)I_n\}\right],$$
$$\sum_{n=0}^{k} Q_{n+1}(t) = L^{-1}\left[\frac{1}{s^{\alpha_5}}L\{\omega I_n - (\mu+\theta+\delta)Q_n\}\right],$$
$$\sum_{n=0}^{k} R_{n+1}(t) = L^{-1}\left[\frac{1}{s^{\alpha_6}}L\{\sigma I_n + \delta Q_n - \omega A_n - (\mu+\eta)R_n\}\right]. \quad (10)$$

The followings were obtained from eq. (10);

$$S_0 = n_1,\ A_0 = n_2,\ E_0 = n_3,\ I_0 = n_4,\ Q_0 = n_5,\ R_0 = n_6.$$

When $n = 0$, from the first equation of eq. (10)

$$\begin{aligned} S_1 &= L^{-1}\left[\frac{1}{s^{\alpha_1}}L\{\pi - \beta X_0 - (\psi+\mu)S_0 + \eta R_0\}\right], \\ &= L^{-1}\left[\frac{1}{s^{\alpha_1}}L\{\pi - \beta n_1 n_4 - (\psi+\mu)n_1 + \eta n_6\}\right], \\ &= L^{-1}\left[\{\pi - \beta n_1 n_4 - (\psi+\mu)n_1 + \eta n_6\}\frac{1}{s^{\alpha_1+1}}\right], \\ &= (\pi - \beta n_1 n_4 - (\psi+\mu)n_1 + \eta n_6)\frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)}. \end{aligned}$$

In the case where $n = 0$, considering the second equation within the set denoted as eq. (10).

$$\begin{aligned} A_1 &= L^{-1}\left[\frac{1}{s^{\alpha_2}}L\{\psi S_0 - \mu A_0 - \phi_2 Q_0 - \phi_1 B_0\}\right], \\ &= L^{-1}\left[\frac{1}{s^{\alpha_2}}L\{\psi n_1 - (\phi+\mu)n_2 - \rho n_2 n_4\}\right], \\ &= L^{-1}\left[\{\psi n_1 - (\phi-\mu)n_2 - \rho n_2 n_4\}\frac{1}{s^{\alpha_2+}}\right], \\ &= (\psi n_1 - (\phi-\mu)n_2 - \rho n_2 n_4)\frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)}. \end{aligned} \quad (11)$$

In instances where $n = 0$, we are examining the third eq. (10).

$$\begin{aligned} E_1 &= L^{-1}\left[\frac{1}{s^{\alpha_3}}L\{\beta X_0 - (\mu+\gamma)E_0 + \rho Y_0\}\right], \\ &= L^{-1}\left[\frac{1}{s^{\alpha_3}}L\{\beta n_1 n_4 - (\mu+\gamma)n_3 + \rho n_2 n_4\}\right], \\ &= L^{-1}\left[\{\beta n_1 n_4 - (\mu+\gamma)n_3 + \rho n_2 n_4\}\frac{1}{s^{\alpha_3+1}}\right], \\ &= (\beta n_1 n_4 - (\mu+\gamma)n_3 + \rho n_2 n_4)\frac{t^{\alpha_3}}{\Gamma(\alpha_3+1)}. \end{aligned}$$

In the scenario where $n = 0$, our focus shifts to the fourth equation found within the specified set, identified as eq. (10).

$$\begin{aligned} I_1 &= L^{-1}\left[\frac{1}{s^{\alpha_4}}L\{\gamma E_0 - (\mu+\theta+\sigma+\omega)I\}\right], \\ &= L^{-1}\left[\frac{1}{s^{\alpha_4}}L\{\gamma n_3 - (\mu+\theta+\sigma+\omega)n_4\}\right], \\ &= L^{-1}\left[\{\gamma n_3 - (\mu+\theta+\sigma+\omega)n_4\}\frac{1}{s^{\alpha_4+1}}\right], \\ &= (\gamma n_3 - (\mu+\theta+\sigma+\omega)n_4)\frac{t^{\alpha_4}}{\Gamma(\alpha_4+1)}. \end{aligned}$$

In the scenario where $n = 0$, our focus shifts to the fifth equation found within the specified set, identified as eq. (10).

$$\begin{aligned}
Q_1 &= L^{-1}\left[\frac{1}{s^{\alpha_5}}L\{\omega I_0 - (\mu+\theta+\sigma+\delta)Q_0\}\right],\\
&= L^{-1}\left[\frac{1}{s^{\alpha_5}}L\{\omega n_4 - (\mu+\theta+\sigma+\delta)n_5\}\right],\\
&= L^{-1}\left[\omega n_4 - (\mu+\theta+\sigma+\delta)n_5\frac{1}{s^{\alpha_5+1}}\right],\\
Q_1 &= (\omega n_4 - (\mu+\theta+\sigma+\delta)n_5)\,\frac{t^{\alpha_5}}{\Gamma(\alpha_5+1)}.
\end{aligned}$$

In the scenario where $n = 0$, our focus shifts to the sixth equation found within the specified set, identified as eq. (10).

$$\begin{aligned}
R_1 &= L^{-1}\left[\frac{1}{s^{\alpha_6}}L\{\sigma I_0 + \delta Q_0 + \omega A_0 - (\mu+\eta)R_0\}\right],\\
&= L^{-1}\left[\frac{1}{s^{\alpha_6}}L\{\sigma n_4 + \delta n_5 + \omega n_2 - (\mu+\eta)n_6\}\right],\\
&= L^{-1}\left[\{\sigma n_4 + \delta n_5 + \omega n_2 - (\mu+\eta)n_6\}\frac{1}{s^{\alpha_6+1}}\right],\\
&= (\sigma n_4 + \delta n_5 + \omega n_2 - (\mu+\eta)n_6)\,\frac{t^{\alpha_6}}{\Gamma(\alpha_6+1)}.
\end{aligned}$$

In the scenario where $n = 1$, our focus shifts to the first equation found within the specified set, identified as eq. (10).

$$\begin{aligned}
S_2 &= L^{-1}\left[\frac{1}{s^{\alpha_1}}L\{\pi - \beta X_1 - (\psi+\mu)S_1 + \eta R_1\}\right],\\
&= L^{-1}\Bigg[\frac{1}{s^{\alpha_1}}L\Bigg\{\pi - \beta(n_1)\frac{t^{\alpha_4}}{\Gamma(\alpha_4+1)}(\gamma n_3 - (\mu+\theta+\sigma+\omega)n_4)\\
&\quad + (n_4)\left((\pi - \beta n_1 n_4 - (\psi+\mu)n_1 + \eta n_6)\frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)}\right)\\
&\quad - (\psi+\mu)\,(\pi - \beta n_1 n_4 - (\psi+\mu)n_1 + \eta n_6)\,\frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)}\\
&\quad + \eta\,(\sigma n_4 + \delta n_5 + \omega n_2 - (\mu+\eta)n_6)\,\frac{t^{\alpha_6}}{\Gamma(\alpha_6+1)}\Bigg\}\Bigg],\\
S_2 &= L^{-1}\Bigg[\frac{1}{s^{\alpha_1}}\Bigg\{\pi - \beta(n_1)\frac{\Gamma(\alpha_4+1)}{\Gamma(\alpha_4+1)s^{\alpha_4+1}}(\gamma n_3 - (\mu+\theta+\sigma+\omega)n_4)\\
&\quad + (n_4)\left((\pi - \beta n_1 n_4 - (\psi+\mu)n_1 + \eta n_6)\frac{\Gamma(\alpha_1+1)}{\Gamma(\alpha_1+1)s^{\alpha_1+1}}\right)\\
&\quad - (\psi+\eta)\,(\pi - \beta n_1 n_4 - (\psi+\mu)n_1 + \eta n_6)\,\frac{\Gamma(\alpha_1+1)^1}{\Gamma(\alpha_1+1)s^{\alpha_1+1}}\\
&\quad + \eta\,(\sigma n_4 + \delta n_5 + \omega n_2 - (\mu+\eta)n_6\,)\frac{\Gamma(\alpha_6+1)}{\Gamma(\alpha_6+1)s^{\alpha_6+1}}\Bigg\}\Bigg],\\
S_2 &= L^{-1}\Bigg[\Bigg\{\pi - \beta(n_1)\frac{1}{s^{\alpha_4+\alpha_1+1}}(\gamma n_3 - (\mu+\theta+\sigma+\omega)n_4)\\
&\quad + (n_4)\left((\pi - \beta n_1 n_4 - (\psi+\mu)n_1 + \eta n_6)\frac{1}{s^{2\alpha_1+1}}\right)\\
&\quad - (\psi+\mu)\,(\pi - \beta n_1 n_4 - (\psi+\mu)n_1 + \eta n_6)\,\frac{1}{s^{2\alpha_1+1}}\\
&\quad + \eta\,(\sigma n_4 + \delta n_5 + \omega n_2 - (\mu+\eta)n_6)\,\frac{1}{s^{\alpha_6+\alpha_1+1}}\Bigg\}\Bigg],\\
S_2 &= \pi - \beta(n_1)\,\Bigg((\gamma n_3 - (\mu+\theta+\sigma+\omega)n_4)\frac{t^{\alpha_1+\alpha_4}}{\Gamma(\alpha_1+\alpha_4)}\\
&\quad + (n_4)(\pi - \beta n_1 n_4 - (\psi+\mu)n_1 + \eta n_6)\frac{t^{2\alpha_1}}{\Gamma(2\alpha_1+1)}\Bigg)\\
&\quad - (\mu+\psi)(\pi - \beta n_1 n_4 - (\psi+\mu)n_1 + \eta n_6)\frac{t^{2\alpha_1}}{\Gamma(2\alpha_1+1)}\\
&\quad + \eta(\sigma n_4 + \delta n_5 + \omega n_2 - (\mu+\eta)n_6)\frac{t^{\alpha_1+\alpha_6}}{\Gamma(\alpha_1+\alpha_6+1)}.
\end{aligned}$$

