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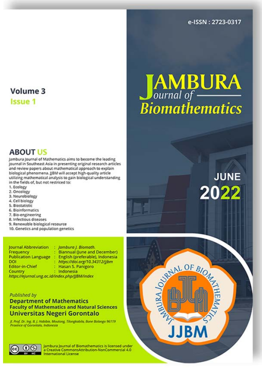

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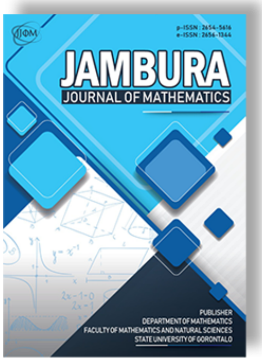

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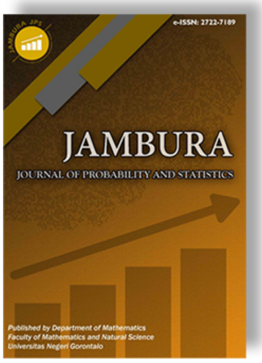

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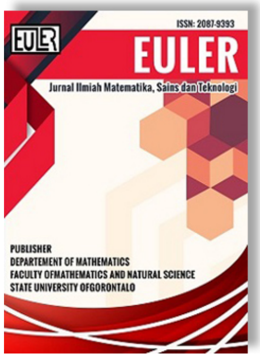

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Research Article

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Energy and Laplacian Energy of Pythagorean Intuitionistic Fuzzy Graphs with Applications in Medical Diagnosis Networks

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ABSTRACT. *This study extends fuzzy graph energy analysis by introducing energy and Laplacian energy for Pythagorean Intuitionistic Fuzzy Graphs (PIFGs), a powerful generalization of intuitionistic fuzzy graphs capable of representing higher degrees of uncertainty. A novel connection matrix for PIFGs is defined, and new formulations for energy and Laplacian energy are established, along with sharp lower and upper bounds. Beyond theoretical contributions, the approach is applied to medical diagnosis networks, where vertices represent symptoms, diagnostic tests, and diseases, and edges encode Pythagorean intuitionistic fuzzy relationships. These measures quantify both the overall strength of associations (energy) and their structural irregularity (Laplacian energy), offering interpretable indicators for diagnostic certainty or ambiguity. The framework provides a robust mathematical basis for decision-making in biomedical contexts where data are uncertain, imprecise, or conflicting.*

1. Introduction

In biomedical decision-making, particularly in medical diagnosis, data often contain uncertainty, vagueness, and even conflicting information. Symptoms may be subjective, diagnostic tests may yield ambiguous results, and different clinical indicators may suggest multiple possible conditions. Mathematical tools capable of representing and analyzing such uncertainty are therefore essential [1–7].

Krassimir Atanassov pioneered the concept of intuitionistic fuzzy sets [8], which capture uncertainty by simultaneously considering membership and non-membership degrees. Building on this foundation, Yager [9] introduced Pythagorean fuzzy sets, which relax the constraints of intuitionistic fuzzy sets to allow greater expressive power. The integration of these two theoretical frameworks led to Pythagorean intuitionistic fuzzy sets [10], offering a richer mathematical structure for modeling hesitation and imprecision in complex systems such as diagnostic networks.

Within graph theory, fuzzy and intuitionistic fuzzy graph models have been widely explored for representing uncertain relationships between entities. Nagarani and Vimala [11] investigated the energy of fuzzy regular graphs, while Anitha and Jayalakshmi [12] studied degrees and regularity conditions in Pythagorean intuitionistic fuzzy graphs (PIFGs). Earlier, Anjali and Mathew [13] examined the energy properties of fuzzy graphs, and Sharbaf and Fayazi [14] introduced the concept of Laplacian energy for fuzzy graphs. Building upon these works, Praba et al. [15] extended the energy framework to intuitionistic fuzzy graphs (IFGs), and Basha and Kartheek [16] generalized Laplacian energy analysis for IFGs.

Recent advances in spectral graph theory and fuzzy graph models further motivate the present study. The signless Laplacian energy of intuitionistic fuzzy graphs has been shown to play a crucial role in group decision–making frameworks, demonstrating how spectral quantities can guide collective judgments under uncertainty [17]. Classical works on Laplacian energy, Laplacian–energy–like invariants and the Kirchhoff index provide powerful tools to relate structural properties of graphs to their spectral characteristics, and have been extensively developed in both general and specialized graph classes [18–20]. In parallel, applications of Pythagorean fuzzy sets and related structures have expanded rapidly, from multi–criteria decision making such as GIS–based solar power plant site selection using Pythagorean fuzzy AHP to soft graph models capturing complex, parameter–dependent relationships [21, 22]. Within the broader fuzzy graph setting, properties such as fuzzy chromatic numbers in intuitionistic fuzzy graphs further illustrate how enriched uncertainty representations can yield more nuanced combinatorial and optimization results [23]. These developments collectively indicate that combining spectral invariants (energy and Laplacian energy) with advanced fuzzy and Pythagorean models is a promising direction,

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and they provide a strong theoretical basis for extending such concepts to Pythagorean intuitionistic fuzzy graphs with applications in biomedical decision–making.

1.1. Research gap

In biomedical decision–making, particularly in the diagnosis of heart diseases, existing fuzzy and intuitionistic fuzzy graph models are limited in their ability to capture situations where both supportive and contradictory evidence co–exist with high intensity under substantial hesitation [8, 13, 15, 24] . Classical intuitionistic fuzzy graphs impose the linear constraint

$$T(a_i) + F(a_i) \leq 1,$$

which restricts the simultaneous representation of strong truth and falsity degrees and therefore underestimates the actual level of diagnostic ambiguity present in complex clinical networks [8, 24].

