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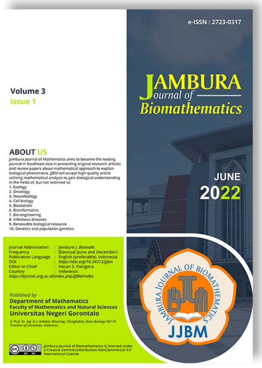
Volume 5, Issue 2, Pages 71–82, December 2024

Received 9 September 2024, Revised 4 November 2024, Accepted 14 November 2024, Published Online 26 December 2024

To Cite this Article : D. Mukherjee,“Complex dynamics in a discrete-time model of two competing prey with a shared predator”, *Jambura J. Biomath*, vol. 5, no. 2, pp. 71–82, 2024, *https://doi.org/10.37905/jjbm.v5i2.27453*

JOURNAL INFO • JAMBURA JOURNAL OF BIOMATHEMATICS



Homepage	:	http://ejurnal.ung.ac.id/index.php/JJBM/index
Journal Abbreviation	:	Jambura J. Biomath.
Frequency	:	Biannual (June and December)
Publication Language	:	English
DOI	:	https://doi.org/10.37905/jjbm
Online ISSN	:	2723-0317
Editor-in-Chief	:	Hasan S. Panigoro
Publisher	:	Department of Mathematics, Universitas Negeri Gorontalo
Country	:	Indonesia
OAI Address	:	http://ejurnal.ung.ac.id/index.php/jjbm/oai
Google Scholar ID	:	XzYgeKQAAAAJ
Email	:	editorial.jjbm@ung.ac.id

Jambura Journal of Biomathematics, Volume 5, Issue 2, Pages 71–82, December 2024 https://doi.org/10.37905/jjbm.v5i2.27453

Research Article

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Complex dynamics in a discrete-time model of two competing prey with a shared predator

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ARTICLE HISTORY
Received 9 September 2024
Revised 4 November 2024
Accepted 14 November 2024
Published 26 December 2024

KEYWORDS
Predator-prey model
Stability
Bifurcation
Chaos Control

ABSTRACT. *This paper concentrates on the study of a discrete time model of two competing prey with a shared predator. The condition for the existence and local stability of positive fixed point are derived. By using an iteration scheme and the comparison principle of difference equations, it is possible to obtain the sufficient condition for global stability of the positive fixed point. The sufficient criterion for Neimark-Sacker bifurcation and flip bifurcation are established. The system admits chaotic dynamics for a certain choice of the system parameters which is controlled by applying hybrid control method. The intra-specific competition among predators and the intrinsic growth rate of prey species have major impact for different bifurcation.*
For continuous system, handling time spent for prey population plays an important role for obtaining limit cycle behaviour. The decrease amount of this rate makes the system stable. Global convergence of the solutions to the coexistence equilibrium point is possible for a particular choice of system parameters. The obtained results for discrete system are verified through numerical simulations. Also some diagrams are presented for continuous system.

1. Introduction

In ecology, predation and competition are two important inter-specific interactions. These two factors motivated some researchers to develop mathematical models for investigating the predation effect on the competitive interactions [1–5]. It is mostly observed that predator attacks weak competitor. In that case, strong competitor plays a significant role in structuring communities. Also, the species coexistence in predator-prey system depends on the choice of functional response. Initially, Lotka and Volterra first showed in predator-prey model after that numerous problems have been studied by considering different types of functional response [6]. In [7], the authors observed chaotic dynamics in two prey- one predator model with a shared predator. The effect of delay maintaining the persistence of two preys and one predator system is studied in [8]. Tripathi et al. [9] studied the local and global behaviour of a two prey and one predator system with help. Bhattacharya and Pal [10] analysed a delay induced two prey and one predator system with Beddington DeAngelis response function. Khalif and Majeed [11] investigated the dynamical behaviour of two prey and one predator mode. Most of the previous studies mentioned above are done on continuous models. However, some literatures indicate that discrete models are more reasonable than continuous systems when the population have non-overlapping generations. In nature, mathematical model for discrete system can be observed in fish populations which reproduce at specific timed moments or for insect populations where non-overlapping generations always take place.

Discrete-time models also permits additional efficient computational outcomes for numerical computations and reveal a rich dynamics in respect of continuous systems [12–15]. One can observe chaotic behaviour in discrete models [12, 13]. Several works on discrete prey-predator models may be found in [16–27].

Most of the previous studies are dealt with into two species. But the dynamics will be more complex when multiple species occur. In this paper, we propose a discrete-time two competing prey with a shared predator. We study the existence and local as well as global stability of the positive fixed point. After then, we identify the system parameters that give Neimark-Sacker bifurcation and flip bifurcation. Chaos control of the system will be investigated.

The model studied in [8], is modified in the following form:

$$\begin{aligned} \frac{dx}{dt} &= x\left(r_1 - a_{11}x - a_{12}y - \frac{p_1\lambda_1 z}{1+p_1h_1\lambda_1 x + p_2h_2\lambda_2 y}\right), \\ \frac{dy}{dt} &= y\left(r_2 - a_{21}x - a_{22}y - \frac{p_2\lambda_2 z}{1+p_1h_1\lambda_1 x + p_2h_2\lambda_2 y}\right), \\ \frac{dz}{dt} &= z\left(-d + \frac{p_1e_1\lambda_1 x + p_2e_2\lambda_2 y}{1+p_1h_1\lambda_1 x + p_2h_2\lambda_2 y} - hz\right). \end{aligned} \tag{1}$$

Here x and y represent the densities of two competing prey, and z stands for the density of the shared predator. When there is no predator, each prey grows logistically. Here r_1 and r_2 denote the intrinsic growth rate of the two competing prey species. a_{11} and a_{22} represent the intra-specific competition among the prey species x and y respectively. a_{12} and a_{21} are the inter-specific competition. Inter-specific competition is a controlling force in

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Homepage : http://ejurnal.ung.ac.id/index.php/JJBM/index / E-ISSN : 2723-0317

interacting species that induces niche shifts in ecological and evolutionary time. The predation process follows Holling type II response function. p_1 and p_2 represent the probability that a predator will attack prey x and y upon an encounter respectively, λ_1 and λ_2 are the search rate of a predator for the prey x and y respectively. e_1 and e_2 are the expected net energy gained from the prey x and y, respectively, and h_1 and h_2 are the expected handling time spent with the prey x and y respectively. h stands for intra-specific competition among the predator species. d is the death rate of the predator species. Specific example illustrates the above situation. In real world, zebras and cattle are two competing species both of them are shared by lions. Here, all the parameters are assumed to be positive.

For qualitative analysis, including global stability, bifurcation analysis and chaos control for a discrete analogue system (1), a piecewise constant argument is introduced to describe the following exponential form of non-linear difference equations:

$$\begin{aligned} x_{n+1} &= x_n \exp\left\{r_1 - \frac{p_1\lambda_1 z_n}{1+p_1h_1\lambda_1x_n+p_2h_2\lambda_2y_n} - a_{11}x_n - a_{12}y_n\right\},\\ y_{n+1} &= y_n \exp\left\{r_2 - a_{21}x_n - a_{22}y_n - \frac{p_2\lambda_2 z_n}{1+p_1h_1\lambda_1x_n+p_2h_2\lambda_2y_n}\right\},\\ z_{n+1} &= z_n \exp\left\{-d + \frac{p_1e_1\lambda_1x_n+p_2e_2\lambda_2y_n}{1+p_1h_1\lambda_1x_n+p_2h_2\lambda_2y_n} - hz_n\right\}, \end{aligned} \tag{2}$$

where x_n and y_n represent the densities of two competing preys at generation $n \in \mathbb{N}$ and z_n denotes predator density at a discrete time step $n \in \mathbb{N}$.

The rest of the paper is structured as follows: The existence and stability of the interior fixed point are discussed in Section 2. Global stability criterion is derived in Section 3. Neimark-Sacker bifurcation and flip bifurcation are described in Section 4. Chaos control mechanism is presented in Section 5. Numerical examples are given in Section 6. Section 7 concludes the paper.

2. Existence of interior fixed point

Clearly, system (2) has six boundary fixed points $E_0 = (0,0,0)$ and $E_1 = (\frac{r_1}{a_{11}},0,0)$, $E_2 = (0,\frac{r_2}{a_{22}},0)$, $E_{12} = (\bar{x},\bar{y},0)$ where $\bar{x} = \frac{r_1a_{22}-r_2a_{12}}{a_{11}a_{22}-a_{12}a_{21}}$, $\bar{y} = \frac{r_2a_{11}-r_1a_{21}}{a_{11}a_{22}-a_{12}a_{21}}$ provided, either $\frac{a_{11}}{a_{21}} > \frac{r_1}{r_2} > \frac{a_{12}}{a_{22}}$ or $\frac{a_{11}}{a_{21}} < \frac{r_1}{r_2} < \frac{a_{12}}{a_{22}}$.

There exits a unique second prey free fixed point $E_{13}(\hat{x},0,\hat{z})$ where $\hat{x}$ is a positive root of the equation

$$\begin{aligned} &B_1x^3 + B_2x^2 + B_3x - B_4 = 0,\\ B_1 &= a_{11}hp_1^2h_1^2\lambda_1^2,\\ B_2 &= hh_1p_1\lambda_1(2a_{11} - r_1p_1h_1\lambda_1),\\ B_3 &= ha_{11} - 2hr_1p_1\lambda_1 + p_1^2\lambda_1^2e_1 - dp_1^2\lambda_1h_1,\\ B_4 &= hr_1 - dp_1\lambda_1, \end{aligned}$$

and $\hat{z} = \frac{(r_1-a_{11}\hat{x})(1+p_1h_1\lambda_1\hat{x})}{p_1\lambda_1}$ provided $r_1 > a_{11}\hat{x}$.

There exists a unique first prey free fixed point $E_{23} = (0,\tilde{y},\tilde{z})$ where $\tilde{y}$ is a positive root of the equation

$$\begin{aligned} &C_1y^3 + C_2y^2 + C_3y - C_4 = 0,\\ C_1 &= a_{22}hp_2^2h_2^2\lambda_2^2,\\ C_2 &= hh_2p_2\lambda_2(2a_{11} - r_1p_1h_1\lambda_1),\\ C_3 &= ha_{11} - 2hr_1p_2\lambda_2 + p_2^2\lambda_2^2e_2 - dp_2^2\lambda_2h_2,\\ C_4 &= hr_2 - dp_2\lambda_2, \end{aligned}$$

and $\tilde{z} = \frac{(r_2-a_{22}\tilde{y})(1+p_2h_2\lambda_2\tilde{y})}{p_2\lambda_2}$ provided $r_2 > a_{22}\tilde{y}$.