In the scenario where $n = 1$, our focus shifts to the second equation found within the specified set, identified as eq. (10).

$$\begin{aligned}
A_2 &= L^{-1}\left[\frac{1}{s^{\alpha_2}}L\{\psi S_1 - (\phi+\mu)A_1 - \rho Y_1\}\right],\\
&= L^{-1}\Bigg[\frac{1}{s^{\alpha_2}}L\Bigg\{\psi(\pi - \beta n_1 n_4 - (\psi+\mu)n_1 + \eta n_6)\frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)}\\
&\quad - (\phi+\mu)(\psi n_1 - (\phi-\mu)n_2 - \rho n_2 n_4)\frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)}\\
&\quad - \rho(n_2)(\gamma n_3 - (\mu+\theta+\sigma+\omega)n_4)\frac{t^{\alpha_4}}{\Gamma(\alpha_4+1)}\\
&\quad + (n_4)(\psi n_1 - (\phi-\mu)n_2 - \rho n_2 n_4)\frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)}\Bigg\}\Bigg],\\
&= L^{-1}\Bigg[\frac{1}{s^{\alpha_2}}\Bigg\{\psi(\pi - \beta n_1 n_4 - (\psi+\mu)n_1 + \eta n_6)\frac{\Gamma(\alpha_1+1)^1}{\Gamma(\alpha_1+1)s^{\alpha_1+1}}\\
&\quad - (\phi+\mu)(\psi n_1 - (\phi-\mu)n_2 - \rho n_2 n_4)\frac{\Gamma(\alpha_2+1)}{\Gamma(\alpha_2+1)s^{\alpha_2+1}}\\
&\quad - \rho(n_2)(\gamma n_3 - (\mu+\theta+\sigma+\omega)n_4)\frac{\Gamma(\alpha_4+1)}{\Gamma(\alpha_4+1)s^{\alpha_2+1}}\\
&\quad + (n_4)(\psi n_1 - (\phi-\mu)n_2 - \rho n_2 n_4)\frac{\Gamma(\alpha_2+1)}{\Gamma(\alpha_2+1)s^{\alpha_2+1}}\Bigg\}\Bigg],\\
&= L^{-1}\Bigg[\frac{1}{s^{\alpha_2}}\Bigg\{\psi(\pi - \beta n_1 n_4 - (\psi+\mu)n_1 + \eta n_6)\frac{1}{s^{\alpha_1+1}}\\
&\quad - (\phi+\mu)(\psi n_1 - (\phi-\mu)n_2 - \rho n_2 n_4)\frac{1}{s^{\alpha_2+1}}\\
&\quad - \rho(n_2)(\gamma n_3 - (\mu+\theta+\sigma+\omega)n_4)\frac{1}{s^{\alpha_2+1}}\\
&\quad + (n_4)(\psi n_1 - (\phi-\mu)n_2 - \rho n_2 n_4)\frac{1}{s^{\alpha_2+1}}\Bigg\}\Bigg],\\
&= L^{-1}\Bigg[\Bigg\{\psi(\pi - \beta n_1 n_4 - (\psi+\mu)n_1 + \eta n_6)\frac{1}{s^{\alpha_1+\alpha_2+1}}\\
&\quad - (\phi+\mu)(\psi n_1 - (\phi-\mu)n_2 - \rho n_2 n_4)\frac{1}{s^{2\alpha_2+1}}\\
&\quad - \rho(n_2)(\gamma n_3 - (\mu+\theta+\sigma+\omega)n_4)\frac{1}{s^{2\alpha_2+1}}\\
&\quad + (n_4)(\psi n_1 - (\phi-\mu)n_2 - \rho n_2 n_4)\frac{1}{s^{\alpha_2+\alpha_4+1}}\Bigg\}\Bigg],\\
A_2 &= \psi(\pi - \beta n_1 n_4 - (\psi+\mu)n_1 + \eta n_6)\frac{t^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}\\
&\quad - (\phi+\mu)(\psi n_1 - (\phi-\mu)n_2 - \rho n_2 n_4)\frac{t^{2\alpha_2}}{\Gamma(2\alpha_2+1)}\\
&\quad - \rho((n_2)(\gamma n_3 - (\mu+\theta+\sigma+\omega)n_4))\frac{t^{\alpha_4+\alpha_2}}{\Gamma(\alpha_4+\alpha_2+1)}\\
&\quad + (n_4)(\psi n_1 - (\phi-\mu)n_2 - \rho n_2 n_4)\frac{t^{2\alpha_2}}{\Gamma(2\alpha_2+1)}.
\end{aligned}$$

In the scenario where $n = 1$, our focus shifts to the third equation found within the specified set, identified as eq. (10).

$$\begin{aligned}
E_2 &= L^{-1}\left[\frac{1}{s^{\alpha_3}}L\{\beta X_1 - (\mu-\gamma)E_1 + \rho Y_1\}\right],\\
&= L^{-1}\Bigg[\frac{1}{s^{\alpha_3}}L\Bigg\{\beta(n_1)\frac{t^{\alpha_4}}{\Gamma(\alpha_4+1)}(\gamma n_3 - (\mu+\theta+\sigma+\omega)n_4)\\
&\quad + (n_4)\frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)}(\pi - \beta n_1 n_4 - (\psi+\mu)n_1 + \eta n_6) - (\mu\\
&\quad + \gamma)(\beta n_1 n_4 - (\mu+\gamma)n_3 + \rho n_2 n_4)\frac{t^{\alpha_3}}{\Gamma(\alpha_3+1)} + \rho(n_2)(\gamma n_3\\
&\quad - (\mu+\theta+\sigma+\omega)n_4)\frac{t^{\alpha_4}}{\Gamma(\alpha_4+1)} + (n_4)(\psi n_1 - (\phi-\mu)n_2
\end{aligned}$$

$$- \rho n_2 n_4) \frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)} \Big\} \Big],$$

$$\begin{aligned}
E_2 = L^{-1}\Big[& \frac{1}{s^{\alpha_3}} \beta(n_1)(\gamma n_3 - (\mu+\theta+\sigma+\omega) n_4 \frac{\Gamma(\alpha_4+1)}{\Gamma(\alpha_4+1)s^{\alpha_4+1}} \\
& + (n_4)(\pi - \beta n_1 n_4 - (\psi+\mu) n_1 + \eta n_6) \frac{\Gamma(\alpha_1+1)}{\Gamma(\alpha_1+1)s^{\alpha_1+1}} \\
& - (\mu+\gamma)(\beta n_1 n_4 - (\mu+\gamma) n_3 + \rho n_2 n_4) \frac{\Gamma(\alpha_3+1)}{\Gamma(\alpha_3+1)s^{\alpha_3+1}} \\
& - \rho(n_2)(\gamma n_3 - (\mu+\theta+\sigma+\omega) n_4) \frac{\Gamma(\alpha_4+1)}{\Gamma(\alpha_4+1)s^{\alpha_2+1}} \\
& + (n_4)(\psi n_1 - (\phi-\mu) n_2 - \rho n_2 n_4) \frac{\Gamma(\alpha_2+1)}{\Gamma(\alpha_2+1)s^{\alpha_2+1}} \Big], \\
= L^{-1}\Big[& \frac{1}{s^{\alpha_3}} \beta(n_1)(\gamma n_3 - (\mu+\theta+\sigma+\omega) n_4) \frac{1}{s^{\alpha_4+1}} \\
& + (n_4)(\pi - \beta n_1 n_4 - (\psi+\mu) n_1 + \eta n_6) \frac{1}{s^{\alpha_1+1}} \\
& - (\mu+\gamma)(\beta n_1 n_4 - (\mu+\gamma) n_3 + \rho n_2 n_4) \frac{1}{s^{\alpha_3+1}} \\
& - \rho(n_2)(\gamma n_3 - (\mu+\theta+\sigma+\omega) n_4) \frac{1}{s^{\alpha_2+1}} \\
& + (n_4)(\psi n_1 - (\phi-\mu) n_2 - \rho n_2 n_4) \frac{1}{s^{\alpha_2+1}} \Big], \\
= L^{-1}\Big[& \beta(n_1)(\gamma n_3 - (\mu+\theta+\sigma+\omega) n_4) \frac{1}{s^{\alpha_4+\alpha_3+1}} \\
& + (n_4)(\pi - \beta n_1 n_4 - (\psi+\mu) n_1 + \eta n_6) \frac{1}{s^{\alpha_1+\alpha_3+1}} \\
& - (\mu+\gamma)(\beta n_1 n_4 - (\mu+\gamma) n_3 + \rho n_2 n_4) \frac{1}{s^{2\alpha_3+1}} \\
& - \rho(n_2)(\gamma n_3 - (\mu+\theta+\sigma+\omega) n_4) \frac{1}{s^{\alpha_2+\alpha_3+1}} \\
& + (n_4)(\psi n_1 - (\phi-\mu) n_2 - \rho n_2 n_4) \frac{1}{s^{\alpha_2+\alpha_3+1}} \Big],
\end{aligned}$$