Moreover, most available energy and Laplacian energy formulations for fuzzy or intuitionistic fuzzy graphs focus only on single–parameter uncertainty and do not explicitly incorporate the richer hesitation structure required for modern clinical data, where symptoms, test results and comorbidities interact in multi–valued, conflicting ways [11, 14, 16, 25]. Existing studies on graph energy in medical applications mainly treat uncertainty qualitatively, lacking a spectral framework that can jointly quantify both interaction strength and structural irregularity under Pythagorean–type constraints [1, 2].

Consequently, there is a clear need for a generalized graph model that (i) employs the Pythagorean intuitionistic condition

$$T^2(a_i) + F^2(a_i) \leq 1$$

to allow higher expressive power for hesitation, and (ii) extends energy and Laplacian energy concepts to this setting so that the magnitude and uniformity of uncertain relationships in diagnostic networks can be rigorously measured and interpreted [9, 10, 12]. This work addresses this gap by formulating energy and Laplacian energy for Pythagorean intuitionistic fuzzy graphs and demonstrating their usefulness in heart–disease diagnosis networks [10, 12].

This study extends the energy and Laplacian energy concepts to PIFGs, establishing new formulations and corresponding bounds. Beyond theoretical development, the framework is applied to medical diagnosis networks, where vertices represent symptoms, diagnostic tests, and diseases, and edges encode Pythagorean intuitionistic fuzzy relationships. The proposed measures—graph energy as an indicator of overall interaction strength, and Laplacian energy as a measure of structural irregularity—offer quantitative insights into the certainty or ambiguity of diagnostic conclusions, making them valuable tools in the biomathematics of medical decision-making.

2. Preliminaries

Definition 1. [10] Let $G = (\mathbb{A}, \mathbb{B})$ is a Pythagorean intuitionistic fuzzy graph (PIFG) of a graph $G^* = (\mathbb{V},\mathbb{E})$ which is a crisp graph, where $\mathbb{A} = \langle \mathfrak{T}_{\mathbb{A}}, \mathfrak{F}_{\mathbb{A}} \rangle$ is an intuitionistic set on vertex $\mathbb{V}$ and $\mathbb{B} = \langle \mathfrak{T}_{\mathbb{B}}, \mathfrak{F}_{\mathbb{B}} \rangle$ is an intuitionistic relation on edge $\mathbb{E}$ with $\mathfrak{T}_{\mathbb{A}}$ and $\mathfrak{F}_{\mathbb{A}}$ from $\mathbb{V}$ to $[0, 1]$ signifying truth membership and falsity membership functions of $\mathbb{V}$, and it satisfies

1. For every vertex $\mathbb{A} \subset \mathbb{V}$
$$0 \leq \mathfrak{T}_{\mathbb{A}}(a_i) + \mathfrak{F}_{\mathbb{A}}(a_i) \leq 1, \ a_i \in \mathbb{V}, \ \text{(Intutionistic Condition)}.$$
2. For every edge $\mathbb{B} \subset \mathbb{E}$
$$\mathfrak{T}_{\mathbb{B}}(a_i a_j) \leq \mathfrak{T}_{\mathbb{A}}(a_i) \wedge \mathfrak{T}_{\mathbb{A}}(a_j), \ \mathfrak{F}_{\mathbb{B}}(a_i a_j) \geq \mathfrak{F}_{\mathbb{A}}(a_i) \vee \mathfrak{F}_{\mathbb{A}}(a_j),$$
and
$$0 \leq \mathfrak{T}^2_{\mathbb{B}}(a_i a_j) + \mathfrak{F}^2_{\mathbb{B}}(a_i a_j) \leq 1, \ \text{(Pythogorean Condition)},$$
$$0 \leq \mathfrak{T}_{\mathbb{B}}(a_i a_j) + \mathfrak{F}_{\mathbb{B}}(a_i a_j) \leq 1, \ \text{(Intutionistic Condition)},$$
for all $a_i a_j \in \mathbb{E}$.

where $\mathfrak{T}_{\mathbb{B}}, \mathfrak{F}_{\mathbb{B}}$ from $\mathbb{V} \times \mathbb{V}$ to $[0, 1]$ stand for the truth membership and falsity membership functions of $\mathbb{E}$.

3. Energy of PIFG

Definition 2. The connection matrix of a PIFG $G = (\mathbb{A}, \mathbb{B})$ is defined as a square matrix

$$\mathcal{A} = \mathcal{A}(G) = [a_{ij}] = \left(\mathcal{A}\mathfrak{T}_{\mathbb{B}}(a_i a_j), \ \mathcal{A}\mathfrak{F}_{\mathbb{B}}(a_i a_j)\right),$$

where $\mathcal{A}\mathfrak{T}_{\mathbb{B}}(a_i a_j)$ is a matrix containing the truth membership values, and $\mathcal{A}\mathfrak{F}_{\mathbb{B}}(a_i a_j)$ is a matrix containing the falsity membership values.