To locate the interior fixed point $E^* = (x^*,y^*,z^*)$ of system (2), we apply isocline method. x^*, y^* and z^* are the positive solutions of the following system of equations:

$$r_1 - a_{11}x - a_{12}y - \frac{p_1\lambda_1 z}{1+p_1h_1\lambda_1x+p_2h_2\lambda_2y} = 0, \tag{3}$$

$$r_2 - a_{21}x - a_{22}y - \frac{p_2\lambda_2 z}{1+p_1h_1\lambda_1x+p_2h_2\lambda_2y} = 0, \tag{4}$$

$$-d + \frac{p_1e_1\lambda_1x+p_2e_2\lambda_2y}{1+p_1h_1\lambda_1x+p_2h_2\lambda_2y} - hz = 0. \tag{5}$$

From equation eq. (3), we get

$$z = \frac{1}{h}\left\{-d + \frac{p_1e_1\lambda_1x+p_2e_2\lambda_2y}{1+p_1h_1\lambda_1x+p_2h_2\lambda_2y}\right\} = z_e(\text{say}).$$

$z > 0$ if $p_1\lambda_1e_1x + p_2\lambda_2e_2y > d(1+p_1h_1\lambda_1x+p_2\lambda_2h_2)$. Now, we substitute the value of z in eqs. (3) and (4) and obtained

$$f_1(x,y) = r_1 - a_{11}x - a_{12}y - \frac{p_1\lambda_1 z_e}{1+p_1h_1\lambda_1x+p_2h_2\lambda_2y} = 0, \tag{6}$$

$$f_2(x,y) = r_2 - a_{21}x - a_{22}y - \frac{p_2\lambda_2 z_e}{1+p_1h_1\lambda_1x+p_2h_2\lambda_2y} = 0. \tag{7}$$

Now, we have

$$\frac{dx}{dy} = -\frac{\partial f_1}{\partial y} \Big/ \frac{\partial f_1}{\partial x} = -\frac{M_1}{N_1},$$

where

$$\begin{aligned} M_1 &= -a_{12} - \frac{p_1\lambda_1(z_{ey}M - p_2h_2\lambda_2z_e)}{M^2},\\ N_1 &= -a_{11} - \frac{p_1\lambda_1(z_{ex}M - p_1h_1\lambda_1z_e)}{M^2},\\ M &= 1 + p_1h_1\lambda_1x + p_2h_2\lambda_2y. \end{aligned}$$

It is evident that $\frac{dx}{dy} > 0$ if either (i) $M_1 > 0$ and $N_1 < 0$ or (ii) $M_1 < 0$ and $N_1 > 0$ hold. Also we get

$$\frac{dx}{dy} = -\frac{\partial f_2}{\partial y} \Big/ \frac{\partial f_2}{\partial x} = -\frac{M_2}{N_2},$$

where

$$\begin{aligned} M_2 &= -a_{22} - \frac{p_2\lambda_2(z_{ey}M - p_2h_2\lambda_2z_e)}{M^2},\\ N_2 &= -a_{21} - \frac{p_2\lambda_2(z_{ex}M - p_1h_1\lambda_1z_e)}{M^2},\\ M &= 1 + p_1h_1\lambda_1x + p_2h_2\lambda_2y. \end{aligned}$$

We note that $\frac{dx}{dy} < 0$ if either (i) $M_2 > 0$ and $N_2 > 0$ or (ii) $M_2 < 0$ and $N_2 < 0$ hold. From the above analysis, we conclude that the two isoclines eqs. (6) and (7) intersect at the point (x^*,y^*) under certain restrictions. Throughout this work, we assume that E^* exists.

2.1. Stability of interior fixed point

Let $E^* = (x^*, y^*, z^*)$ be the interior fixed point of system (2). The Jacobian matrix for system (2) at E^* is given by:

$$J(x^*, y^*, z^*) = \begin{pmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{pmatrix},$$

$$\begin{aligned}
j_{11} &= 1 - x^* \left(a_{11} - \frac{p_1^2 \lambda_1^2 h_1 z^*}{M^2} \right), \quad j_{13} = -\frac{p_1 \lambda_1 x^*}{M}, \\
j_{12} &= x^* \left(a_{12} - \frac{p_1 \lambda_1 p_2 \lambda_2 h_2 z^*}{M^2} \right), \quad j_{23} = -\frac{p_2 \lambda_2 y^*}{M}, \\
j_{21} &= -y^* \left(a_{21} - \frac{p_1 \lambda_1 p_2 \lambda_2 h_1 z^*}{M^2} \right), \quad j_{33} = 1 - hz^*, \\
j_{22} &= 1 - y^* \left(a_{22} - \frac{p_2^2 \lambda_2^2 h_2 z^*}{M^2} \right), \\
j_{31} &= \frac{z^* \{ p_1 e_1 \lambda_1 + p_1 \lambda_1 p_2 \lambda_2 y^* (e_1 h_2 - e_2 h_1) \}}{M^2}, \\
j_{32} &= \frac{z^* \{ p_2 e_2 \lambda_2 + p_1 \lambda_1 p_2 \lambda_2 x^* (e_2 h_1 - e_1 h_2) \}}{M^2}.
\end{aligned}$$

The characteristic polynomial of $J(E^*)$ is given by

$$Q(\lambda) = \lambda^3 + q_1 \lambda^2 + q_2 \lambda + q_3 \tag{8}$$

where

$$\begin{aligned}
q_1 &= x^* \left(a_{11} - \frac{p_1^2 \lambda_1^2 h_1 z^*}{M^2} \right) + y^* \left(a_{22} - \frac{p_2^2 \lambda_2^2 h_2 z^*}{M^2} \right) + hz^* - 3, \\
q_2 &= (1 - hz^*) \left\{ 1 - y^* \left(a_{22} - \frac{p_2^2 \lambda_2^2 h_2 z^*}{M^2} \right) \right\} + \left\{ 1 - x^* \left(a_{11} \right.\right. \\
&\quad \left.\left. - \frac{p_1^2 \lambda_1^2 h_1 z^*}{M^2} \right) \right\} \left\{ 2 - y^* \left(a_{22} - \frac{p_2^2 \lambda_2^2 h_2 z^*}{M^2} \right) - hz^* \right\} \\
&\quad - x^* y^* \left(a_{12} - \frac{p_1 \lambda_1 p_2 \lambda_2 h_2 z^*}{M^2} \right) \left(a_{21} - \frac{p_1 \lambda_1 p_2 \lambda_2 h_1 z^*}{M^2} \right) \\
&\quad + \frac{p_1 \lambda_1 x^* z^*}{M^3} \{ p_1 e_1 \lambda_1 + p_1 \lambda_1 p_2 \lambda_2 y^* (e_1 h_2 - e_2 h_1) \}, \\
q_3 &= \left(1 - x^* \left[a_{11} - \frac{p_1^2 \lambda_1^2 h_1 z^*}{M^2} \right] \right) \left(1 - y^* \left[a_{22} - \frac{p_2^2 \lambda_2^2 h_2 z^*}{M^2} \right] (hz^* \right. \\
&\quad \left. - 1) \right) - x^* \left(a_{12} - \frac{p_1 \lambda_1 p_2 \lambda_2 h_2 z^*}{M^2} \left[y^* (hz^* - 1) \left(a_{21} \right.\right.\right. \\
&\quad \left.\left. - \frac{p_1 \lambda_1 p_2 \lambda_2 h_1 z^*}{M^2} \right) + \frac{p_2 \lambda_2 y^* z^*}{M^3} \{ p_1 e_1 \lambda_1 + p_1 \lambda_1 p_2 \lambda_2 y^* (e_1 h_2 \right. \\
&\quad \left.\left. - e_2 h_1) \} \right] \right) - \frac{p_1 \lambda_1 x^*}{M} \left[y^* z^* \left(a_{11} - \frac{p_1 \lambda_1 p_2 \lambda_2 h_1 z^*}{M^2} \right) \left(\frac{p_2 e_2 \lambda_2}{M^2} \right.\right. \\
&\quad \left. + \frac{p_1 \lambda_1 p_2 \lambda_2 x^* (e_2 h_1 - e_1 h_2)}{M^2} \right) + z^* \left\{ 1 - y^* \left(a_{22} \right.\right. \\
&\quad \left.\left.\left. - \frac{p_2^2 \lambda_2^2 h_2 z^*}{M^2} \right) \right\} \left\{ \frac{p_1 e_1 \lambda_1 + p_1 \lambda_1 p_2 \lambda_2 y^* (e_1 h_2 - e_2 h_1)}{M^2} \right\} \right], \\
M &= 1 + p_1 h_1 \lambda_1 x^* + p_2 h_2 \lambda_2 y^*.
\end{aligned} \tag{9}$$

To determine local stability of the interior fixed point E^*, we require the following lemma.

Lemma 1 ([28]). *Consider the cubic equation*

$$\lambda^3 + q_1 \lambda^2 + q_2 \lambda + q_3 = 0$$

where q_1, q_2 and q_3 are real numbers. Then necessary and sufficient conditions that all the roots of eq. (8) lie in an open disk $|\lambda| < 1$ are:
(i). $|q_1 + q_3| < 1 + q_2$,
(ii). $|q_1 - 3q_3| < 3 - q_2$, and
(iii). $q_3^2 + q_2 - q_3 q_1 < 1$.
We now use Lemma 1 to investigate stability of E^*.

Lemma 2. *E^* is locally asymptotically stable if and only if the following conditions are satisfied:*
(i). $|q_1 + q_3| < 1 + q_2$,
(ii). $|q_1 - 3q_3| < 3 - q_2$, and
(iii). $q_3^2 + q_2 - q_3 q_1 < 1$.
where q_1, q_2 and q_3 are defined in eq. (9).

3. Global stability

In this section, we will utilize the process of iteration scheme and the comparison principle of difference equation to investigate the global stability of the positive fixed point of system (2). To establish global stability result, we require the following lemmas.