$$\begin{aligned}
E_2 = & \beta(n_1)(\gamma n_3 - (\mu+\theta+\sigma+\omega) n_4) \frac{t^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4)} \\
& + (n_4)(\pi - \beta n_1 n_4 - (\psi+\mu) n_1 + \eta n_6) \frac{t^{\alpha_3+\alpha_1}}{\Gamma(\alpha_3+\alpha_1)} \\
& - (\mu+\gamma)(\beta n_1 n_4 - (\mu+\gamma) n_3 + \rho n_2 n_4) \frac{t^{2\alpha_3}}{\Gamma(2\alpha_3+1)} \\
& - \rho(n_2)(\gamma n_3 - (\mu+\theta+\sigma+\omega) n_4) \frac{t^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4)} \\
& + (n_4)(\psi n_1 - (\phi-\mu) n_2 - \rho n_2 n_4) \frac{t^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2+\alpha_3)}.
\end{aligned}$$

In the scenario where $n = 1$, our focus shifts to the forth equation found within the specified set, identified as equation eq. (10).

$$\begin{aligned}
I_2 = & L^{-1}\Big[\frac{1}{s^{\alpha_4}} L\{\gamma E_1 - (\mu+\theta+\sigma+\omega) I_1\} \Big], \\
= & L^{-1}\Big[\frac{1}{s^{\alpha_4}} L\Big\{ \gamma(\beta n_1 n_4 - (\mu+\gamma) n_3 + \rho n_2 n_4) \frac{t^{\alpha_3}}{\Gamma(\alpha_3+1)} \\
& - (\mu+\theta+\sigma+\omega)(\gamma n_3 - (\mu+\theta+\sigma+\omega) n_4) \frac{t^{\alpha_4}}{\Gamma(\alpha_4+1)} \Big\} \Big], \\
= & L^{-1}\Big[\frac{1}{s^{\alpha_4}} \Big\{ \gamma(\beta n_1 n_4 - (\mu+\gamma) n_3 + \rho n_2 n_4) \frac{\Gamma(\alpha_3+1)}{\Gamma(\alpha_3+1)s^{\alpha_3+1}} \\
& - (\mu+\theta+\sigma+\omega)(\gamma n_3 - (\mu+\theta+\sigma+\omega) n_4) \frac{\Gamma(\alpha_4+1)}{\Gamma(\alpha_4+1)s^{\alpha_4+1}} \Big\} \Big], \\
= & L^{-1}\Big[\frac{1}{s^{\alpha_4}} \Big\{ \gamma(\beta n_1 n_4 - (\mu+\gamma) n_3 + \rho n_2 n_4) \frac{1}{s^{\alpha_3+1}} - (\mu+\theta+\sigma \\
& + \omega)(\gamma n_3 - (\mu+\theta+\sigma+\omega) n_4) \frac{1}{s^{\alpha_3+1}} \Big\} \Big], \\
= & L^{-1}\Big[\gamma(\beta n_1 n_4 - (\mu+\gamma) n_3 + \rho n_2 n_4) \frac{1}{s^{\alpha_3+\alpha_4+1}} - (\mu+\theta+\sigma \\
& + \omega)(\gamma n_3 - (\mu+\theta+\sigma+\omega) n_4) \frac{1}{s^{4\alpha_4+1}} \Big],
\end{aligned}$$

$$\begin{aligned}
I_2 = & \gamma(\beta n_1 n_4 - (\mu+\gamma) n_3 + \rho n_2 n_4) \frac{t^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4)} - (\mu+\theta+\sigma \\
& + \omega)(\gamma n_3 - (\mu+\theta+\sigma+\omega) n_4) \frac{t^{2\alpha_4}}{\Gamma(2\alpha_4+1)}
\end{aligned}$$

In the scenario where $n = 1$, our focus shifts to the fifth equation found within the specified set, identified as eq. (10).

$$\begin{aligned}
Q_2 = & L^{-1}\Big[\frac{1}{s^{\alpha_5}} L\{\omega I_1 - (\mu+\theta+\delta) Q_1\} \Big], \\
= & L^{-1}\Big[\frac{1}{s^{\alpha_5}} L\Big\{ \omega(\gamma n_3 - (\mu+\theta+\sigma+\omega) n_4) \frac{t^{\alpha_4}}{\Gamma(\alpha_4+1)} \\
& - (\mu+\theta+\delta)(\omega n_4 - (\mu+\theta+\sigma+\delta) n_5) \frac{t^{\alpha_5}}{\Gamma(\alpha_5+1)} \Big\} \Big], \\
= & L^{-1}\Big[\frac{1}{s^{\alpha_5}} \omega(\gamma n_3 - (\mu+\theta+\sigma+\omega) n_4) \frac{\Gamma(\alpha_4+1)}{\Gamma(\alpha_4+1)s^{\alpha_4+1}} \\
& - (\mu+\theta+\delta)(\omega n_4 - (\mu+\theta+\sigma+\delta) n_5) \frac{\Gamma(\alpha_5+1)}{\Gamma(\alpha_5+1)s^{\alpha_5+1}} \Big], \\
= & L^{-1}\Big[\frac{1}{s^{\alpha_5}} \omega(\gamma n_3 - (\mu+\theta+\sigma+\omega) n_4) \frac{1}{s^{\alpha_4+1}} - (\mu+\theta \\
& + \delta)(\omega n_4 - (\mu+\theta+\sigma+\delta) n_5) \frac{1}{s^{\alpha_5+1}} \Big], \\
= & L^{-1}\Big[\omega(\gamma n_3 - (\mu+\theta+\sigma+\omega) n_4) \frac{1}{s^{\alpha_4+\alpha_5+1}} - (\mu+\theta \\
& + \delta)(\omega n_4 - (\mu+\theta+\sigma+\delta) n_5) \frac{1}{s^{2\alpha_5+1}} \Big],
\end{aligned}$$

$$\begin{aligned}
Q_2 = & \omega(\gamma n_3 - (\mu+\theta+\sigma+\omega) n_4) \frac{t^{\alpha_4+\alpha_5}}{\Gamma(\alpha_4+\alpha_5+1)} - (\mu+\theta \\
& + \delta)(\omega n_4 - (\mu+\theta+\sigma+\delta) n_5) \frac{t^{2\alpha_5}}{\Gamma(2\alpha_5+1)}.
\end{aligned}$$

In the scenario where $n = 1$, our focus shifts to the Sixth equation found within the specified set, identified as eq. (10).