Definition 3. The spectrum of the connection matrix $\mathcal{A}(G)$ of a PIFG is defined as $(E_{\mathfrak{T}}, E_{\mathfrak{F}})$, where:

$$E_{\mathfrak{T}} = \{\lambda_1, \lambda_2, \lambda_3, \dots\}$$

is the set of characteristic values of $\mathcal{A}\mathfrak{T}_{\mathbb{B}}(a_i a_j)$, and

$$E_{\mathfrak{F}} = \{\delta_1, \delta_2, \delta_3, \dots\}$$

is the set of characteristic values of $\mathcal{A}\mathfrak{F}_{\mathbb{B}}(a_i a_j)$.

Definition 4. The energy of a PIFG $G = (\mathbb{A}, \mathbb{B})$ is defined as the sum of the absolute values of the characteristic values of $\mathcal{A}\mathfrak{T}_{\mathbb{B}}(a_i a_j)$, and of $\mathcal{A}\mathfrak{F}_{\mathbb{B}}(a_i a_j)$:

$$\begin{aligned}\mathfrak{E}(G) &= \left(\mathfrak{E}(\mathfrak{T}_{\mathbb{B}}(a_i a_j)), \ \mathfrak{E}(\mathfrak{F}_{\mathbb{B}}(a_i a_j))\right) \\ &= \left(\sum_{\lambda_i \in E_{\mathfrak{T}}} |\lambda_i|, \ \sum_{\delta_i \in E_{\mathfrak{F}}} |\delta_i|\right).\end{aligned}$$

Example 1. Consider PIFG (see Figure 1). The Connection

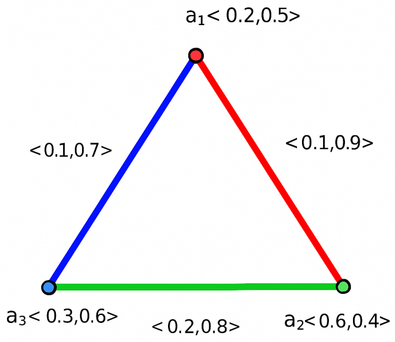

Figure 1. Pythagorean intuitionistic fuzzy graph (PIFG)

matrix for the above PIFG is

$$\mathcal{A}(G) = ((\mathcal{A}\mathfrak{T}_{\mathbb{B}}(a_i a_j)), (\mathcal{A}\mathfrak{F}_{\mathbb{B}}(a_i a_j))),$$

where

$$\mathcal{A}\mathfrak{T}_{\mathbb{B}}(a_i a_j) = \begin{pmatrix} 0 & 0.1 & 0.1 \\ 0.1 & 0 & 0.2 \\ 0.1 & 0.2 & 0 \end{pmatrix},$$

$$\mathcal{A}\mathfrak{F}_{\mathbb{B}}(a_i a_j) = \begin{pmatrix} 0 & 0.9 & 0.7 \\ 0.9 & 0 & 0.8 \\ 0.7 & 0.8 & 0 \end{pmatrix},$$

$$\mathcal{A}(G) = \begin{pmatrix} <0,0> & <0.1,0.9> & <0.1,0.7> \\ <0.1,0.9> & <0,0> & <0.2,0.8> \\ <0.1,0.7> & <0.2,0.8> & <0,0> \end{pmatrix}.$$

The spectrum of the above PIFG is $(E_{\mathfrak{T}}, E_{\mathfrak{F}})$, where

$$E_{\mathfrak{T}}(\mathfrak{T}_{\mathbb{B}}(a_i a_j)) = \{-0.2, -0.0732051, 0.273205\},$$
$$E_{\mathfrak{F}}(\mathfrak{F}_{\mathbb{B}}(a_i a_j)) = \{-0.9168, -0.6860, 1.6028\}$$

The Energy of the above PIFG is

$$\mathfrak{E}(G) = (\mathfrak{E}((\mathfrak{T}_{\mathbb{B}}(a_i a_j)), \mathfrak{E}(\mathfrak{F}_{\mathbb{B}}(a_i a_j))),$$

with

$$\mathfrak{E}(\mathfrak{T}_{\mathbb{B}}(a_i a_j)) = 0.2 + 0.0732051 + 0.273205 = 0.5464101,$$
$$\mathfrak{E}(\mathfrak{F}_{\mathbb{B}}(a_i a_j)) = 0.9168 + 0.6860 + 1.6028 = 3.2056$$

Theorem 1. *If $G = (\mathbb{A}, \mathbb{B})$ be a PIFG and its Connection matrix is $\mathcal{A}(G)$.If the set of characteristic values are in non-increasing sequence then: (i) Sum of characteristic values of truth in the connection matrix is zero. (ii) Sum of the square of characteristic values of truth in the connection matrix is twice the sum of the square of the truth values of the edges(without loop) $1 \le a_i < a_j \le p$.*

Proof.

1. The connection matrix $\mathcal{A}(G)$ is a symmetric matrix with zero trace since the values in the diagonal entries are zero. Its characteristic values are real and their sum is zero

$$\text{(i.e)} \sum_{\substack{i=1 \\ \lambda_i \in E_{\mathfrak{T}}}}^{p} \lambda_i = 0$$

2. Using the properties of trace of a matrix,

$$\text{tr}\left((\mathcal{A}\mathfrak{T}_{\mathbb{B}}(a_i a_j))^2\right) = \sum_{i=1 \lambda_i \in E_{\mathfrak{T}}}^{p} \lambda_i^2.$$