Lemma 3 ([29]). *Let $f(u) = u\exp(\delta - \eta u)$, where δ and η are positive constants. Then $f(u)$ is non-decreasing for $u \in \left(0, \frac{1}{\eta}\right]$.*

Lemma 4 ([29]). *Assume that the sequence u_n satisfies*

$$u_{n+1} = u_n \exp(\delta - \eta u_n), \ n = 1, 2, 3, ...,$$

where δ and η are positive constants and $u_0 > 0$. Then:
(i). If $\delta < 2$, then $\lim_{n\to\infty} u_n = \frac{\delta}{\eta}$.
(ii). If $\delta \le 1$, then $u_n \le \frac{1}{\eta}$, $n = 2, 3, ...$

Lemma 5 ([30]). *Suppose that functions $f, g : \mathbb{Z}_+ \times [0, \infty)$ satisfy $f(n, x) \le g(n, x)$ $(f(n, x) \ge g(n, x))$ for $n \in \mathbb{Z}_+$ and $g(n, x)$ is non-decreasing with respect to x. If u_n are the non-negative solutions of the difference equations $x_{n+1} = f(n, x_n)$, $u_{n+1} = g(n, u_n)$ respectively, and $x_0 \le u_0$ $(x_0 \ge u_0)$ then $x_n \le u_n$ $(x_n \ge u_n)$ for all $n \ge 0$.*

Theorem 1. *Assume that $r_1 \le 1, r_2 \le 1$ and $p_1 e_1 \lambda_1 r_1 a_{22} + p_2 e_2 \lambda_2 r_2 a_{11} < a_{11} a_{22}(1 + d)$ then the fixed point $E^*(x^*, y^*, z^*)$ of system (2) is globally asymptotically stable.*

Proof. Assume that (x_n, y_n, z_n) is any solution of system (2) with initial values $x_0 > 0, y_0 > 0, z_0 > 0$. Let

$$U_1 = \limsup_{n\to\infty} x_n, \ V_1 = \liminf_{n\to\infty} x_n,$$

$$U_2 = \limsup_{n\to\infty} y_n,\ V_2 = \liminf_{n\to\infty} y_n,$$
$$U_3 = \limsup_{n\to\infty} z_n,\ V_3 = \liminf_{n\to\infty} z_n.$$

In the following, we will prove that $U_1 = V_1 = x^*, U_2 = V_2 = y^*, U_3 = V_3 = z^*$. First we show that $U_1 \le M_1^x$, $U_2 \le M_1^y$, $U_3 \le M_1^z$. From the first equation of system (2), we find

$$x_{n+1} \le x_n \exp(r_1 - a_{11}x_n),\ n = 0, 1, 2, \ldots.$$

Considering the auxiliary equation

$$u_{n+1} = u_n \exp(r_1 - a_{11}u_n), \tag{10}$$

by Lemma 4 (ii), because of $r_1 \le 1$, we get $u_n \le \frac{1}{a_{11}}$ for all $n \ge 2$. By Lemma 3, we obtain $f(u) = u\exp(r_1 - a_{11}u)$ is non-decreasing for $u \in (0, \frac{1}{a_{11}}]$. Thus from Lemma 5, we get $x_n \le u_n$ for all $n \ge 2$, where u_n is the solution of eq. (10) with initial value $u_2 = x_2$. By Lemma 4 (i), we get

$$U_1 = \limsup_{n\to\infty} x_n \le \lim_{n\to\infty} u_n = \frac{r_1}{a_{11}} \triangleq M_1^x.$$

Hence, for any $\epsilon > 0$, there exists a $n_1 > 2$ such that if $n \ge n_1$, then $x_n \le M_1^x + \epsilon$. In the same way, using the second equation of system (2), we get,

$$U_2 = \limsup_{n\to\infty} y_n \le \lim_{n\to\infty} u_n = \frac{r_2}{a_{22}} \triangleq M_1^y.$$

Hence, for any $\epsilon > 0$, there exists a $n_2 > n_1$ such that if $n \ge n_2$, then $y_n \le M_1^y + \epsilon$. The third equation of system (2), yields

$$z_{n+1} \le z_n \exp\{-d + p_1e_1\lambda_1(M_1^x + \epsilon) + p_2e_2\lambda_2(M_1^y + \epsilon) - hz_n\}.$$

Again taking the auxiliary equation

$$u_{n+1} = u_n \exp\{-d + p_1e_1\lambda_1(M_1^x + \epsilon) + p_2e_2\lambda_2(M_1^y + \epsilon) - hu_n\},$$

by Lemma 4 (ii), because of

$$p_1e_1\lambda_1(M_1^x + \epsilon) + p_2e_2\lambda_2(M_1^y + \epsilon) \le 1 + d,$$

we get $u_n \le \frac{1}{h}$ for all $n \ge 2$. From Lemma 3, we get

$$f(u) = u \exp\{-d + p_1e_1\lambda_1(M_1^x + \epsilon) + p_2e_2\lambda_2(M_1^y + \epsilon) - hu\}$$

is non-decreasing for $u \in (0, \frac{1}{h}]$. Lemma 5 yields $z_n \le u_n$ for all $n \ge 2$. Consequently

$$U_3 = \limsup_{n\to\infty} z_n \le \lim_{n\to\infty} u_n = \psi_1 \triangleq M_1^z,$$
$$\psi_1 = \frac{-d + p_1e_1\lambda_1(M_1^x + \epsilon) + p_2e_2\lambda_2(M_1^y + \epsilon)}{h}.$$

Thus for given any $\epsilon > 0$, there exists $n_3 > n_2$ such that for $n \ge n_3$, $z_n \le M_1^z + \epsilon$. Now we prove that $V_1 \ge N_1^x$, $V_2 \ge N_1^y$, $V_3 \ge N_1^z$. Using the first equation of system (2), we derive

$$x_{n+1} \ge x_n \exp[r_1 - a_{11}x_n - a_{12}(M_1^y + \epsilon) - p_1\lambda_1(M_1^z + \epsilon)],\ n \ge n_3.$$

Consider the auxiliary equation

$$u_{n+1} = u_n \exp[r_1 - a_{11}u_n - a_{12}(M_1^y + \epsilon) - p_1\lambda_1(M_1^z + \epsilon)].$$

Since we have

$$r_1 - a_{11}u_n - a_{12}(M_1^y + \epsilon) - p_1\lambda_1(M_1^z + \epsilon) < 1,$$

by Lemma 4 (ii), we have $u_n \le \frac{1}{a_{11}}$ for $n \ge n_3$. By Lemma 3, we obtain

$$f(u) = u\exp(r_1 - a_{11}u - a_{12}(M_1^y + \epsilon) - p_1\lambda_1(M_1^z + \epsilon))$$

is non-decreasing for $u \in (0, \frac{1}{a_{11}}]$. Thus from Lemma 5, we get $x_n \ge u_n$ for all $n \ge n_3$. By Lemma 4 (i), we get

$$V_1 = \liminf_{n\to\infty} x_n \ge \lim_{n\to\infty} u_n = \psi_2,$$
$$\psi_2 = \frac{r_1 - a_{12}(M_1^y + \epsilon) - p_1\lambda_1(M_1^z + \epsilon)}{a_{11}}.$$

Since $\epsilon > 0$ is arbitrary, we have $V_1 \ge N_1^x = \psi_2$. So for any $\epsilon > 0$, one can find a $n_4 > n_3$ such that for $n \ge n_4$, $x_n \ge N_1^x - \epsilon$.

Considering the second equation of system (2), we get

$$y_{n+1} \ge y_n \exp[r_2 - a_{22}y_n - a_{21}(M_1^x + \epsilon) - p_2\lambda_2(M_1^z + \epsilon)],\ n \ge n_4.$$

By the same way, we can get

$$V_2 = \liminf_{n\to\infty} y_n \ge \lim_{n\to\infty} u_n = \psi_3,$$
$$\psi_3 = \frac{r_2 - a_{21}(M_1^x + \epsilon) - p_2\lambda_2(M_1^z + \epsilon)}{a_{22}}.$$

As $\epsilon > 0$ is arbitrary, we have $V_2 \ge N_1^y = \psi_3$. So for given any $\epsilon > 0$, there exists $n_5 > n_4$ such that for $n \ge n_5, y_n \ge N_1^y - \epsilon$. Similarly, from the third equation of system (2), we get

$$z_{n+1} \ge z_n \exp\left[\psi_4 - d - hz_n\right], \qquad n \ge n_5$$
$$\psi_4 = \frac{p_1e_1\lambda_1(N_1^x - \epsilon) + p_2e_2\lambda_2(N_1^y - \epsilon)}{1 + p_1h_1\lambda_1(M_1^x + \epsilon) + p_2h_2\lambda_2(M_1^y + \epsilon)}.$$

with

$$V_3 = \liminf_{n\to\infty} z_n \ge \lim_{n\to\infty} u_n = \frac{1}{h}\{\psi_4 - d\}.$$

As $\epsilon > 0$ is arbitrary, we have $V_3 \ge N_1^z = \frac{1}{h}\{\psi_4 - d\}$. So for any given $\epsilon > 0$, there exists $n_6 > n_5$ such that for $n \ge n_6, z_n \ge N_1^z - \epsilon$.