$$\begin{aligned}
R_2 = & L^{-1}\Big[\frac{1}{s^{\alpha_6}} L\{\sigma I_1 + \delta Q_1 - \omega A_1 - (\mu+\eta) R_1\} \Big], \\
= & L^{-1}\Big[\frac{1}{s^{\alpha_6}} L\Big\{ \sigma(\gamma n_3 - (\mu+\theta+\sigma+\omega) n_4) \frac{t^{\alpha_4}}{\Gamma(\alpha_4+1)} \\
& + \delta(\omega n_4 - (\mu+\theta+\sigma+\delta) n_5) \frac{t^{\alpha_5}}{\Gamma(\alpha_5+1)} - \omega(\psi n_1 \\
& - (\phi-\mu) n_2 - \rho n_2 n_4) \frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)} - (\mu+\eta)(\sigma n_4 + \delta n_5 + \omega n_2 \\
& - (\mu+\eta) n_6) \frac{t^{\alpha_6}}{\Gamma(\alpha_6+1)} \Big\} \Big],
\end{aligned}$$

$$= L^{-1}\left[\frac{1}{s^{\alpha_6}}\left\{\sigma(\gamma n_3-(\mu+\theta+\sigma+\omega)n_4)\frac{\Gamma(\alpha_4+1)}{\Gamma(\alpha_4+1)s^{\alpha_4+1}} + \delta(\omega n_4-(\mu+\theta+\sigma+\delta)n_5)\frac{\Gamma(\alpha_5+1)}{\Gamma(\alpha_5+1)s^{\alpha_5+1}} - \omega(\psi n_1 - (\phi-\mu)n_2-\rho n_2 n_4)\frac{\Gamma(\alpha_2+1)}{\Gamma(\alpha_2+1)s^{\alpha_2+1}} - (\mu+\eta)(\sigma n_4 + \delta n_5+\omega n_2-(\mu+\eta)n_6)\frac{\Gamma(\alpha_6+1)}{\Gamma(\alpha_6+1)s^{\alpha_6+1}}\right\}\right],$$

$$= L^{-1}\left[\frac{1}{s^{\alpha_6}}\left\{(\gamma n_3-(\mu+\theta+\sigma+\omega)n_4)\frac{\sigma}{s^{\alpha_4+1}} + (\omega n_4-(\mu+\theta+\sigma+\delta)n_5)\frac{\delta}{s^{\alpha_5+1}} - (\psi n_1-(\phi-\mu)n_2-\rho n_2 n_4)\frac{\omega}{s^{\alpha_2+1}} - (\mu+\eta)(\sigma n_4+\delta n_5+\omega n_2-(\mu+\eta)n_6)\frac{1}{s^{\alpha_6+1}}\right\}\right],$$

$$= L^{-1}\left[(\gamma n_3-(\mu+\theta+\sigma+\omega)n_4)\frac{\sigma}{s^{\alpha_4+\alpha_6+1}} + (\omega n_4-(\mu+\theta+\sigma+\delta)n_5)\frac{\delta}{s^{\alpha_5+\alpha_6+1}} - \frac{\omega}{s^{\alpha_2+\alpha_6+1}}(\psi n_1-(\phi-\mu)n_2 - \rho n_2 n_4) - (\mu+\eta)(\sigma n_4+\delta n_5+\omega n_2-(\mu+\eta)n_6)\frac{1}{s^{2\alpha_6+1}}\right],$$

$$R_2 = \sigma(\gamma n_3-(\mu+\theta+\sigma+\omega)n_4)\frac{t^{\alpha_4+\alpha_5}}{\Gamma(\alpha_4+\alpha_6+1)} + \delta(\omega n_4-(\mu+\theta+\sigma+\delta)n_5)\frac{t^{\alpha_5+\alpha_5}}{\Gamma(\alpha_6+\alpha_5+1)} - \omega(\psi n_1-(\phi-\mu)n_2 - \rho n_2 n_4)\frac{t^{\alpha_4+\alpha_5}}{\Gamma(\alpha_2+\alpha_6+1)} - (\mu+\eta)(\sigma n_4+\delta n_5+\omega n_2 - (\mu+\eta)n_6)\frac{1}{s^{2\alpha_6+1}}]\frac{t^{2\alpha_6}}{\Gamma(2\alpha_6+1)}.$$

$$S_1 = (\pi-\beta n_1 n_4-(\psi+\mu)n_1+\eta n_6)\frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)},$$

$$A_1 = (\psi n_1-(\phi+\mu)n_2-\rho n_2 n_4)\frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)},$$

$$E_1 = (\beta n_1 n_4-(\mu+\gamma)n_3+\rho n_2 n_4)\frac{t^{\alpha_3}}{\Gamma(\alpha_3+1)},$$

$$I_1 = (\gamma n_3-(\mu+\theta+\sigma+\omega)n_4)\frac{t^{\alpha_4}}{\Gamma(\alpha_4+1)},$$

$$Q_1 = (\omega n_4-(\mu+\theta+\sigma+\delta)n_5)\frac{t^{\alpha_5}}{\Gamma(\alpha_5+1)},$$

$$R_1 = (\sigma n_4+\delta n_5+\omega n_2-(\mu+\eta)n_6)\frac{t^{\alpha_6}}{\Gamma(\alpha_6+1)}.$$

Hence, eqs. (12) and (13) are obtained.

$$S_2 = \pi-\beta(n_1)\left((\gamma n_3-(\mu+\theta+\sigma+\omega)n_4)\frac{t^{\alpha_1+\alpha_4}}{\Gamma(\alpha_1+\alpha_4)} + (n_4)(\pi-\beta n_1 n_4-(\psi+\mu)n_1+\eta n_6)\frac{t^{2\alpha_1}}{\Gamma(2\alpha_1+1)}\right) - (\mu+\psi)(\pi-\beta n_1 n_4-(\psi+\mu)n_1+\eta n_6)\frac{t^{2\alpha_1}}{\Gamma(2\alpha_1+1)} + \eta(\sigma n_4+\delta n_5+\omega n_2-(\mu+\eta)n_6)\frac{t^{\alpha_1+\alpha_6}}{\Gamma(\alpha_1+\alpha_6+1)},$$

$$A_2 = \psi(\pi-\beta n_1 n_4-(\psi+\mu)n_1+\eta n_6)\frac{t^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} \tag{12}$$

$$- (\phi+\mu)(\psi n_1-(\phi-\mu)n_2-\rho n_2 n_4)\frac{t^{2\alpha_2}}{\Gamma(2\alpha_2+1)} - \rho(n_2)(\gamma n_3-(\mu+\theta+\sigma+\omega)n_4)\frac{t^{\alpha_4+\alpha_2}}{\Gamma(\alpha_4+\alpha_2+1)} + (n_4)(\psi n_1-(\phi-\mu)n_2-\rho n_2 n_4)\frac{t^{2\alpha_2}}{\Gamma(2\alpha_2+1)},$$

$$E_2 = \beta(n_1)(\gamma n_3-(\mu+\theta+\sigma+\omega)n_4)\frac{t^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4)} + (n_4)(\pi-\beta n_1 n_4-(\psi+\mu)n_1+\eta n_6)\frac{t^{\alpha_3+\alpha_1}}{\Gamma(\alpha_3+\alpha_1)} - (\mu+\gamma)(\beta n_1 n_4-(\mu+\gamma)n_3+\rho n_2 n_4)\frac{t^{2\alpha_3}}{\Gamma(2\alpha_3+1)} - \rho((n_2)(\gamma n_3-(\mu+\theta+\sigma+\omega)n_4))\frac{t^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4)} + (n_4)(\psi n_1-(\phi-\mu)n_2-\rho n_2 n_4)\frac{t^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2+\alpha_3)},$$

$$I_2 = \gamma(\beta n_1 n_4-(\mu+\gamma)n_3+\rho n_2 n_4)\frac{t^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4)} - (\mu+\theta+\sigma+\omega)(\gamma n_3-(\mu+\theta+\sigma+\omega)n_4)\frac{t^{2\alpha_4}}{\Gamma(2\alpha_4+1)},$$

$$Q_2 = \omega(\gamma n_3-(\mu+\theta+\sigma+\omega)n_4)\frac{t^{\alpha_4+\alpha_5}}{\Gamma(\alpha_4+\alpha_5+1)} - (\mu+\theta+\delta)(\omega n_4-(\mu+\theta+\sigma+\delta)n_5)\frac{t^{2\alpha_5}}{\Gamma(2\alpha_5+1)}. \tag{13}$$

5. Results

Assessing the acquired outcomes utilizing the subsequent parameters [7]: $S_0 = 30$, $A_0 = 5$, $E_0 = 2$, $I_0 = 1$, $Q_0 = 0$, $R_0 = 3$, $\rho = 0.2$, $\phi = 0.3$, $\beta = 0.09$, $\theta = 0.035$, $\sigma = 0.35$, $\omega = 0.3$, $\mu = 0.05$, $\delta = 0.65$, $\eta = 0.01$, $\gamma = 0.45$, $\psi = 0.1$, $\pi = 0.01$. The following series solution was obtained.

$$S = 30. - \frac{14.32t^{\alpha}}{\Gamma(\alpha+1)} + \frac{12.26t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$A = 5 + \frac{0.25t^{\alpha}}{\Gamma(\alpha+1)}\frac{1.0685t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$E = 2 + \frac{2.70t^{\alpha}}{\Gamma(\alpha+1)}\frac{0.41450t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$I = 1 + \frac{0.21^{\alpha}}{\Gamma(\alpha+1)} + \frac{1.067725t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$Q = \frac{0.3t^{\alpha}}{\Gamma(\alpha+1)}\frac{0.2610t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$R = 3 + \frac{1.62t^{\alpha}}{\Gamma(\alpha+1)} + \frac{0.2373t^{2\alpha}}{\Gamma(2\alpha+1)}.$$