Now,

$$\begin{aligned} \text{tr}\left((\mathcal{A}\mathfrak{T}_{\mathbb{B}}(a_i a_j))^2\right) &= (0 + (\mathfrak{T}_{\mathbb{B}}(a_1 a_2))^2 + (\mathfrak{T}_{\mathbb{B}}(a_1 a_3))^2 \\ &\quad + \cdots + (\mathfrak{T}_{\mathbb{B}}(a_1 a_p))^2) + ((\mathfrak{T}_{\mathbb{B}}(a_2 a_1))^2 \\ &\quad + 0 + (\mathfrak{T}_{\mathbb{B}}(a_2 a_3))^2 + \ldots \\ &\quad + (\mathfrak{T}_{\mathbb{B}}(a_2 a_p))^2) + \cdots + ((\mathfrak{T}_{\mathbb{B}}(a_p a_1))^2 \\ &\quad + (\mathfrak{T}_{\mathbb{B}}(a_p a_2))^2 + \cdots + 0), \\ &= 2 \sum_{1 \le i < j \le p} \mathfrak{T}_{\mathbb{B}}(a_i a_j)^2, \end{aligned}$$

$$\Rightarrow \sum_{\substack{i=1 \\ \lambda_i \in E\mathfrak{T}}}^{p} \lambda_i^2 = 2 \sum_{1 \le i < j \le p} \mathfrak{T}_{\mathbb{B}}(a_i a_j)^2.$$

□

Note:
The above Theorem 1 holds for falsity values also

$$\text{(i.e)} \sum_{\substack{i=1 \\ \delta_i \in E_{\mathfrak{F}}}}^{p} \delta_i = 0,$$

$$\sum_{\substack{i=1 \\ \delta_i \in E_{\mathfrak{F}}}}^{p} \delta_i^2 = 2 \sum_{1 \le i < j \le p} \mathfrak{F}_{\mathbb{B}}(a_i a_j)^2.$$

Example 2. From Figure 1,

$$\sum_{\substack{i=1 \\ \lambda_i \in E_{\mathfrak{T}}}}^{3} \lambda_i = (-0.2) + (-0.0732051) + 0.273205 = 0,$$

$$\sum_{\substack{i=1 \\ \delta_i \in E_{\mathfrak{F}}}}^{3} \delta_i = (-0.9168) + (-0.6860) + 1.6028 = 0,$$

$$\begin{aligned} \sum_{\substack{i=1 \\ \lambda_i \in E_{\mathfrak{T}}}}^{3} \lambda_i^2 &= (-0.2)^2 + (-0.0732051)^2 + (0.273205)^2 \\ &= .12 = 2(0.06), \\ &= 2((0.1)^2 + (0.1)^2 + (0.2)^2), \\ &= 2(\mathfrak{T}_{\mathbb{B}}(a_1 a_2)^2 + \mathfrak{T}_{\mathbb{B}}(a_1 a_3)^2 + \mathfrak{T}_{\mathbb{B}}(a_2 a_3)^2), \\ &= 2 \sum_{1 \le i < j \le 3} \mathfrak{T}_{\mathbb{B}}(a_i a_j)^2, \end{aligned}$$

$$\sum_{\substack{i=1 \\ \delta_i \in E_{\mathfrak{F}}}}^{3} \delta_i^2 = (-0.9168)^2 + (-0.6860)^2 + (1.6028)^2$$

$$= 3.88 = 2(1.94),$$
$$= 2((0.9)^2 + (0.7)^2 + (0.8)^2),$$
$$= 2(\mathfrak{F}_{\mathbb{B}}(a_1a_2)^2 + \mathfrak{F}_{\mathbb{B}}(a_1a_3)^2 + \mathfrak{F}_{\mathbb{B}}(a_2a_3)^2),$$
$$= 2\sum_{1\le i<j\le 3}\mathfrak{F}_{\mathbb{B}}(a_ia_j)^2.$$

Theorem 2. *Let G be an PIFG (without loops) with $|\mathbb{A}| = p$ and $|\mathbb{B}| = q$ and $\mathcal{A} = [a_{ij}] = ((\mathcal{A}\mathfrak{T}_{\mathbb{B}}(a_ia_j)), (\mathcal{A}\mathfrak{F}_{\mathbb{B}}(a_ia_j)))$ is a connection matrix of PIFG then*

(i) $\sqrt{2p\sum_{1\le i<j\le p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)\mathfrak{T}_{\mathbb{B}}(a_ja_i)} \ge E(\mathfrak{T}_{\mathbb{B}}(a_ia_j)) \ge \sqrt{\Phi_1+\Phi_2}$, *and*

(ii) $\sqrt{2p\sum_{1\le i<j\le p}\mathfrak{F}_{\mathbb{B}}(a_ia_j)\mathfrak{F}_{\mathbb{B}}(a_ja_i)} \ge \mathfrak{E}(\mathfrak{F}_{\mathbb{B}}(a_ia_j)) \ge \sqrt{\Phi_3+\Phi_4}$,

where

$$\Phi_1 = 2\sum_{1\le i<j\le p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)\mathfrak{T}_{\mathbb{B}}(a_ja_i),$$
$$\Phi_2 = p(p-1)|\det(\mathcal{A}\mathfrak{T}_{\mathbb{B}}(a_ia_j))|^{\frac{2}{p}},$$
$$\Phi_3 = 2\sum_{1\le i<j\le p}\mathfrak{F}_{\mathbb{B}}(a_ia_j)\mathfrak{F}_{\mathbb{B}}(a_ja_i),$$
$$\Phi_4 = p(p-1)|\det(\mathcal{A}\mathfrak{F}_{\mathbb{B}}(a_ia_j))|^{\frac{2}{p}}.$$