Now we show that $U_1 \le M_2^x$, $U_2 \le M_2^y$, and $U_3 \le M_2^z$, where $M_2^x \le M_1^x$, $M_2^y \le M_1^y$, and $M_2^z \le M_1^z$ respectively. From the first equation of system (2) for $n > n_6$, we get

$$x_{n+1} \le x_n \exp\Bigg[r_1 - a_{11}x_n - a_{12}(N_1^y - \epsilon) - \frac{p_1\lambda_1(N_1^z - \epsilon)}{1 + p_1h_1\lambda_1(M_1^x + \epsilon) + p_2h_2\lambda_2(M_1^y + \epsilon)}\Bigg].$$

Consider the auxiliary equation

$$u_{n+1} = u_n \exp\Bigg[r_1 - a_{11}u_n - a_{12}(N_1^y - \epsilon) - \frac{p_1\lambda_1(N_1^z - \epsilon)}{1 + p_1h_1\lambda_1(M_1^x + \epsilon) + p_2h_2\lambda_2(M_1^y + \epsilon)}\Bigg].$$

Using the same type of argument as in above, we can get

$$U_1 = \limsup_{n\to\infty} x_n \le \frac{1}{a_{11}}\Bigg[r_1 - a_{12}(N_1^y - \epsilon) - \frac{p_1\lambda_1(N_1^z - \epsilon)}{1 + p_1h_1\lambda_1(M_1^x + \epsilon) + p_2h_2\lambda_2(M_1^y + \epsilon)}\Bigg],$$

since

$$r_1 - a_{12}(N_1^y - \epsilon) - \frac{p_1\lambda_1(N_1^z - \epsilon)}{1 + p_1h_1\lambda_1(M_1^x + \epsilon) + p_2h_2\lambda_2(M_1^y + \epsilon)} \leq 1.$$

As $\epsilon > 0$ is arbitrary, we claim that

$$U_1 \leq M_2^x = \frac{1}{a_{11}}\bigg[r_1 - a_{12}(N_1^y - \epsilon) - \frac{p_1\lambda_1(N_1^z - \epsilon)}{1 + p_1h_1\lambda_1(M_1^x + \epsilon) + p_2h_2\lambda_2(M_1^y + \epsilon)}\bigg].$$

Thus for any given $\epsilon > 0$, there exists $n_7 > n_6$ such that for $n \geq n_7$, $x_n \leq M_2^x + \epsilon$. Similarly to the above argument, we get

$$U_2 \leq M_2^y = \frac{1}{a_{22}}\bigg[r_2 - a_{21}(N_1^x - \epsilon) - \frac{p_2\lambda_2(N_1^z - \epsilon)}{1 + p_1h_1\lambda_1(M_1^x + \epsilon) + p_2h_2\lambda_2(M_1^y + \epsilon)}\bigg].$$

So for any $\epsilon > 0$, there exists $n_8 > n_7$ such that for $n \geq n_8$, $y_n \leq M_2^y + \epsilon$. From the third equation of system (2) for $n > n_8$, we get

$$z_{n+1} \leq z_n \exp\bigg[\frac{p_1e_1\lambda_1(M_2^x + \epsilon) + p_2e_2\lambda_2(M_2^y + \epsilon)}{1 + p_1h_1\lambda_1(N_1^x - \epsilon) + p_2h_2\lambda_2(N_1^y - \epsilon)} - d - hz_n\bigg].$$

Similarly to the above argument, we get

$$U_3 \leq M_2^z = \frac{1}{h}\bigg[\frac{p_1e_1\lambda_1(M_2^x + \epsilon) + p_2e_2\lambda_2(M_2^y + \epsilon)}{1 + p_1h_1\lambda_1(N_1^x - \epsilon) + p_2h_2\lambda_2(N_1^y - \epsilon)} - d\bigg].$$

Hence for any sufficiently small $\epsilon > 0$, there exists $n_9 > n_8$ such that for $n \geq n_9$, $z_n \leq M_2^z + \epsilon$.

Now we show that $V_1 \geq N_2^x$, $V_2 \geq N_2^y$, and $V_3 \geq N_2^z$, where $N_2^x \geq N_1^x$, $N_2^y \geq N_1^y$, and $N_2^z \geq N_1^z$ respectively. Further, from the first equation of system (2) for $n > n_9$, we get

$$x_{n+1} \geq x_n \exp\bigg[r_1 - a_{11}x_n - a_{12}(M_2^y + \epsilon) - \frac{p_1\lambda_1(M_2^z + \epsilon)}{1 + p_1h_1\lambda_1(N_1^x - \epsilon) + p_2h_2\lambda_2(N_1^y)}\bigg].$$

Using a similar argument, we get

$$V_1 = \liminf_{n\to\infty} x_n \geq \frac{1}{a_{11}}\bigg[r_1 - a_{12}(M_2^y + \epsilon) - \frac{p_1\lambda_1(M_2^z + \epsilon)}{1 + p_1h_1\lambda_1(N_1^x - \epsilon) + p_2h_2\lambda_2(N_1^y)}\bigg],$$

since

$$r_1 - a_{12}(M_2^y + \epsilon) - \frac{p_1\lambda_1(M_2^z + \epsilon)}{1 + p_1h_1\lambda_1(N_1^x - \epsilon) + p_2h_2\lambda_2(N_1^y)} \leq 1.$$

As $\epsilon > 0$ is arbitrary, we claim that

$$V_1 \geq N_2^x = \frac{1}{a_{11}}\bigg[r_1 - a_{12}(M_2^y + \epsilon) - \frac{p_1\lambda_1(M_2^z + \epsilon)}{1 + p_1h_1\lambda_1(N_1^x - \epsilon) + p_2h_2\lambda_2(N_1^y)}\bigg].$$

So for given $\epsilon > 0$, there exists $n_{10} > n_9$ such that for $n \geq n_{10}$, $x_n \geq N_2^x - \epsilon$. Similarly, from the second equation of system (2) for $n > n_{10}$, we have

$$y_{n+1} \geq y_n \exp\bigg[r_2 - a_{22}y_n - a_{21}(M_2^x + \epsilon) - \frac{p_2\lambda_2(M_2^z + \epsilon)}{1 + p_1h_1\lambda_1(N_1^x - \epsilon) + p_2h_2\lambda_2(N_1^y)}\bigg].$$

with

$$V_2 = \liminf_{n\to\infty} y_n \geq \frac{1}{a_{22}}\bigg[r_2 - a_{21}(M_2^x + \epsilon) \frac{p_2\lambda_2(M_2^z + \epsilon)}{1 + p_1h_1\lambda_1(N_1^x - \epsilon) + p_2h_2\lambda_2(N_1^y)}\bigg].$$

Since $\epsilon > 0$, is arbitrary, we claim that

$$V_2 \geq N_2^y = \frac{1}{a_{22}}\bigg[r_2 - a_{21}(M_2^x + \epsilon) - \frac{p_2\lambda_2(M_2^z + \epsilon)}{1 + p_1h_1\lambda_1(N_1^x - \epsilon) + p_2h_2\lambda_2(N_1^y)}\bigg].$$

Hence for any $\epsilon > 0$, there exists $n_{11} > n_{10}$ such that for $n \geq n_{11}$, $y_n \geq N_2^y - \epsilon$. Similarly, from the third equation of system (2) for $n > n_{11}$, we have

$$z_{n+1} \geq z_n \exp\bigg[\frac{p_1e_1\lambda_1(N_2^x - \epsilon) + p_2e_2\lambda_2(N_2^y - \epsilon)}{1 + p_1h_1\lambda_1(M_2^x + \epsilon) + p_2h_2\lambda_2(M_2^y + \epsilon)} - d - hz_n\bigg].$$

with

$$V_3 = \liminf_{n\to\infty} z_n \geq \frac{1}{h}\bigg[\frac{p_1e_1\lambda_1(N_2^x - \epsilon) + p_2e_2\lambda_2(N_2^y - \epsilon)}{1 + p_1h_1\lambda_1(M_2^x + \epsilon) + p_2h_2\lambda_2(M_2^y + \epsilon)} - d\bigg].$$

Since $\epsilon > 0$ is arbitrary, we conclude that

$$V_3 \geq N_2^z = \frac{1}{h}\bigg[\frac{p_1e_1\lambda_1(N_2^x - \epsilon) + p_2e_2\lambda_2(N_2^y - \epsilon)}{1 + p_1h_1\lambda_1(M_2^x + \epsilon) + p_2h_2\lambda_2(M_2^y + \epsilon)} - d\bigg].$$

So for given $\epsilon > 0$, there exists $n_{12} > n_{11}$ such that for $n \geq n_{12}$, $z_n \geq N_2^x - \epsilon$. Repeating the above process, we ultimately get six sequences $\{M_n^x\}$, $\{M_n^y\}$, $\{M_n^z\}$, $\{N_n^x\}$, $\{N_n^y\}$, and $\{N_n^z\}$ such that for all $n \geq 2$,

$$M_n^x = \frac{1}{a_{11}}\bigg[r_1 - a_{12}(N_{n-1}^y - \epsilon) - \frac{p_1\lambda_1(N_{n-1}^z - \epsilon)}{1 + p_1h_1\lambda_1(M_{n-1}^x + \epsilon) + p_2h_2\lambda_2(M_{n-1}^y + \epsilon)}\bigg],$$

$$M_n^y = \frac{1}{a_{22}}\bigg[r_2 - a_{21}(N_{n-1}^x - \epsilon) - \frac{p_2\lambda_2(N_{n-1}^z - \epsilon)}{1 + p_1h_1\lambda_1(M_{n-1}^x + \epsilon) + p_2h_2\lambda_2(M_{n-1}^y + \epsilon)}\bigg],$$

$$M_n^z = \frac{1}{h}\bigg[\frac{p_1e_1\lambda_1(M_{n-1}^x + \epsilon) + p_2e_2\lambda_2(M_n^y + \epsilon)}{1 + p_1h_1\lambda_1(N_{n-1}^x - \epsilon) + p_2h_2\lambda_2(N_{n-1}^y - \epsilon)} - d\bigg],$$

$$\begin{aligned}N_n^x &= \frac{1}{a_{11}}\Bigg[r_1 - a_{12}(M_n^y+\epsilon) \\ &\quad \frac{p_1\lambda_1(M_n^z+\epsilon)}{1+p_1h_1\lambda_1(N_{n-1}^x-\epsilon)+p_2h_2\lambda_2(N_{n-1}^y-\epsilon)}\Bigg],\\ N_n^y &= \frac{1}{a_{22}}\Bigg[r_2 - a_{21}(M_n^x+\epsilon)\\ &\quad - \frac{p_2\lambda_2(M_n^z+\epsilon)}{1+p_1h_1\lambda_1(N_n^x-\epsilon)+p_2h_2\lambda_2(N_{n-1}^y-\epsilon)}\Bigg],\\ N_n^z &= \frac{1}{h}\left[\frac{p_1e_1\lambda_1(N_n^x-\epsilon)+p_2e_2\lambda_2(N_n^y-\epsilon)}{1+p_1h_1\lambda_1(M_n^x+\epsilon)+p_2h_2\lambda_2(M_n^y+\epsilon)}-d\right].\end{aligned} \tag{11}$$

Clearly, we have for any integer $n>0$, $N_n^x \le V_1 \le U_1 \le M_n^x$, $N_n^y \le V_2 \le U_2 \le M_n^y$, and $N_n^z \le V_3 \le U_3 \le M_n^z$. In the following, we will prove that $\{M_n^x\}$, $\{M_n^y\}$, and $\{M_n^z\}$ are monotonically decreasing and $\{N_n^x\}$, $\{N_n^y\}$, and $\{N_n^z\}$ are monotonically increasing, with the help of inductive method.