6. Numerical Simulation

To assess the influence of the Caputo fractional order within the model's compartment, we perform numerical simulations by altering the order values to $\alpha = 0.55$, $\alpha = 0.75$, $\alpha = 0.95$, and $\alpha = 1$. The results of the simulation procedure are depicted visually Bottom of Form

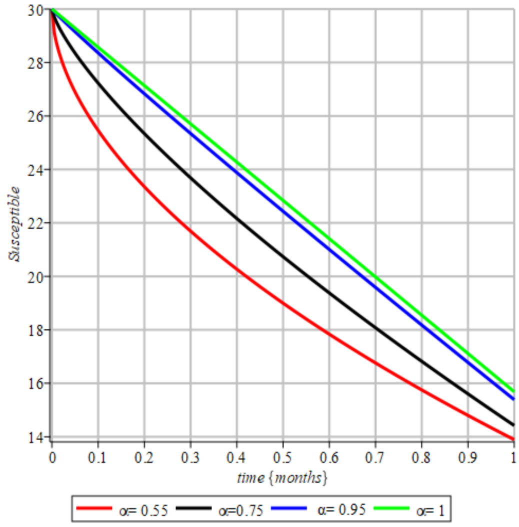


Figure 2. Fractional order effect on susceptible

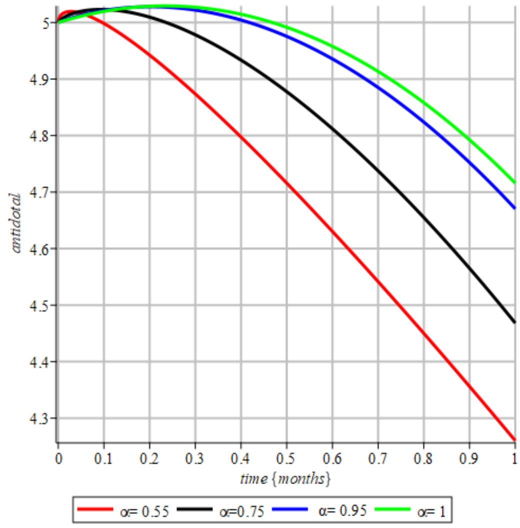


Figure 3. Fractional order effect on antidotal

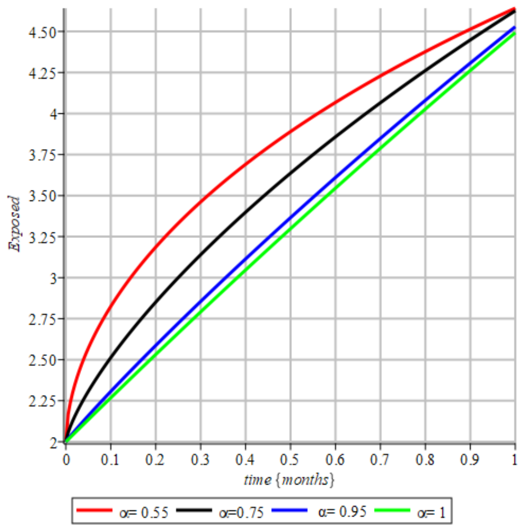


Figure 4. Fractional order effect on exposed

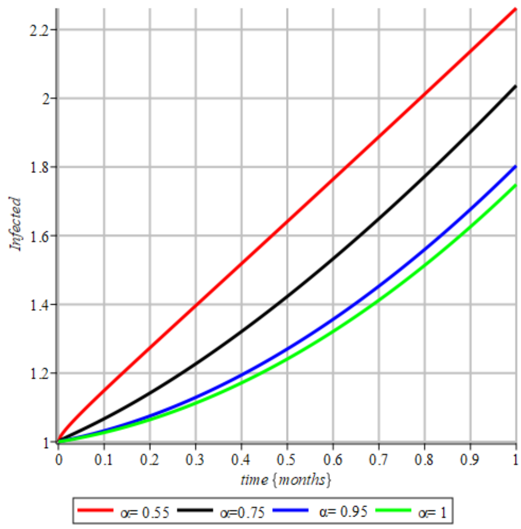


Figure 5. Fractional order effect on infected

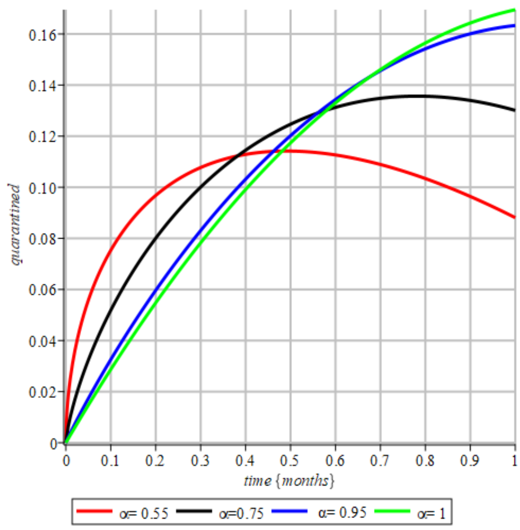


Figure 6. Fractional order effect on quarantined

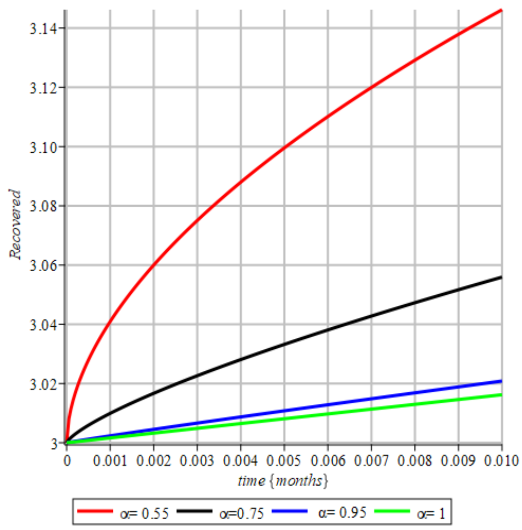
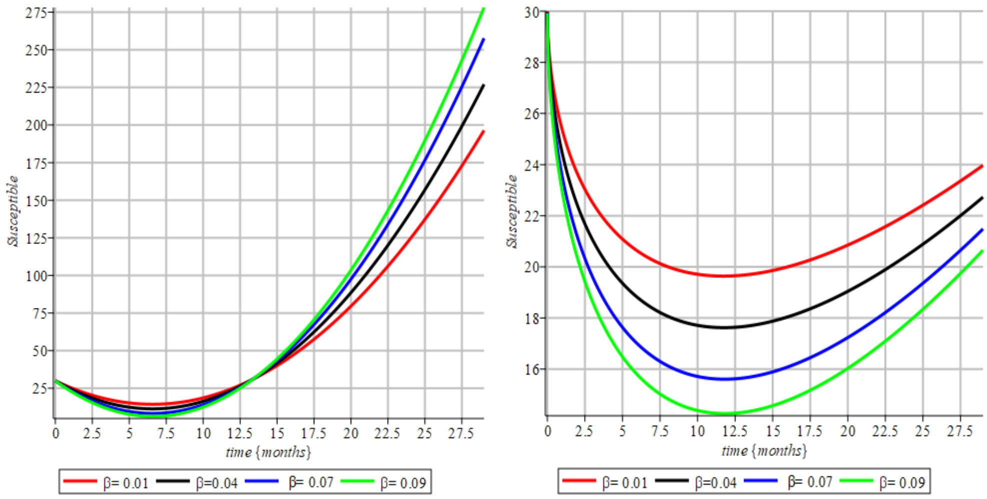
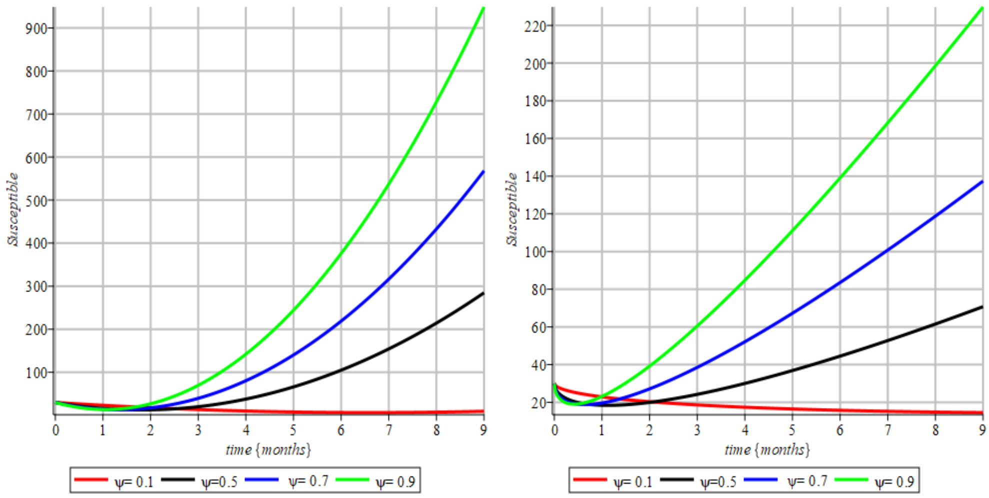