Proof.
Upper Bound:
Applying Cauchy's Schwarts inequality to p numbers $1, 1, \dots, 1$ and $|\lambda_1|, |\lambda_2|, \dots, |\lambda_p|$.

$$\sum_{i=1}^{p}|\lambda_i| \le \sqrt{p}\sqrt{\sum_{i=1}^{p}|\lambda_i|^2}, \tag{1}$$
$$(\sum_{i=1}^{p}\lambda_i)^2 = \sum_{i=1}^{p}|\lambda_i|^2 + 2\sum_{1\le i<j\le p}\lambda_i\lambda_j \tag{2}$$

By equating the coefficient of λ^{p-2} in the characteristic polynomial

$$\prod_{i=1}^{p}(\lambda-\lambda_i) = |\mathcal{A}(G) - \lambda I|,$$
$$\sum_{1\le i<j\le p}\lambda_i\lambda_j = -2\sum_{1\le i<j\le p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)^2 \tag{3}$$

Substitute eq. (3) in eq. (2), we get

$$\sum_{i=1}^{p}|\lambda_i|^2 = 2\sum_{1\le i<j\le p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)^2 \tag{4}$$

Next, by substituting eq. (4) into eq. (1), we obtain

$$\sum_{i=1}^{p}|\lambda_i| \le \sqrt{p}\sqrt{2\sum_{1\le i<j\le p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)^2} = \sqrt{2p\sum_{1\le i<j\le p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)^2}.$$

Therefore,

$$\mathfrak{E}(\mathfrak{T}_{\mathbb{B}}(a_ia_j)) \le \sqrt{2p\sum_{1\le i<j\le p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)\mathfrak{T}_{\mathbb{B}}(a_ja_i)}$$

Lower Bound:

$$\mathfrak{E}(\mathfrak{T}_{\mathbb{B}}(a_ia_j))^2 = \sum_{i=1}^{p}|\lambda_i|^2 = \sum_{i=1}^{p}|\lambda_i|^2 + 2\sum_{1\le i<j\le p}|\lambda_i\lambda_j| \tag{5}$$
$$= 2\sum_{1\le i<j\le p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)^2 + \frac{2p(p-1)}{2}A.M\{|\lambda_i\lambda_j|\}, \tag{6}$$
$$\mathfrak{E}(\mathfrak{T}_{\mathbb{B}}(a_ia_j)) \ge \sqrt{2\sum_{1\le i<j\le p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)^2 + p(p-1)G.M\{|\lambda_i\lambda_j|\}} \tag{7}$$

$$G.M\{|\lambda_i\lambda_j|\} = \left(\prod_{1\le i<j\le p}|\lambda_i\lambda_j|\right)^{\frac{2}{p(p-1)}},$$
$$= \left(\prod_{i=1}^{p}|\lambda_i|^{p-1}\right)^{\frac{2}{p(p-1)}},$$
$$= \left(\prod_{i=1}^{p}|\lambda_i|\right)^{\frac{2}{p}},$$
$$= |\det(\mathcal{A}\mathfrak{T}_{\mathbb{B}}(a_ia_j))|^{\frac{2}{p}}. \tag{8}$$

Substitute eq. (8) in eq. (7), we get

$$\mathfrak{E}(\mathfrak{T}_{\mathbb{B}}(a_ia_j)) \ge \sqrt{\Phi_5}$$

where

$$\Phi_5 = 2\sum_{1\le i<j\le p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)^2 + p(p-1)|A\det(\mathcal{A}\mathfrak{T}_{\mathbb{B}}(a_ia_j))|^{\frac{2}{p}}.$$

Thus,

$$\sqrt{\Phi_1+\Phi_2} \le \mathfrak{E}(\mathfrak{T}_{\mathbb{B}}(a_ia_j)) \le \sqrt{2p\sum_{1\le i<j\le p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)\mathfrak{T}_{\mathbb{B}}(a_ja_i)},$$

Similarly, we can demonstrate that

$$\sqrt{\Phi_3+\Phi_4} \le \mathfrak{E}(\mathfrak{F}_{\mathbb{B}}(a_ia_j)) \le \sqrt{2p\sum_{1\le i<j\le p}\mathfrak{F}_{\mathbb{B}}(a_ia_j)\mathfrak{F}_{\mathbb{B}}(a_ja_i)}.$$

□

Example 3. From Figure 1, $p = 3$, $q = 3$, $\mathfrak{E}(\mathfrak{T}_{\mathbb{B}}(a_ia_j)) = 0.546410$ and $\mathfrak{E}(\mathfrak{F}_{\mathbb{B}}(a_ia_j))$ =3.2056

$$|\det(\mathcal{A}\mathfrak{T}_{\mathbb{B}}(a_ia_j))| = \begin{vmatrix} 0 & 0.1 & 0.1 \\ 0.1 & 0 & 0.2 \\ 0.1 & 0.2 & 0 \end{vmatrix} = 0.004,$$
$$|\det(\mathcal{A}\mathfrak{F}_{\mathbb{B}}(a_ia_j))| = \begin{vmatrix} 0 & 0.9 & 0.7 \\ 0.9 & 0 & 0.8 \\ 0.7 & 0.8 & 0 \end{vmatrix} = 1.008$$

1. Lower limit of truth values:

$$\sqrt{2(0.01+0.01+0.04)+3(3-1)|0.004|^{\frac{2}{3}}}=0.5208.$$

2. Upper limit of truth values:

$$\sqrt{2.3(0.06)}=0.6.$$

3. Lower limit of falsity values:

$$\sqrt{2(0.81+0.49+0.64)+3(3-1)|1.008|^{\frac{2}{3}}}=3.1483.$$

4. Upper limit of falsity values:

$$\sqrt{2.3(1.94)}=3.4112.$$

Theorem 2 is satisfied.