Firstly, it is clear that $M_2^x \le M_1^x$, $M_2^y \le M_1^y$, $M_2^z \le M_1^z$, $N_2^x \ge N_1^x$, $N_2^y \ge N_1^y$, and $N_2^z \ge N_1^z$. For $n=k(k\ge 2)$, we assume that $M_k^x \le M_{k-1}^x$, $M_k^y \le M_{k-1}^y$, $M_k^z \le M_{k-1}^x$, $N_k^x \ge N_{k-1}^x$, $N_k^y \ge N_{k-1}^y$, and $N_k^z \ge N_{k-1}^z$. Now

$$\begin{aligned}M_{k+1}^x - M_k^x &= \frac{1}{a_{11}}\Bigg[r_1 - \frac{p_1\lambda_1(N_k^z-\epsilon)}{1+p_1h_1\lambda_1(M_k^x+\epsilon)+p_2h_2\lambda_2(M_k^y+\epsilon)}\\ &\quad - a_{12}(N_k^y-\epsilon)\Bigg] - \frac{1}{a_{11}}\Bigg[r_1 - a_{12}(N_{k-1}^y-\epsilon)\\ &\quad - \frac{p_1\lambda_1(N_{k-1}^z-\epsilon)}{1+p_1h_1\lambda_1(M_{k-1}^x+\epsilon)+p_2h_2\lambda_2(M_{k-1}^y+\epsilon)}\Bigg],\\ &= -\frac{p_1\lambda_1}{a_{11}}\Bigg[\frac{(N_{k-1}^z-\epsilon)}{1+p_1h_1\lambda_1(M_{k-1}^x+\epsilon)+p_2h_2\lambda_2(M_{k-1}^y+\epsilon)}\\ &\quad - \frac{(N_k^z-\epsilon)}{1+p_1h_1\lambda_1(M_k^x+\epsilon)+p_2h_2\lambda_2(M_k^y+\epsilon)}\Bigg]\\ &\quad - \frac{a_{12}}{a_{11}}[N_{k-1}^y - N_k^y],\\ &\le \frac{p_1\lambda_1(N_{k-1}^z - N_k^z)}{1+p_1h_1\lambda_1(M_k^x+\epsilon)+p_2h_2\lambda_2(M_k^y+\epsilon)}\\ &\quad + \frac{a_{12}}{a_{11}}[N_{k-1}^y - N_k^y],\\ &\le 0.\end{aligned}$$

Similarly we can show that $M_{k+1}^y - M_k^y \le 0$. Again

$$\begin{aligned}M_{k+1}^z - M_k^z &= \frac{1}{h}\left[\frac{p_1e_1\lambda_1(M_k^x+\epsilon)+p_2e_2\lambda_2(M_{k+1}^y+\epsilon)}{1+p_1h_1\lambda_1(N_k^x-\epsilon)+p_2h_2\lambda_2(N_k^y-\epsilon)}-d\right]\\ &\quad - \frac{1}{h}\left[\frac{p_1e_1\lambda_1(M_{k-1}^x+\epsilon)+p_2e_2\lambda_2(M_k^y+\epsilon)}{1+p_1h_1\lambda_1(N_{k-1}^x-\epsilon)+p_2h_2\lambda_2(N_{k-1}^y-\epsilon)}-d\right],\\ &\le \frac{1}{h}\left[\frac{p_1e_1\lambda_1(M_k^x - M_{k-1}^x)+p_2e_2\lambda_2(M_{k+1}^y - M_k^y)}{1+p_1h_1\lambda_1(N_{k-1}^x-\epsilon)+p_2h_2\lambda_2(N_{k-1}^y-\epsilon)}\right],\\ &\le 0,\\ N_{k+1}^x - N_k^x &= \frac{1}{a_{11}}\Bigg[r_1 - \frac{p_1\lambda_1(M_{k+1}^z+\epsilon)}{1+p_1h_1\lambda_1(N_k^x-\epsilon)+p_2h_2\lambda_2(N_k^y-\epsilon)}\\ &\quad - a_{12}(M_{k+1}^y+\epsilon)\Bigg] - \frac{1}{a_{11}}\Bigg[r_1 - a_{12}(M_k^y+\epsilon)\\ &\quad - \frac{p_1\lambda_1(M_k^z+\epsilon)}{1+p_1h_1\lambda_1(N_{k-1}^x-\epsilon)+p_2h_2\lambda_2(N_{k-1}^y-\epsilon)}\Bigg],\\ &= \frac{p_1\lambda_1}{a_{11}}\Bigg[\frac{M_k^z+\epsilon}{1+p_1h_1\lambda_1(N_{k-1}^x-\epsilon)+p_2h_2\lambda_2(N_{k-1}^y-\epsilon)}\\ &\quad - \frac{M_{k+1}^z+\epsilon}{1+p_1h_1\lambda_1(N_k^x-\epsilon)+p_2h_2\lambda_2(N_k^y-\epsilon)}\Bigg] + \frac{a_{12}}{a_{11}}[M_k^y - M_{k+1}^y],\\ &\ge \frac{a_{12}}{a_{11}}[M_k^y - M_{k+1}^y] + \frac{p_1\lambda_1}{a_{11}}\left[\frac{M_k^z - M_{k+1}^z}{1+p_1h_1\lambda_1(N_k^x-\epsilon)+p_2h_2\lambda_2(N_k^y-\epsilon)}\right],\\ &\ge 0.\end{aligned}$$

Similarly, we can show that

$$\begin{aligned}&N_{k+1}^y - N_k^y \ge 0,\\ &N_{k+1}^z - N_k^z = \frac{1}{h}\left[\frac{p_1e_1\lambda_1(N_{k+1}^x-\epsilon)+p_2e_2\lambda_2(N_{k+1}^y-\epsilon)}{1+p_1h_1\lambda_1(M_{k+1}^x+\epsilon)+p_2h_2\lambda_2(M_{k+1}^y+\epsilon)}-d\right]\\ &\qquad - \frac{1}{h}\left[\frac{p_1e_1\lambda_1(N_k^x-\epsilon)+p_2e_2\lambda_2(N_k^y-\epsilon)}{1+p_1h_1\lambda_1(M_k^x+\epsilon)+p_2h_2\lambda_2(M_k^y+\epsilon)}-d\right]\\ &\qquad \ge \frac{1}{h}\left[\frac{p_1e_1\lambda_1(N_{k+1}^x - N_k^x)+p_2e_2\lambda_2(N_{k+1}^y - N_k^y)}{1+p_1h_1\lambda_1(M_k^x+\epsilon)+p_2h_2\lambda_2(M_k^y+\epsilon)}\right]\\ &\qquad \ge 0.\end{aligned}$$

This shows that $\{M_n^x\}$, $\{M_n^y\}$ and $\{M_n^z\}$ are monotonically decreasing and $\{N_n^x\}$, $\{N_n^y\}$ and $\{N_n^z\}$ are monotonically increasing. Therefore, by the criterion of monotonic bounded, we have established that every one of this six sequences has a limit.

Let $\lim_{n\to\infty} M_n^x = x_1$, $\lim_{n\to\infty} M_n^y = x_2$, $\lim_{n\to\infty} M_n^z = x_3$, $\lim_{n\to\infty} N_n^x = y_1$, $\lim_{n\to\infty} N_n^y = y_2$, $\lim_{n\to\infty} N_n^z = y_3$. Passing to the limit as $n\to\infty$ in eq. (11), we get

$$\begin{aligned}x_1 &= \frac{1}{a_{11}}\left[r_1 - a_{12}y_2 - \frac{p_1\lambda_1 y_3}{1+p_1h_1\lambda_1x_1+p_2h_2\lambda_2x_2}\right],\\ x_2 &= \frac{1}{a_{22}}\left[r_2 - a_{21}y_1 - \frac{p_2\lambda_2 y_3}{1+p_1h_1\lambda_1x_1+p_2h_2\lambda_2x_2}\right],\\ x_3 &= \frac{1}{h}\left[\frac{p_1e_1\lambda_1x_1+p_2e_2\lambda_2x_2}{1+p_1h_1\lambda_1y_1+p_2h_2\lambda_2y_2}-d\right],\\ y_1 &= \frac{1}{a_{11}}\left[r_1 - a_{12}x_2 - \frac{p_1\lambda_1 x_3}{1+p_1h_1\lambda_1y_1+p_2h_2\lambda_2y_2}\right],\\ y_2 &= \frac{1}{a_{22}}\left[r_2 - a_{21}x_1 - \frac{p_2\lambda_2 x_3}{1+p_1h_1\lambda_1y_1+p_2h_2\lambda_2y_2}\right],\\ y_3 &= \frac{1}{h}\left[\frac{p_1e_1\lambda_1y_1+p_2e_2\lambda_2y_2}{1+p_1h_1\lambda_1x_1+p_2h_2\lambda_2x_2}-d\right].\end{aligned} \tag{12}$$

It is clear that $x_1=y_1$, $x_2=y_2$, and $x_3=y_3$. Thus we obtain $x_1=x^*$, $x_2=y^*$, $x_3=z^*$ as a solution of eq. (12). Hence, the global asymptotic stability of (x^*,y^*,z^*) is obtained. This completes the proof of the theorem. □

4. Bifurcation analysis

In this section, we investigate Neimark-Sacker bifurcation and flip bifurcation at the interior fixed point E^* of system (2).

4.1. Neimark-Sacker bifurcation

To examine Neimark-Sacker bifurcation in system (2), we need the following result [31].