Figure 7. Fractional order effect on recovered



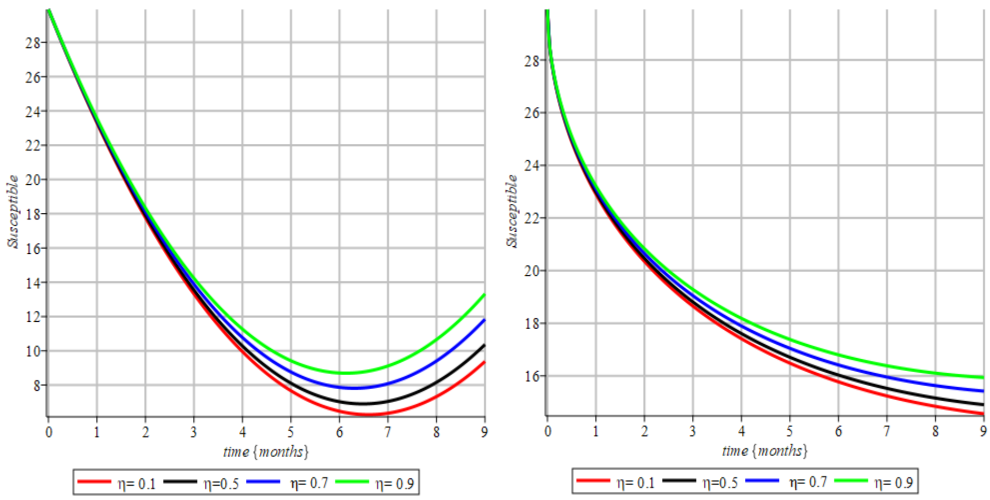
(a) Classical order (b) Fractional order

Figure 8. Effect of transmission rate on susceptible compartment.



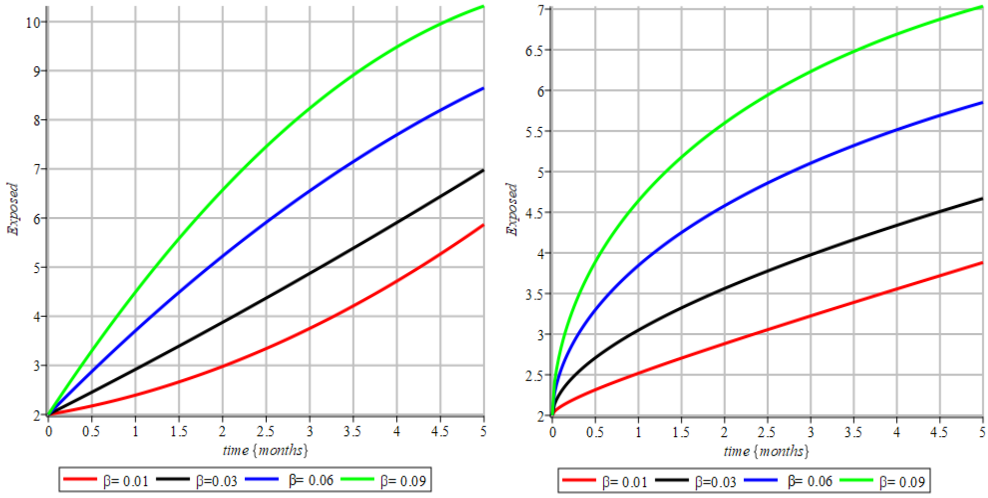
(a) Classical order (b) Fractional order

Figure 9. Effect of antidotal process rate on susceptible compartment.



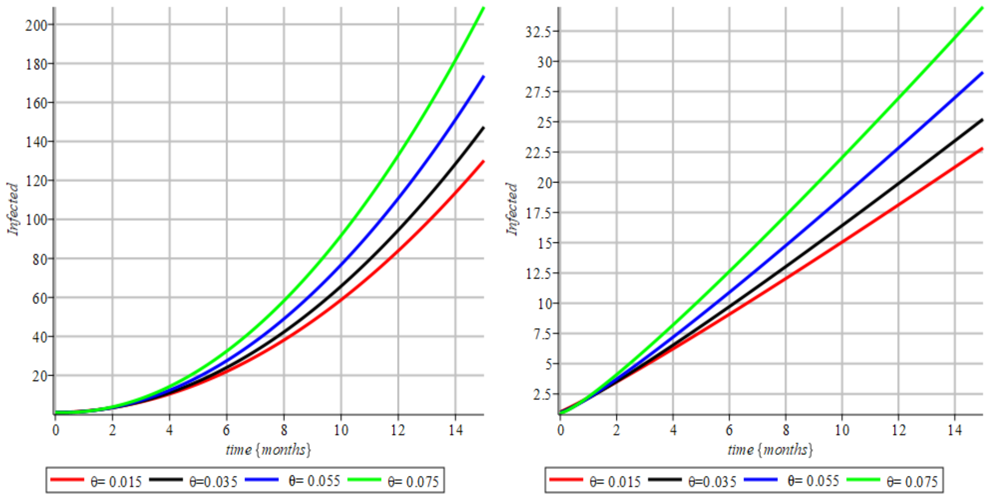
(a) Classical order (b) Fractional order

Figure 10. Effect rate of transition from the recovery class to the susceptible class (R to S) on susceptible compartment.



(a) Classical order (b) Fractional order

Figure 11. Effect of transmission rate on exposed compartment.



(a) Classical order (b) Fractional order

Figure 12. Crashing rate of computers due to the attack of virus on infected compartment.

7. Discussion

The research employed advanced methodologies, utilizing the Next Generation Matrix Method for calculating the basic reproduction number (R_0) and performing sensitivity analysis on it, and the Laplace-Adomian decomposition method to approximate the numerical solution of the model. To comprehensively evaluate the results, a simulation analysis was performed using MAPLE 21 software, showcasing the practical application of the research and enhancing its credibility.

Figure 1 shows the sensitivity chart of essential parameters contained in model. Figures 2 to 7 highlight the enhanced flexibility of the fractional-order computer virus model compared to traditional derivative-based models. Notably, the research predicted relatively low initial values, necessitating short time periods for the simulations, as illustrated by integral curves representing state variables across various alpha values (ranging from 0.55 to 1), and revealing distinct patterns. In Figure 2, we observe a gradual reduction in susceptibility to infection as alpha values increase, primarily attributed to the implementation.

Figures 2, 4 and 6 emphasize the importance of selecting sufficiently large initial data for extended time intervals to prevent the population from becoming negative. Furthermore, all compartments exhibited significant increases from zero, as indicated by the curves in the figures. These observations provided valuable insights into virus transmission dynamics, contributing to a better understanding of the research findings.

The study also aimed to evaluate transmission rates within susceptible and exposed populations, illustrated in Figures 8 and 12, respectively, under classical and fractional-order scenarios. Additionally, the study assessed transmission rates and computer crash rates in the infected compartment during the process of eradicating computer viruses. The primary goal was to minimize the emergence of new cases by comprehending the impact model and other contributing factors to virus prevalence. Consequently, numerical experiments were conducted to explore how these factors affected critical state variables within the proposed model, with the results depicted visually.

8. Convergence Analysis

The solution obtained is a series that rapidly converges and consistently approaches the exact solution. We employ traditional methods to assess the convergence of the series, building upon the concept presented in reference [41].