Theorem 3. *Let G be an PIFG (without loops) with $|\mathbb{A}|=p$ and $|\mathbb{B}|=q$ and $\mathcal{A}(G)=[a_{ij}]$. $a_{ij}=((\mathfrak{T}_{\mathbb{B}}(a_ia_j)),(\mathfrak{F}_{\mathbb{B}}(a_ia_j)))$ is a connection matrix of PIFG G. If*

$$p\leq 2\sum_{1\leq i<j\leq p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)^2 and p\leq 2\sum_{1\leq i<j\leq p}\mathfrak{F}_{\mathbb{B}}(a_ia_j)^2,$$

then

(i) $\mathfrak{E}(\mathfrak{T}_{\mathbb{B}}(a_ia_j))\leq\frac{2\sum_{1\leq i<j\leq p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)^2}{p}\sqrt{(p-1)\{\Phi_6-\Phi_7\}}$ *with*

$$\Phi_6=2\sum_{1\leq i<j\leq p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)^2,$$

$$\Phi_7=\left[\frac{2\sum_{1\leq i<j\leq p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)^2}{p}\right]^2.$$

(ii) $\mathfrak{E}(\mathfrak{F}_{\mathbb{B}}(a_ia_j))\leq\frac{2\sum_{1\leq i<j\leq p}\mathfrak{F}_{\mathbb{B}}(a_ia_j)^2}{p}\sqrt{(p-1)\{\Phi_8-\Phi_9\}}$ *with*

$$\Phi_8=2\sum_{1\leq i<j\leq p}\mathfrak{F}_{\mathbb{B}}(a_ia_j)^2,$$

$$\Phi_9=\left[\frac{2\sum_{1\leq i<j\leq p}\mathfrak{F}_{\mathbb{B}}(a_ia_j)^2}{p}\right]^2.$$

Proof. The connection matrix $\mathcal{A}(G)$ is a symmetric matrix with zero trace, then $\max(\lambda)\geq\frac{2\sum_{1\leq i<j\leq p}a_{ij}}{p}$ which is a maximum of eigen value. $\lambda_1\geq\frac{2\sum_{1\leq i<j\leq p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)}{p}$, where the set of characteristic values are in non-increasing sequence. Also

$$\sum_{i=1}^{p}\lambda_i^2=2\sum_{1\leq i<j\leq p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)^2,$$

$$\sum_{i=2}^{p}\lambda_i^2=|2\sum_{1\leq i<j\leq p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)^2-\lambda_1^2. \quad (9)$$

Using Cauchy-Schwarz inequality for the vectors $(1,1\ldots1)$ and $(|\lambda_1|,|\lambda_2|,\ldots,|\lambda_p|)$ with p-1 entries,

$$\mathfrak{E}(\mathfrak{T}_{\mathbb{B}}(a_ia_j))-\lambda_1=\sum_{i=2}^{p}|\lambda_i|\leq\sqrt{(p-1)\sum_{i=2}^{p}|\lambda_i|^2},$$

$$\mathfrak{E}(\mathfrak{T}_{\mathbb{B}}(a_ia_j))-\lambda_1\leq\sqrt{(p-1)(2\sum_{1\leq i<j\leq p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)^2-\lambda_1^2)},$$

$$\mathfrak{E}(\mathfrak{T}_{\mathbb{B}}(a_ia_j))\leq\lambda_1+\sqrt{(p-1)(2\sum_{1\leq i<j\leq p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)^2-\lambda_1^2)}. \quad (10)$$

The function,

$$F(z)=z+\sqrt{(p-1)(2\sum_{1\leq i<j\leq p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)^2-z^2)}$$

decreases on the interval

$$\left[\sqrt{\frac{2\sum_{1\leq i<j\leq p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)^2}{p}},\sqrt{2\sum_{1\leq i<j\leq p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)^2}\right]$$

also,

$$p\leq 2\sum_{1\leq i<j\leq p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)^2,$$

$$1\leq\frac{2\sum_{1\leq i<j\leq p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)^2}{p}.$$

Therefore,

$$\sqrt{2\sum_{1\leq i<j\leq p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)^2}\geq\lambda_1\geq\frac{2\sum_{1\leq i<j\leq p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)}{p}$$
$$\geq\frac{2\sum_{1\leq i<j\leq p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)^2}{p}$$
$$\geq\sqrt{\frac{2\sum_{1\leq i<j\leq p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)^2}{p}}.$$

From eq. (10), we get

$$\mathfrak{E}(\mathfrak{T}_{\mathbb{B}}(a_ia_j))\leq\frac{2\sum_{1\leq i<j\leq p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)^2}{p}\sqrt{(p-1)\{\Phi_6-\Phi_7\}},$$

$$\Phi_6=2\sum_{1\leq i<j\leq p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)^2,$$

$$\Phi_7=\left[\frac{2\sum_{1\leq i<j\leq p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)^2}{p}\right]^2.$$

Same way it is easy to prove

$$\mathfrak{E}(\mathfrak{F}_{\mathbb{B}}(a_ia_j))\leq\frac{2\sum_{1\leq i<j\leq p}\mathfrak{F}_{\mathbb{B}}(a_ia_j)^2}{p}\sqrt{(p-1)\{\Phi_8-\Phi_9\}},$$

$$\Phi_8=2\sum_{1\leq i<j\leq p}\mathfrak{F}_{\mathbb{B}}(a_ia_j)^2,$$

$$\Phi_9=\left[\frac{2\sum_{1\leq i<j\leq p}\mathfrak{F}_{\mathbb{B}}(a_ia_j)^2}{p}\right]^2.$$