Proposition 1. *Suppose an n-dimensional system $v_{k+1} = g_m(v_k)$ where $m\in\mathbb{R}$ is a bifurcation parameter. Let v^* be fixed point of g_m and the characteristic polynomial for Jacobian matrix*

$J(v^*) = (d_{ij})_{n\times n}$ of n-dimensional map $g_m(v_k)$ is given by

$$P_m(x) = x^n + d_1x^{n-1} + \cdots + d_{n-1}x + d_n \tag{13}$$

where $d_i = d_i(m,\alpha)$, $i = 1,2,3,\cdots,n$ and α is a control parameter or another parameter to be deduced. Let $\Delta_0^{\pm}(m,\alpha) = 1, \Delta_1^{\pm}(m,\alpha),\cdots,\Delta_n^{\pm}(m,\alpha)$ be a sequence of determinants defined by $\Delta_i^{\pm}(m,\alpha) = \det(D_1 \pm D_2), i = 1,2,3,\cdots,n$ where

$$D_1 = \begin{pmatrix} 1 & d_1 & b_2 & \cdots & d_{i-1} \\ 0 & 1 & d_1 & \cdots & d_{i-2} \\ 0 & 0 & 1 & \cdots & d_{i-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

$$D_2 = \begin{pmatrix} b_{n-i+1} & d_{n-i+2} & \cdots & d_{n-1} & d_n \\ d_{n-i+2} & d_{n-i+3} & \cdots & d_n & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ d_{n-1} & d_n & \cdots & 0 & 0 \\ d_n & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Further assume that

L1 *Eigenvalue assignment $\Delta_{n-1}^{-}(m_0,\alpha) = 0$, $\Delta_{n-1}^{+}(m_0,\alpha) > 0$, $P_{m_0}(1) > 0$, $(-)^nP_{m_0}(-1) > 0$, $\Delta_i^{\pm}(m_0,\alpha) > 0$, $i = n-3, n-5, \cdots, 1$(or 2), when n is even or odd, respectively.*

L2 *Transversality condition: $[\frac{d(\Delta_{n-1}^{-}(m,\alpha))}{dm}]_{m=m_0} \neq 0$.*

L3 *Non-resonance condition: $\cos(2\pi/j) \neq \phi$, or resonance condition $\cos(2\pi/j) = \phi$ where $j = 3, 4, 5, \cdots$ and $\phi = 1 - 0.5P_{m_0}(1)\Delta_{n-3}^{-}(m_0,\alpha)/\Delta_{n-2}^{+}(m_0,\alpha)$.*

Then Neimark-Sacker bifurcation occurs at m_0.

Now we state bifurcation result by taking h as a bifurcation parameter of system (2), we found the bifurcation phenomenon in the following theorem.

Theorem 2. *The fixed point E^* of system (2) exhibits Neimark-Sacker bifurcation if the following conditions hold:*

$$\begin{aligned} 1 - q_2 + q_3(q_1 - q_3) &= 0, \\ 1 + q_2 - q_3(q_1 + q_3) &> 0, \\ 1 + q_1 + q_2 + q_3 &> 0, \\ 1 - q_1 + q_2 - q_3 &> 0, \end{aligned} \tag{14}$$

where q_1, q_2 and q_3 are defined in eq. (9).

Proof. Using Proposition 1, we have found the following equalities and inequalities:

$$\begin{aligned} \Delta_2^{-}(h^*) &= 1 - q_2 + q_3(q_1 - q_3) = 0, \\ \Delta_2^{+}(h^*) &= 1 + q_2 - q_3(q_1 + q_3) > 0, \\ P_{h^*}(1) &= 1 + q_1 + q_2 + q_3 > 0, \\ (-1)^3P_{h^*}(-1) &= 1 - q_1 + q_2 - q_3 > 0. \end{aligned} \tag{15}$$

□

4.2. Flip bifurcation

Now we present the criterion for which a unique positive fixed point of system (2) undergoes a flip bifurcation. Before stating our result, we require the following result:

Proposition 2 ([32]). *In stead of the conditions* **L1**, **L2** *and* **L3** *in Proposition 1, all other conditions hold. Further it is assumed that the following conditions are satisfied:*

H1 *$P_{m_0}(-1) = 0, \Delta_{n-1}^{\pm}(m_0,\alpha) > 0$, $\Delta_i^{\pm}(m_0,\alpha) > 0$, $i = n-2, n-4, \cdots, 1$(or 2), when n is even or odd, respectively.*

H2 *$\frac{\sum_{i=1}^{n}(-1)^{n-i}d_i'}{\sum_{i=1}^{n}(-1)^{n-i}(n-i+1)d_{i-1}} \neq 0$ where d_i' denotes the derivative of $d_i(m)$ at $m = m_0$. Then a flip bifurcation occurs at m_0.*

Theorem 3. *The fixed point E^* of system (2) admits flip bifurcation at $r_1 = r_1^*$ if the following conditions are fulfilled:*

$$\begin{aligned} 1 - q_2 + q_3(q_1 - q_3) &> 0, \\ 1 + q_2 - q_3(q_1 + q_3) &> 0, \\ 1 + q_1 + q_2 + q_3 &> 0, \\ 1 - q_1 + q_2 - q_3 &= 0, \\ 1 \pm q_3 &> 0, \\ \frac{q_1' - q_2' + q_3'}{3 - 2q_1 + q_2} &\neq 0., \end{aligned}$$

where q_1, q_2, and q_3 are defined in eq. (9) and q_i' stands for the derivative of $q_i(r_1)$ with respect to r_1 at $r_1 = r_1^*$.

Proof. Proof follows from Proposition 2. □

5. Chaos control

Here, we show chaos control for system (2). It is more reasonable for model involving biological population. In studying discrete-time models, one can find chaotic and complicated dynamics than the continuous systems. So it is justifiable to apply control strategy to overcome any uncertainty. We consider hybrid control method developed in [33]. This method uses a single control parameter which lies in the open unit interval. In literature,several types of methods are available for controlling chaos in discrete systems, for example, state feedback method, pole-placement technique and hybrid control method [34–36]. Among these, hybrid control method is very easy to apply. Applying hybrid control method for system (2), we have

$$\begin{aligned} x_{n+1} &= \rho x_n \exp\left\{r_1 - \frac{p_1\lambda_1 z_n}{1 + p_1h_1\lambda_1x_n + p_2h_2\lambda_2y_n}\right. \\ &\qquad \left. - a_{11}x_n - a_{12}y_n\right\} + (1-\rho)x_n, \\ y_{n+1} &= \rho y_n \exp\left\{r_2 - \frac{p_2\lambda_2 z_n}{1 + p_1h_1\lambda_1x_n + p_2h_2\lambda_2y_n}\right. \\ &\qquad \left. - a_{21}x_n - a_{22}y_n\right\} + (1-\rho)y_n, \end{aligned}$$

$$z_{n+1} = \rho z_n \exp\left\{\frac{p_1 e_1 \lambda_1 x_n + p_2 e_2 \lambda_2 y_n}{1 + p_1 h_1 \lambda_1 x_n + p_2 h_2 \lambda_2 y_n} - h z_n - d\right\} + (1-\rho) z_n, \tag{16}$$

where $0 < \rho < 1$ is taken as a control parameter. The Jacobian matrix of controlled system (16) evaluated at E^* is given by

$$\begin{aligned}
J(x^*, y^*, z^*) &= \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{pmatrix}, \\
l_{11} &= 1 - \rho x^* \left(a_{11} - \frac{p_1^2 \lambda_1^2 h_1 z^*}{M^2}\right), \quad l_{13} = -\rho \frac{p_1 \lambda_1 x^*}{M}, \\
l_{12} &= \rho x^* \left(a_{12} - \frac{p_1 \lambda_1 p_2 \lambda_2 h_2 z^*}{M^2}\right), \quad l_{23} = -\rho \frac{p_2 \lambda_2 y^*}{M}, \\
l_{21} &= -\rho y^* \left(a_{21} - \frac{p_1 \lambda_1 p_2 \lambda_2 h_1 z^*}{M^2}\right), \quad l_{33} = 1 - \rho h z^*, \\
l_{22} &= 1 - \rho y^* \left(a_{22} - \frac{p_2^2 \lambda_2^2 h_2 z^*}{M^2}\right), \\
l_{31} &= \frac{\rho z^* \{p_1 e_1 \lambda_1 + p_1 \lambda_1 p_2 \lambda_2 y^* (e_1 h_2 - e_2 h_1)\}}{M^2}, \\
l_{32} &= \frac{\rho z^* \{p_2 e_2 \lambda_2 + p_1 \lambda_1 p_2 \lambda_2 x^* (e_2 h_1 - e_1 h_2)\}}{M^2}.
\end{aligned} \tag{17}$$

The fixed point E^* of the controlled system (16) is locally asymptotically stable if all the roots of the characteristic polynomial of eq. (17) lie in an unit open disk.

6. Numerical Simulation

In this section, we present some numerical simulations to validate our analytical findings. First of all, some phase portraits and time series plots of continuous system (1) are given to compare the results derived in discrete system (2). We show the role of the parameters r_1, h_1 and h on the dynamical behaviour of the system.

Example 1. Suppose $r_1 = 3.5$, $r_2 = 2$, $a_{11} = 0.1$, $a_{12} = 0.2$, $a_{21} = 0.01$, $a_{22} = 0.1$, $p_1 = 2$, $p_2 = 2$, $c_1 = 1$, $c_2 = 1.5$, $h_1 = 0.5$, $h_2 = 0.05$, $h = 0.2$, $d = 1.3$, $\lambda_1 = 1$, and $\lambda_2 = 1$ for system (1). Then limit cycle appears around the equilibrium point $E^* = (12.1061, 4.64, 7.7201)$ (see Figure 1). If we decrease the value of h_1 from 0.5 to 0.35, then system (1) becomes stable around the equilibrium point $E^* = (14.4933, 2.2523, 9.1789)$ (see Figure 2).

Example 2. Suppose $r_1 = 2.842$, $r_2 = 1.6$, $a_{11} = 10$, $a_{12} = 0.2$, $a_{21} = 2$, $a_{22} = 1$, $p_1 = 2$, $p_2 = 2$, $c_1 = 1$, $c_2 = 1.5$, $h_1 = 0.5$, $h_2 = 0.5$, $h = 0.35$, $d = 1.3$, $\lambda_1 = 1$, and $\lambda_2 = 1$ for system (1). Then system (1) becomes stable around the equilibrium point $E^* = (0.2358, 0.8059, 0.3291)$ (see Figure 3).