Theorem 4. *Let V be a Banach space and $H : V \to V$ be a contractive nonlinear operator such that for all $v, v^1 \in V$, $||T(V) - T(V^1)|| \le k||v - v^1||$, $0 < k < 1$. Then T has a unique point y such that $Ty = y$, where $y = (S, A, E, I, Q, R)$.*

The series given in eq. (11) can be written by applying Adomian decomposition method as:

$$v_m = Tv_{m-1},\ v_{m-1} = \sum v_i,\ m = 1, 2, 3, \ldots,$$

and assume that $v_0 \in\ B_r(v)$ where

$$B_r(v) = \left\{ v^1 \in V\ :\ (i) v_m \in B_r(v);\ (ii) \lim_{n\to\infty} v_m = v \right\}$$

Proof. For using mathematical induction for $m = 1$, we have

$$\|v_0 - v\| = \|T(v_0) - T(v)\| \le k \|v_0 - v\|.$$

Let the result is true for $n = 1$, then

$$\|v_0 - v\| \le k^{n-1} \|v_0 - v\|.$$

We have

$$\begin{aligned} \|v_n - v\| =&\ \|T(v_{n-1}) - T(v)\| \\ &\le k \|v_{n-1} - v\| \\ &\le k^n \|v_n - v\|. \end{aligned}$$

i.e.

$$\begin{aligned} \|v_n - v\| &\le k^n \|v_0 - v\| \\ &\le k^n r \\ &< r. \end{aligned}$$

Implies that $v_m \in B$. Since $\|v_n - v\| \le k^n \|v_0 - v\|$ and as $\lim\limits_{n\to\infty} k^n = 0$, we have

$$\lim_{n\to\infty} \|v_n - v\| = 0 \Rightarrow \lim_{n\to\infty} v_m = v$$

□

9. Conclusion

A new six-compartment fractional model is presented in this study to address the ongoing difficulties in controlling malware and computer viruses in both networked and standalone systems. The model is demonstrated to be both mathematically and biologically well-posed by expanding upon traditional frameworks, offering a dependable and novel tool for comprehending the dynamics of cyberthreats. Sensitivity analysis and the calculation of the fundamental reproduction number reveal important variables impacting virus spread, providing useful information for efficient mitigation techniques. The model's resilience in numerical analysis is demonstrated by the application of the Laplace Adomian Decomposition Method, which also demonstrates how it has the ability to completely transform network management and cyber security procedures. Additionally, investigating fractional-order dynamics and associated memory effects broadens our knowledge of how viruses behave over time and opens the door to the creation of complex defenses. These results lay the groundwork for further studies in digital security while also providing useful remedies for virus control and prevention. The knowledge acquired here is crucial for safeguarding network system integrity, improving cyber security tactics, and defending digital infrastructure. Future research might concentrate on improving the model for real-time use and examining how well it can adjust to new cyber threats.Future research should enhance models by incorporating real-world features like variable transmission rates and stochastic processes. Improved computational methods, such as adaptive time-stepping, are needed to maintain accuracy over extended time spans. Integrating real-world data and machine learning will refine predictions, while broader validation across populations will extend the model's applicability to other diseases.

10. Recommendations

We present the findings and conclusions from a meticulous study that explored the use of Caputo fractional order derivatives and the Laplace Adomian decomposition method. This study not only provided invaluable insights into transmission dynamics but also proposed novel solutions for addressing complex systems. Reflecting on these outcomes, numerous recommendations for future investigations emerge.

- Firstly, researchers should explore alternative fractional derivatives beyond the familiar Caputo fractional order derivative. Investigating derivatives such as Antangana-Baleanu and Caputo-Fabrizo can offer a comprehensive understanding of diverse scenarios, enriching analytical capabilities.
- Secondly, there is a need to augment our numerical toolkit by incorporating diverse methods for solving fractional differential equations, such as the homotopy perturbation method or the homotopy analysis method. These methods promise enhanced efficiency and accuracy, potentially revolutionizing problem-solving in complex dynamics.
- Moreover, refining theoretical models by incorporating real-world data from relevant domains can bolster the validity and practical applicability of findings, bridging the gap between theory and reality.
- Embracing interdisciplinary applications of fractional calculus principles beyond virus transmission dynamics can yield remarkable insights and contribute to solutions in various fields.
- Collaborating with policymakers, epidemiologists, and cybersecurity experts can translate research insights into practical strategies, forming proactive measures in public health and cybersecurity.
- A rigorous investigation into the long-term effects of pro-

posed strategies and interventions is crucial for establishing sustainable and resilient systems, while addressing potential risks and ethical implications is imperative.

- Promoting fractional calculus concepts and methodologies in educational curricula can nurture a skilled workforce capable of addressing real-world challenges effectively.
- Ensuring robustness and reproducibility through the replication and validation of study results by fellow researchers is essential for scientific progress.
- Lastly, fostering international collaboration among researchers from diverse backgrounds can enrich the field with varied perspectives and expertise, advancing understanding and practical impact.
- In conclusion, the innovative utilization of Caputo fractional order derivatives and the Laplace Adomian decomposition method in the current study opens up exciting future research avenues. By exploring alternative derivatives, numerical methods, and real-world applications, we have the potential to revolutionize strategies for managing complex transmission dynamics in various domains, leaving behind a trail of possibilities for future exploration.

Author Contributions. **Yunus, A. O.:** Conceptualization, methodology, formal analysis, software, investigation, writing—review and editing, original draft preparation. **Olayiwola, M. O.:** Methodology, investigation software, writing—review and editing, visualization, supervision. **Ajileye, A. M.:** Conceptualization, investigation, writing—review and editing.

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Data availability. Not applicable.

References

[1] M. Farman *et al.*, "*Modeling and analysis of computer virus fractional order model*," in: Methods of Mathematical Modelling, H. Singh, H. M. Srivastava, and D. Baleanu, Eds. Academic Press, 2022, pp. 137–157. ISBN:978-0-323-99888-8. DOI:10.1016/B978-0-323-99888-8.00010-3

[2] P. M. Beach *et al.*, "Analysis of Systems Security Engineering Design Principles for the Development of Secure and Resilient Systems," *IEEE Access*, Vol. 7, pp. 101741–101757, 2019. DOI:10.1109/ACCESS.2019.2930718

[3] E. Balcı, İ. Öztürk, and S. Kartal, "Dynamical behaviour of fractional order tumor model with Caputo and conformable fractional derivative," *Chaos, Solitons & Fractals*, Vol. 123, pp. 43–51, 2019. DOI:10.1016/j.chaos.2019.03.032

[4] A. D. Lazarov, "Mathematical Modelling of Malware Intrusion in Computer Networks," *Cybernetics and Information Technologies*, Vol. 22, no. 3, pp. 29–47, 2022. DOI:10.2478/cait-2022-0026

[5] A. O. Yunus and M. O. Olayiwola, "Mathematical modeling of malaria epidemic dynamics with enlightenment and therapy intervention using the Laplace-Adomian decomposition method and Caputo fractional order," *Franklin Open*, Vol. 8, p. 100147, 2024. DOI:10.1016/j.fraope.2024.100147

[6] T. Korkiatsakul, S. Koonprasert, and K. Neamprem, "Analytical Solutions of Computer Virus Propagation Model with Anti-virus Software and Time Dependent Connecting Network in Caputo Fractional Derivative Sense," *International Journal of Engineering Research and Technology*, Vol. 12, no. 12, pp. 3006–3017, 2019. ISSN:0974-3154

[7] A. K. Chakraborty, P. Shahrear, and Md. A. Islam, "Analysis Of Epidemic Model By Differential Transform Method," *Journal of Multidisciplinary Engineering Science and Technology*, Vol. 4, no. 2, pp. 6574–6581, 2017. ISSN:2458-9403

[8] A. O. Oladapo *et al.*, "Optimal control analysis on mathematical model of dynamical transmission of hiv-malaria co-infection," *Journal of Southwest Jiaotong University*, Vol. 58, no. 1, 2023. DOI:10.35741/issn.0258-2724.58.1.44

[9] A. O. Yunus and M. A. Omoloye, "Mathematical Analysis of Efficacy of Condom as a Contraceptive on the Transmission of Chlamydia Disease," *International Journal of Computer Science and Mobile Applications*, Vol. 10, no. 2, pp. 22–37, 2023. DOI:10.47760/ijcsma.2022.v10i02.002

[10] H. Panigoro and E. Rahmi, "Global stability of a fractional-order logistic growth model with infectious disease," *Jambura Journal of Biomathematics (JJBM)*, Vol. 1, no. 2, pp. 49–56, 2020. DOI:10.34312/jjbm.v1i2.8135

[11] R. R. Musafir *et al.*, "Comparison of Fractional-Order Monkeypox Model with Singular and Non-Singular Kernels," *Jambura Journal of Biomathematics (JJBM)*, Vol. 5, no. 1, pp. 1–9, 2024. DOI:10.37905/jjbm.v5i1.24920

[12] F. Haq *et al.*, "Numerical solution of fractional order smoking model via laplace Adomian decomposition method," *Alexandria Engineering Journal*, Vol. 57, no. 2, pp. 1061–1069, 2018. DOI:10.1016/j.aej.2017.02.015

[13] A. O. Yunus *et al.*, "A fractional order model of Lassa disease using the Laplace-Adomian Decomposition Method," *Healthcare Analytics*, Vol. 3, p. 100167, 2023. DOI:10.1016/j.health.2023.100167