□

Definition 5. The degree matrix of an PIFG, G $(\mathbb{A}, \mathbb{B})$ is defined as a $n \times n$ diagonal matrix,

$$\mathfrak{D}(G) = (\mathfrak{D}((\mathfrak{T}_{\mathbb{B}}(a_i a_j)), \mathfrak{D}(\mathfrak{F}_{\mathbb{B}}(a_i a_j))) = [d_{ij}],$$

where

$$d_{ij} = \begin{cases} do(ai), & ififi = j, \\ 0, & \text{otherwise} \end{cases}$$

Definition 6. The Laplacian matrix of an PIFG, G $(\mathbb{A}, \mathbb{B})$ is defined as,

$$\mathfrak{L}(G) = (\mathfrak{L}(\mathfrak{T}_{\mathbb{B}}(a_i a_j)), \mathfrak{L}(\mathfrak{F}_{\mathbb{B}}(a_i a_j))) = \mathfrak{D}(G) - \mathcal{A}(G).$$

Example 4. Consider Figure 1 from Example 1,

$$\mathfrak{D}(G) = \begin{pmatrix} <0.2, 1.6> & <0,0> & <0,0> \\ <0,0> & <0.3, 1.7> & <0,0> \\ <0,0> & <0,0> & <0.3, 1.5> \end{pmatrix},$$

$$\mathcal{A}(G) = \begin{pmatrix} <0,0> & <0.1, 0.9> & <0.1, 0.7> \\ <0.1, 0.9> & <0,0> & <0.2, 0.8> \\ <0.1, 0.7> & <0.2, 0.8> & <0,0> \end{pmatrix},$$

$$\mathfrak{L}(G) = \begin{pmatrix} <0.2, 1.6> & <-0.1, -0.9> & <-0.1, -0.7> \\ <-0.1, -0.9> & <0.3, 1.7> & <-0.2, -0.8> \\ <-0.1, -0.7> & <-0.2, -0.8> & <0.3, 1.5> \end{pmatrix}.$$

The degree matrix $\mathfrak{D}(G)$, connection matrix $\mathcal{A}(G)$ and Laplacian matrix $\mathfrak{L}(G)$ were found.

Definition 7. The spectrum of Laplacian matrix of a PIFG $\mathfrak{L}(G)$ is defined as $(\mathfrak{L}E_{\mathfrak{T}}, \mathfrak{L}E_{\mathfrak{F}})$, where $\mathfrak{L}E_{\mathfrak{T}}$ is the set of Laplacian characteristic values of $\mathfrak{L}(\mathfrak{T}_{\mathbb{B}}(a_i a_j))$ and $\mathfrak{L}E_{\mathfrak{F}}$ is the set of Laplacian characteristic values of $\mathfrak{L}(\mathfrak{F}_{\mathbb{B}}(a_i a_j))$.

Theorem 4. *Let $G = (\mathbb{A}, \mathbb{B})$ be a PIFG and $\mathfrak{L}(G)$ be the Laplacian matrix of G. If the Laplacian characteristic values of $\mathfrak{L}(\mathfrak{T}_{\mathbb{B}}(a_i a_j))$ and $\mathfrak{L}(\mathfrak{F}_{\mathbb{B}}(a_i a_j))$ are $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_p$ and $\beta_1 \geq \beta_2 \geq \beta_3 \geq \cdots \geq \beta_p$ respectively, then*

(i) $\sum_{\substack{i=1 \\ \alpha_i \in \mathfrak{L}E_{\mathfrak{T}}}}^{p} \alpha_i = 2\sum_{1 \leq i < j \leq p} \mathfrak{T}_{\mathbb{B}}(a_i, a_j)$ *and* $\sum_{\substack{i=1 \\ \beta_i \in \mathfrak{L}E_{\mathfrak{F}}}}^{p} \beta_i = 2\sum_{1 \leq i < j \leq p} \mathfrak{F}_{\mathbb{B}}(a_i, a_j)$.

(ii) $\sum_{\substack{i=1 \\ \alpha_i \in \mathfrak{L}E_{\mathfrak{T}}}}^{p} \alpha_i^2 = 2\sum_{1 \leq i < j \leq p} (\mathfrak{T}_{\mathbb{B}}(a_i, a_j))^2 + \sum_{i=1}^{p} d_o^2 \mathfrak{T}_{\mathbb{A}}(a_i)$ *and* $\sum_{\substack{i=1 \\ \beta_i \in \mathfrak{L}E_{\mathfrak{F}}}}^{p} \beta_i^2 = 2\sum_{1 \leq i < j \leq p} (\mathfrak{F}_{\mathbb{B}}(a_i, a_j))^2 + \sum_{i=1}^{p} d_o^2 \mathfrak{F}_{\mathbb{A}}(a_i)$

Proof.