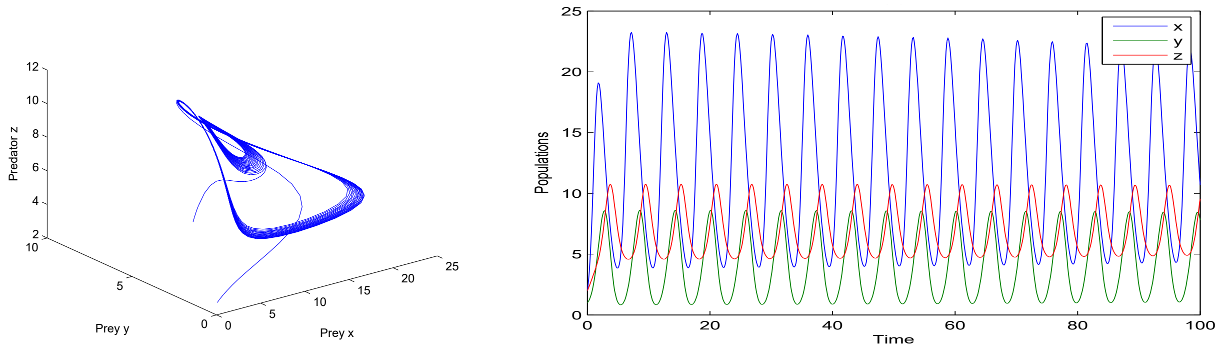
Example 3. Suppose $r_1 = 2.8$, $r_2 = 1.6$, $a_{11} = 10$, $a_{12} = 0.2$, $a_{21} = 0.1$, $a_{22} = 1$, $p_1 = 2$, $p_2 = 2$, $c_1 = 1$, $c_2 = 1.5$, $h_1 = 0.5$, $h_2 = 0.5$, $d = 1.3$, $\lambda_1 = 1$, $\lambda_2 = 1$, initial points $(2, 1, 2)$, and $h \in (0.1, 1)$ in system (2) with the initial condition $(x_0, y_0, z_0) = (2, 1, 2)$. When h is considered as a bifurcation parameter, then at $h = h^* = 0.36$, the interior fixed point $E^* = (0.1994, 0.9827, 0.6658)$ becomes unstable and system (2) undergoes Neimark-Sacker bifurcation. It shows that the Theorem 2 is true. Bifurcation diagrams and maximum Lyapunov exponents (MLE) with respect to the parameter h of system (2) are depicted in Figure 4. As h increases, we observe that a transition from unstable to stable and then bifurcation within a limit cycle to a periodic window and finally to chaos.

Example 4. Suppose $r_2 = 1.6$, $a_{11} = 10$, $a_{12} = 0.2$, $a_{21} = 2$, $a_{22} = 1$, $p_1 = 2$, $p_2 = 2$, $c_1 = 1$, $c_2 = 1.5$, $h_1 = 0.5$, $h_2 = 0.5$, $h = 0.35$, $d = 1.3$, $\lambda_1 = 1$, $\lambda_2 = 1$, initial points $(0.5, 0.5, 0.5)$, and $r_1 \in (2.6, 3.6)$ in system (2) with the initial condition $(x_0, y_0, z_0) = (2, 1, 2)$. When r_1 is considered as a bifurcation parameter, then at $r_1 = 2.842$, the interior fixed point $E^* = (0.2358, 0.8059, 0.3291)$ becomes unstable and system (2) undergoes flip bifurcation. It shows that Theorem 3 is true. Bifurcation diagrams and MLE with respect to the parameter r_1 of system (2) are shown in Figure 5. From Figure 5, it is observed that the system is stable as log as $r_1 < 2.842$ and as r_1 increases, a series of period-doubling bifurcation wherein a 2^k- cycle loses its stability.

Example 5. Suppose $r_1 = 3.3$, $r_2 = 3.2$, $a_{11} = 0.01$, $a_{12} = 0.02$, $a_{21} = 0.01$, $a_{22} = 0.01$, $p_1 = 2$, $p_2 = 2$, $c_1 = 1$, $c_2 = 1.5$, $h_1 = 0.35$, $h_2 = 0.4$, $h = 1$, $d = 1.3$, $\lambda_1 = 1$, $\lambda_2 = 1$, and initial points $(2, 1, 2)$ for system (1). Then the conditions of Lemma 2 are violated. Thus the fixed point $E^* = (0.2627, 317.7, 2.434)$ is unstable. Moreover, system (2) admits chaotic behaviour (see Figure 6a). Suppose $\rho = 0.2$ and other parameters are same as in Example 5. Then the chaotic orbit of system (2) is stabilized at the fixed point $E^* = (0.2627, 317.7, 2.434)$ (see Figure 6b).

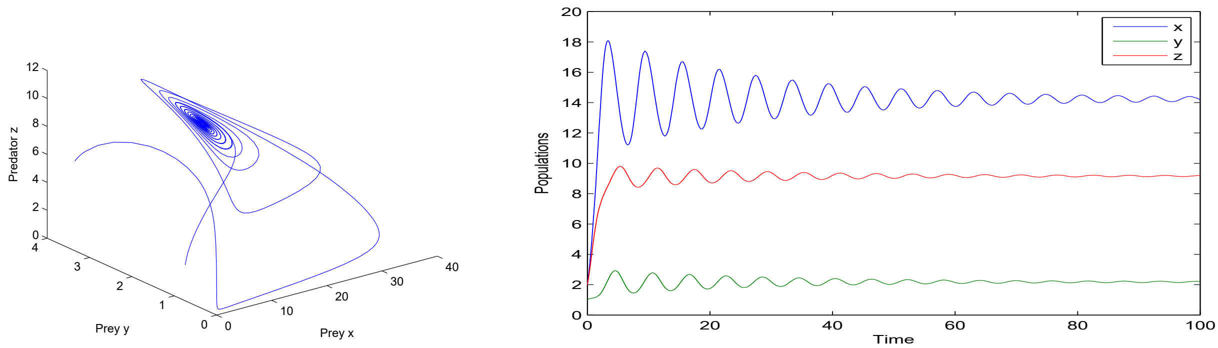
7. Discussion

The relationship with two rival prey and their common predator plays an important role in structuring communities. switching of prey increases the predators ability to consume more prey instead of sticking upto one particular prey and it increases their growth rate with a higher rate. Sometimes it is difficult to predict the actual picture in continuous model of interacting species. The main reason behind this is non-overlapping generation of population. In such occurrence, we have proposed and investigated a three dimensional discrete-time system consisting of two competing preys with a shared predator. System



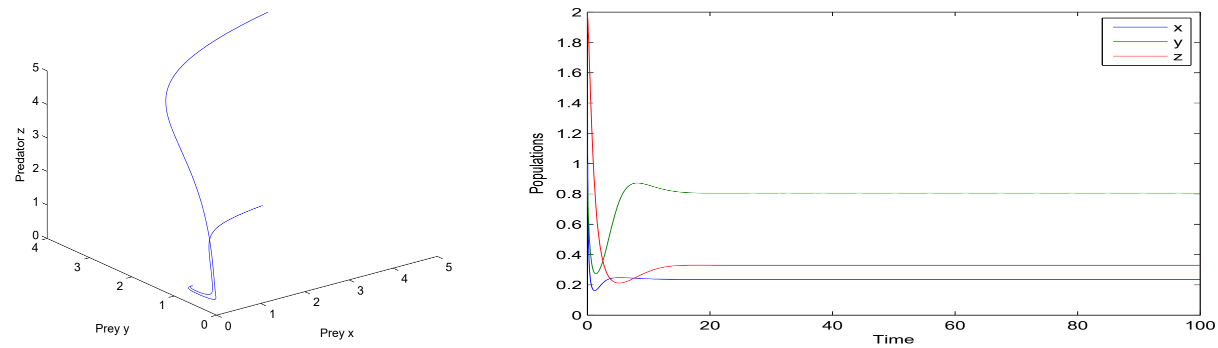
(a) (b)

Figure 1. Phase portrait along with time series plots of system (1) with parameter values $r_1 = 3.5$, $r_2 = 2$, $a_{11} = 0.1$, $a_{12} = 0.2$, $a_{21} = 0.01$, $a_{22} = 0.1$, $p_1 = 2$, $p_2 = 2$, $c_1 = 1$, $c_2 = 1.5$, $h_1 = 0.5$, $h_2 = 0.05$, $h = 0.2$, $d = 1.3$, $\lambda_1 = 1$, $\lambda_2 = 1$, and initial points $(2, 1, 2)$ and $(5, 4, 5)$.



(a) (b)

Figure 2. Phase portrait along with time series plots of system (1) with parameter values $r_1 = 3.5$, $r_2 = 2$, $a_{11} = 0.1$, $a_{12} = 0.2$, $a_{21} = 0.01$, $a_{22} = 0.1$, $p_1 = 2$, $p_2 = 2$, $c_1 = 1$, $c_2 = 1.5$, $h_1 = 0.35$, $h_2 = 0.05$, $h = 0.2$, $d = 1.3$, $\lambda_1 = 1$, $\lambda_2 = 1$, and initial points $(2, 1, 2)$ and $(5, 4, 5)$.



(a) (b)

Figure 3. Phase portrait along with time series plots of system (1) with parameter values $r_1 = 2.842$, $r_2 = 1.6$, $a_{11} = 10$, $a_{12} = 0.2$, $a_{21} = 2$, $a_{22} = 1$, $p_1 = 2$, $p_2 = 2$, $c_1 = 1$, $c_2 = 1.5$, $h_1 = 0.5$, $h_2 = 0.5$, $h = 0.35$, $d = 1.3$, $\lambda_1 = 1$, $\lambda_2 = 1$, and initial points $(2, 1, 2)$ and $(5, 4, 5)$.