[14] A. I. Alaje and M. O. Olayiwola, "A fractional-order mathematical model for examining the spatiotemporal spread of COVID-19 in the presence of vaccine distribution," *Healthcare Analytics*, Vol. 4, p. 100230, 2023. DOI:10.1016/j.health.2023.100230

[15] M. O. Olayiwola *et al.*, "A caputo fractional order epidemic model for evaluating the effectiveness of high-risk quarantine and vaccination strategies on the spread of COVID-19," *Healthcare Analytics*, Vol. 3, p. 100179, 2023. DOI:10.1016/j.health.2023.100179

[16] M. Sinan, "Analytic Approximate Solution of Rabies Transmission Dynamics using Homotopy Perturbation Method," *Matrix Science Mathematic*, Vol. 4, no. 1, pp. 01–05, 2020. DOI:10.26480/msmk.01.2020.01.05

[17] M. O. Olayiwola, A. I. Alaje, and A. O. Yunus, "A caputo fractional order financial mathematical model analyzing the impact of an adaptive minimum interest rate and maximum investment demand," *Results in Control and Optimization*, Vol. 14, p. 100349, 2024. DOI:10.1016/j.rico.2023.100349

[18] A. O. Yunus and M. O. Olayiwola, "The analysis of a novel COVID-19 model with the fractional-order incorporating the impact of the vaccination campaign in Nigeria via the Laplace-Adomian Decomposition Method," *Journal of the Nigerian Society of Physical Sciences*, Vol. 6, no. 2, p. 1830, 2024. DOI:10.46481/jnsps.2024.1830

[19] K. O. Kareem *et al.*, "On the Solution of Volterra Integro-differential Equations using a Modified Adomian Decomposition Method," *Jambura Journal of Mathematics*, Vol. 5, no. 2, pp. 265–277, 2023. DOI:10.34312/jjom.v5i2.19029

[20] M. O. Olayiwola and A. O. Yunus, "Mathematical analysis of a within-host dengue virus dynamics model with adaptive immunity using Caputo fractional-order derivatives," *Journal of Umm Al-Qura University for Applied Sciences*, pp. 1–20, 2024. DOI:10.1007/s43994-024-00151-z

[21] A. M. S. Mahdy, M. Higazy, and M. S. Mohamed, "Optimal and Memristor-Based Control of A Nonlinear Fractional Tumor-Immune Model," *Computers, Materials and Continua*, vol. 67, no. 3, pp. 3463–3486, 2021. DOI:10.32604/cmc.2021.015161

[22] A. O. Yunus *et al.*, "Mathematical analysis of fractional-order Caputo's derivative of coronavirus disease model via Laplace Adomian decomposition method," *Beni-Suef University Journal of Basic and Applied Sciences*, vol. 11, no. 1, p. 144, 2022. DOI:10.1186/s43088-022-00326-9

[23] A. Mahata *et al.*, "Dynamics of Caputo Fractional Order SEIRV Epidemic Model with Optimal Control and Stability Analysis," *International Journal of Applied and Computational Mathematics*, vol. 8, no. 1, p. 28, 2022. DOI:10.1007/s40819-021-01224-x

[24] A. O. Yunus and M. O. Olayiwola, "The Analysis of a Co-Dynamic Ebola and Malaria Transmission Model Using the Laplace Adomian Decomposition Method with Caputo Fractional-Order," *Tanzania Journal of Science*, vol. 50, no. 2, pp. 224–243, 2024. DOI:10.4314/tjs.v50i2.5

[25] K. Shah, F. Jarad, and T. Abdeljawad, "On a nonlinear fractional order model of dengue fever disease under Caputo-Fabrizio derivative," *Alexandria Engineering Journal*, vol. 59, no. 4, pp. 2305–2313, 2020. DOI:10.1016/j.aej.2020.02.022

[26] M. M. El-Dessoky and M. A. Khan, "Modeling and analysis of an epidemic model with fractal-fractional Atangana-Baleanu derivative,"

Alexandria Engineering Journal, vol. 61, no. 1, pp. 729–746, 2022. DOI:10.1016/j.aej.2021.04.103

[27] M. Farman *et al.*, "Fractal fractional-order derivative for HIV/AIDS model with Mittag-Leffler kernel," *Alexandria Engineering Journal*, vol. 61, no. 12, pp. 10965–10980, 2022. DOI:10.1016/j.aej.2022.04.030

[28] R. P. Agarwal *et al.*, "Fractional calculus and fractional differential equations in nonreflexive Banach spaces," *Communications in Nonlinear Science and Numerical Simulation*, vol. 20, no. 1, pp. 59–73, 2015. DOI:10.1016/j.cnsns.2013.10.010

[29] A. M. S. Mahdy, K. Lotfy, and A. A. El-Bary, "Use of optimal control in studying the dynamical behaviors of fractional financial awareness models," *Soft Computing*, vol. 26, no. 7, pp. 3401–3409. 2022. DOI:10.1007/s00500-022-06764-y

[30] G. M. Bahaa, "Fractional optimal control problem for variable-order differential systems," *Fractional Calculus and Applied Analysis*, vol. 20, no. 6, pp. 1447–1470, 2017. DOI:10.1515/fca-2017-0076

[31] S. Chakraverty, R. M. Jena, and S. K. Jena, "*Time-Fractional Model of HIV-I Infection of CD4^{+} T Lymphocyte Cells in Uncertain Environment*," in: Time-Fractional Order Biological Systems with Uncertain Parameters. Synthesis Lectures on Mathematics & Statistics. Springer Cham, 2020, pp. 75–104. ISBN:978-3-031-02423-8. DOI:10.1007/978-3-031-02423-8_6

[32] G. M. Bahaa, "Fractional optimal control problem for variable-order differential systems," *Fractional Calculus and Applied Analysis*, vol. 20, no. 6, pp. 1447—1470, 2017. DOI:10.1515/fca-2017-0076

[33] Y. A. Amer *et al.*, "Laplace transform method for solving nonlinear biochemical reaction model and nonlinear Emden Fowler system," *Journal of Engineering and Applied Sciences*, vol. 13, no. 17, pp. 7388-7394, 2018. DOI:10.3923/jeasci.2018.7388.7394

[34] S. Hasan *et al.*, "Solution of Fractional SIR Epidemic Model Using Residual Power Series Method," *Applied Mathematics & Information Sciences*, vol. 13, no. 2, pp. 153-161, 2019. DOI:10.18576/amis/130202

[35] G. M. Bahaa, "Fractional optimal control problem for differential system with delay argument," *Advances in Difference Equations*, vol. 2017, no. 1, p. 69, 2017. DOI:10.1186/s13662-017-1121-6

[36] G. M. Bahaa, "Fractional optimal control problem for variational inequalities with control constraints," *IMA Journal of Mathematical Control and Information*, vol. 35, no. 1, pp. 107–122, 2016. DOI:10.1093/imamci/dnw040

[37] A. Abdelrazec and D. Pelinovsky, "Convergence of the Adomian decomposition method for initial-value problems," *Numerical Methods for Partial Differential Equations*, vol. 27, no. 4, pp. 749–766, 2011. DOI:10.1002/num.20549

[38] S. Bhatter *et al.*, "Determining glucose supply in blood using the incomplete I-function," *Partial Differential Equations in Applied Mathematics*, vol. 10, p. 100729, 2024. DOI:10.1016/j.padiff.2024.100729

[39] S. Kumawat, "Comparative implementation of fractional blood alcohol model by numerical approach," *Critical Reviews in Biomedical Engineering*, vol. 53, no. 2, pp. 11–19, 2024. DOI:10.1615/CritRevBiomedEng.2024055033

[40] Shyamsunder, S. D. Purohit, and D. L. Suthar, "A novel investigation of the influence of vaccination on pneumonia disease," *International Journal of Biomathematics*, p. 2450080, 2024. DOI:10.1142/S1793524524500803

[41] M. Meena *et al.*, "A novel fractionalized investigation of tuberculosis disease," *Applied Mathematics in Science and Engineering*, vol. 32, no. 1, 2024. DOI:10.1080/27690911.2024.2351229

[42] Shyamsunder and S. D. Purohit, "A novel study of the impact of vaccination on pneumonia via fractional approach," *Partial Differential Equations in Applied Mathematics*, vol. 10, p. 100698, 2024. DOI:10.1016/j.padiff.2024.100698

[43] S. Bhatter *et al.*, "A study of incomplete I-functions relating to certain fractional integral operators," *Applied Mathematics in Science and Engineering*, vol. 31, no. 1, 2023. DOI:10.1080/27690911.2023.2252996

[44] M. Meena, M. Purohit, and Shyamsunder, "Mathematical analysis using fractional operator to study the dynamics of dengue fever," *Physica Scripta*, vol. 99, no. 9, p. 095206, 2024. DOI:10.1088/1402-4896/ad671b