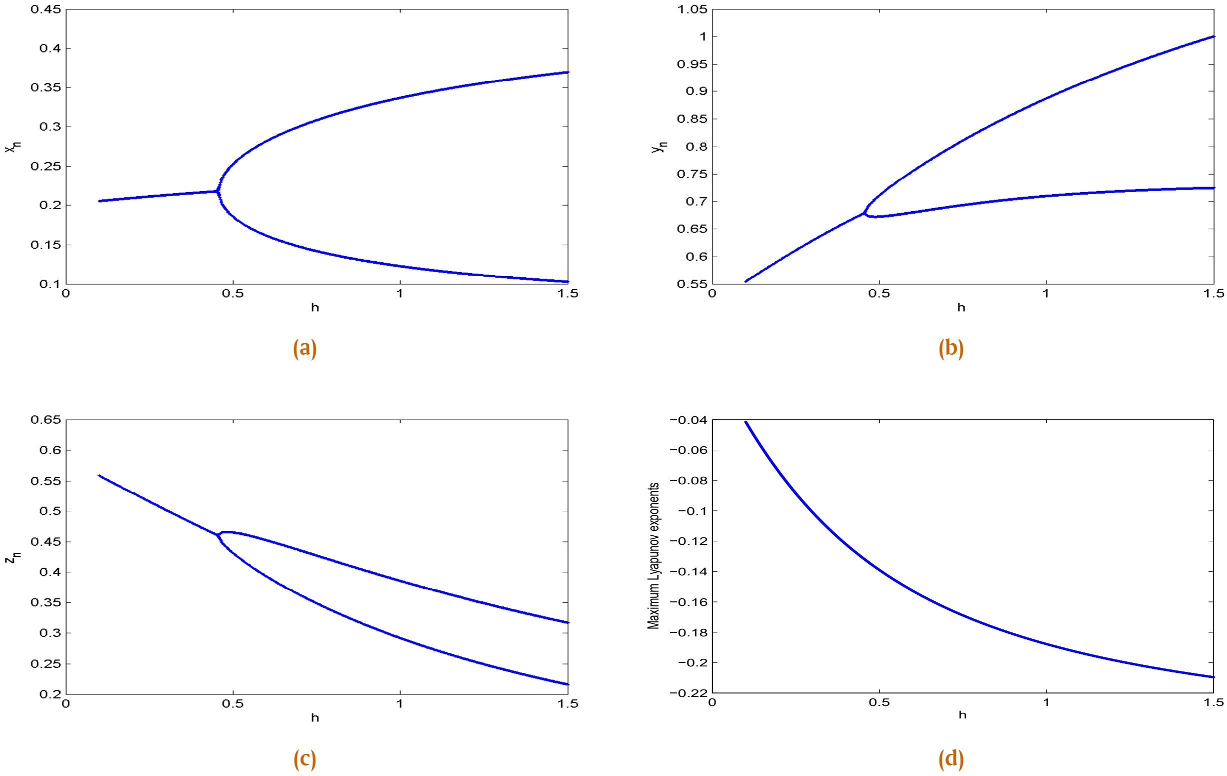


Figure 4. Bifurcation diagrams and MLE for system (2) with parameter values $r_1 = 2.8$, $r_2 = 1.6$, $a_{11} = 10$, $a_{12} = 0.2$, $a_{21} = 0.1$, $a_{22} = 1$, $p_1 = 2$, $p_2 = 2$, $c_1 = 1$, $c_2 = 1.5$, $h_1 = 0.5$, $h_2 = 0.5$, $d = 1.3$, $\lambda_1 = 1$, $\lambda_2 = 1$, $h \in (0.1, 1)$, and initial point $(2, 1, 2)$.

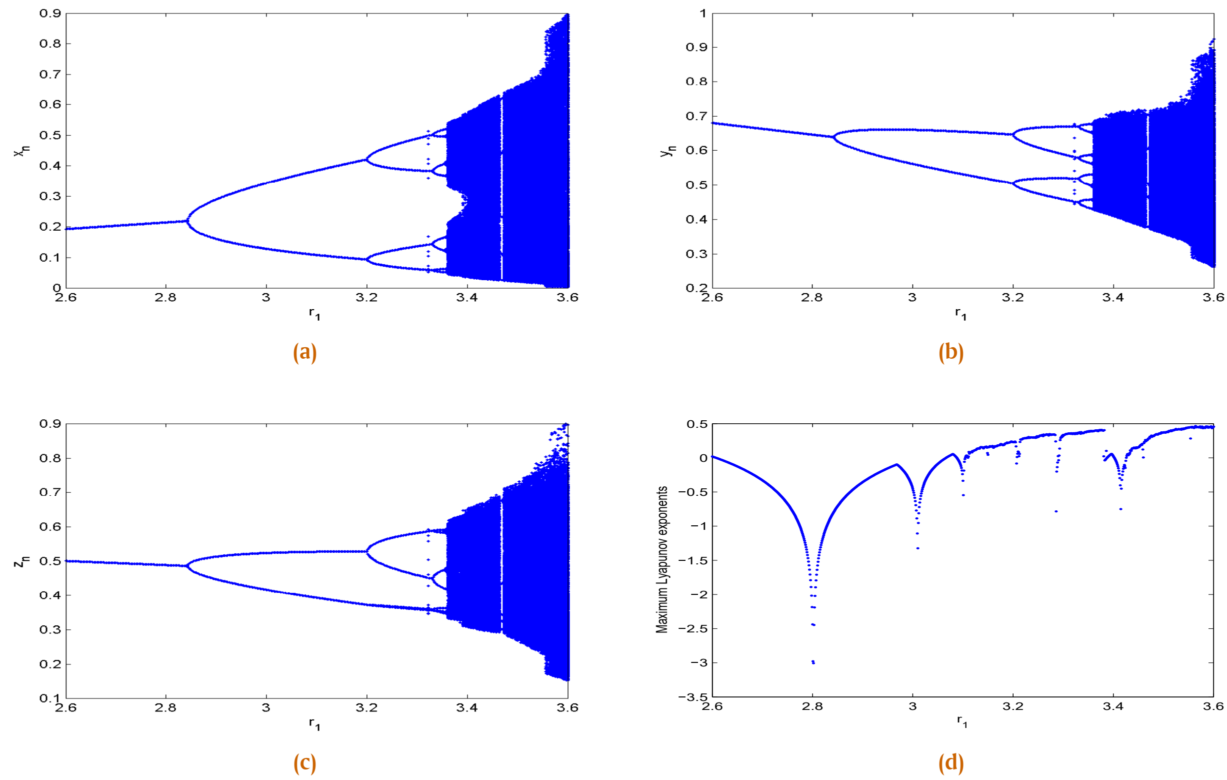
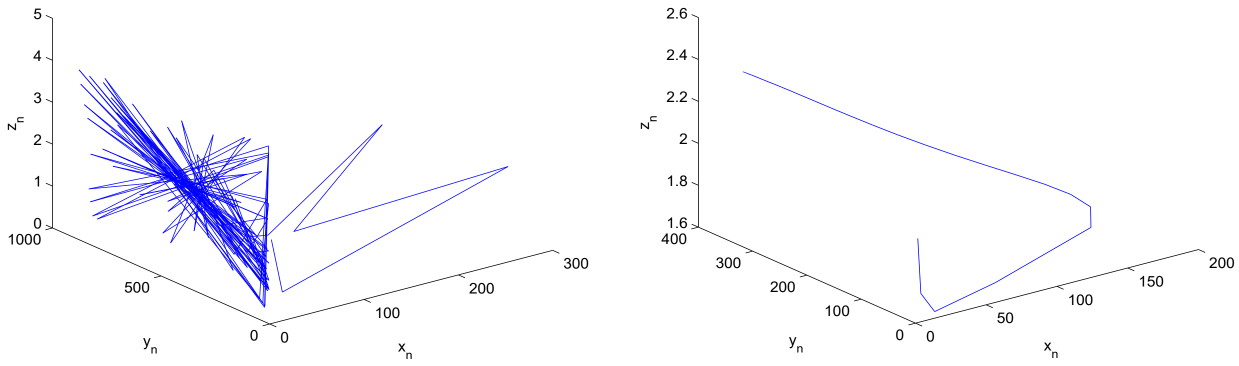


Figure 5. Bifurcation diagrams and MLE for system (2) with parameter values $r_2 = 1.6$, $a_{11} = 10$, $a_{12} = 0.2$, $a_{21} = 2$, $a_{22} = 1$, $p_1 = 2$, $p_2 = 2$, $c_1 = 1$, $c_2 = 1.5$, $h_1 = 0.5$, $h_2 = 0.5$, $d = 1.3$, $\lambda_1 = 1$, $\lambda_2 = 1$, $r_1 \in (2.6, 3.6)$, and initial point (2, 1, 2).



(a) **(b)**

Figure 6. (a). Phase portrait of system (2) with parameter values $r_1 = 3.3$, $r_2 = 3.2$, $a_{11} = 0.01$, $a_{12} = 0.02$, $a_{21} = 0.01$, $a_{22} = 0.01$, $p_1 = 2$, $p_2 = 2$, $c_1 = 1$, $c_2 = 1.5$, $h_1 = 0.35$, $h_2 = 0.4$, $h = 1$, $d = 1.3$, $\lambda_1 = 1$, $\lambda_2 = 1$, and initial points $(2, 1, 2)$ and (b). phase portrait of controlled system (16) for $\rho = 0.2$.

(2) is constructed on the basis of continuous system (1), which was studied in [8]. The stability (local as well as global) of interior fixed point, Neimark-Sacker bifurcation, flip bifurcation and chaos control are examined. The basic results of the model have been studied through phase portrait, bifurcation diagrams and maximum Lyapunov exponents.

In the continuous system (1), it is observed from Figure 2 that when the time spent in handling time with the prey x is decreased, the system becomes stable. Furthermore, the system can be stabilized by decreasing the birth rate of the first prey and the intrinsic growth rate of the second prey (see Figure 3). Figures 3 and 5, one can observe that the system which is stable for continuous case but not stable in the discrete case.

It is identified that the parameter h, representing the intra-specific competition rate among the predator species z is more relevant to the emergence of Neimark-Sacker bifurcation whenever it is varied in some appropriate interval. In examining bifurcation, we have observed that the parameter r_1, representing the birth rate of the prey species x may result flip bifurcation. In real world, it is observed that shrimps posses a regular structure with four long narrow tails joined to a central head. Over the outer surface of a shrimp, the system shows chaos via tangent bifurcation, whereas in inner surface, period doubling transition to chaos appears through flip bifurcation [37]. The proposed model admits more rich characteristics and more complicated dynamics than that exist in the continuous case. We have derived the condition for global stability of the positive fixed point by applying the iteration scheme and comparison principle of difference equation. Conditions of Theorem 3.1 indicate that when the birth rate of prey x and intrinsic growth rate of prey y remain below one and with other restrictions on system parameters then the positive fixed point is globally asymptotically stable.

Occasionally bifurcation and chaos are in fact undesirable problem in discrete dynamical systems, because population may face extinction due to chaos. Hence chaos control is important in dynamical system. To overcome this situation, we have employed the hybrid control method so that stability of the system can be regained.

To our understanding, the dynamical study of discrete time model considering two competing prey with a shared predator has not been studied yet.

The major drawback of discrete-time models is related to data accuracy. In many instances relatively large discretization step lengths are chosen for tractability, resulting in approximation errors in the converted problem data, which may lead to inferior solutions.

Author Contributions. **Mukherjee, D.**: Conceptualization, methodology, software, validation, formal analysis, investigation, writing—original draft preparation, writing—review and editing.

Acknowledgement. The authors is thankful the editors and reviewers who have supported us in improving this manuscript.

Funding. This research received no external funding.

Conflict of interest. The author declare no conflict of interest.

Data availability. Not applicable.

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