(i) With $\mathfrak{L}(G)$ being a symmetric matrix, whose Laplacian characteristic values are non-negative, the implication is:

$$\begin{aligned} \sum_{\substack{i=1 \\ \alpha_i \in \mathfrak{L}E_{\mathfrak{T}}}}^{p} \alpha_i &= \text{tr}(\mathfrak{L}(G)) \\ &= \sum_{i=1}^{p} d_o \mathfrak{T}_{\mathbb{A}}(a_i) \\ &= 2 \sum_{1 \leq i < j \leq p} \mathfrak{T}_{\mathbb{B}}(a_i a_j). \end{aligned}$$

Therefore,

$$\sum_{\substack{i=1 \\ \alpha_i \in \mathfrak{L}E_{\mathfrak{T}}}}^{p} \alpha_i = 2 \sum_{1 \leq i < j \leq p} \mathfrak{T}_{\mathbb{B}}(a_i a_j).$$

Similarly, it is easy to show that

$$\sum_{\substack{i=1 \\ \beta_i \in \mathfrak{L}E_{\mathfrak{F}}}}^{p} \beta_i = 2 \sum_{1 \leq i < j \leq p} \mathfrak{F}_{\mathbb{B}}(a_i a_j).$$

(ii)

$$\mathfrak{L}(\mathfrak{T}_{\mathbb{B}}(a_i a_j)) = \begin{pmatrix} d_o \mathfrak{T}_{\mathbb{A}}(a_1) & -\mathfrak{T}_{\mathbb{B}}(a_1 a_2) & \cdots & -\mathfrak{T}_{\mathbb{B}}(a_1 a_p) \\ -\mathfrak{T}_{\mathbb{B}}(a_2 a_1) & d_o \mathfrak{T}_{\mathbb{A}}(a_2) & \cdots & -\mathfrak{T}_{\mathbb{B}}(a_2\ a_p) \\ \vdots & \vdots & \ddots & \vdots \\ -\mathfrak{T}_{\mathbb{B}}(a_p a_1) & -\mathfrak{T}_{\mathbb{B}}(a_p a_1) & \cdots & d_o \mathfrak{T}_{\mathbb{A}}(a_p) \end{pmatrix}.$$

Using properties of trace in a matrix,

$$\text{Tr}((\mathfrak{L}(\mathfrak{T}_{\mathbb{B}}(a_i a_j)))^2 = \sum_{\substack{i=1 \\ \alpha_i \in \mathfrak{L}E_{\mathfrak{T}}}}^{p} \alpha_i^2.$$

Also

$$\begin{aligned} \text{Tr}(\mathfrak{L}(\mathfrak{T}_{\mathbb{B}}(a_i a_j)))^2) &= (d_o^2 \mathfrak{T}_{\mathbb{A}}(a_1) + \mathfrak{T}_{\mathbb{B}}^2(a_1 a_2) + \cdots \\ &\quad + \mathfrak{T}_{\mathbb{B}}^2(a_1 a_p)) + (\mathfrak{T}_{\mathbb{B}}^2(a_2 a_1) \\ &\quad + d_o^2 \mathfrak{T}_{\mathbb{A}}(a_2) + \cdots + \mathfrak{T}_{\mathbb{B}}^2(a_2 a_p)) + \cdots \\ &\quad + (\mathfrak{T}_{\mathbb{B}}^2(a_p a_1) + \mathfrak{T}_{\mathbb{B}}^2(a_p a_2) + \cdots \\ &\quad + d_o^2 \mathfrak{T}_{\mathbb{A}}(a_p)), \\ &= 2 \sum_{1 \leq i < j \leq p} (\mathfrak{T}_{\mathbb{B}}(a_i a_j))^2 + \sum_{i=1}^{p} d_o^2 \mathfrak{T}_{\mathbb{A}}(a_i). \end{aligned}$$

Therefore,

$$\sum_{\substack{i=1 \\ \alpha_i \in \mathfrak{L}E_{\mathfrak{T}}}}^{p} \alpha_i^2 = \sum_{1 \leq i < j \leq p} (\mathfrak{T}_{\mathbb{B}}(a_i a_j))^2 + \sum_{i=1}^{p} d_o^2 \mathfrak{T}_{\mathbb{A}}(a_i).$$

Analogously, it is easy to prove that

$$\sum_{\substack{i=1 \\ \beta_i \in \mathfrak{L}E_{\mathfrak{F}}}}^{p} \beta_i^2 = 2 \sum_{1 \leq i < j \leq p} (\mathfrak{F}_{\mathbb{B}}(a_i\ a_j))^2 + \sum_{i=1}^{p} d_o^2 \mathfrak{F}_{\mathbb{A}}(a_i).$$

□

Definition 8. The Laplacian energy of PIFG G $(\mathbb{A}, \mathbb{B})$ is defined as

$$\mathfrak{LE}(G) = (\mathfrak{LE}((\mathfrak{T}_{\mathbb{B}}(a_ia_j)), \mathfrak{LE}((\mathfrak{F}_{\mathbb{B}}(a_ia_j))) = (\sum_{i=1}^{p}|\rho_i|, \sum_{i=1}^{p}|\zeta_i|),$$

where

$$\rho_i = a_i - \frac{\sum_{1\leq i<j\leq p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)}{p}, \quad \zeta_i = \beta_i - \frac{\sum_{1\leq i<j\leq p}\mathfrak{F}_{\mathbb{B}}(a_ia_j)}{p}.$$

Example 5. Consider Figure 1 from Example 1

$$\mathfrak{L}(G) = \begin{pmatrix} <0.2, 1.6> & <-0.1,-0.9> & <-0.1,-0.7> \\ <-0.1,-0.9> & <0.3, 1.7> & <-0.2,-0.8> \\ <-0.1,-0.7> & <-0.2,-0.8> & <0.3, 1.5> \end{pmatrix}$$

The spectrum of the above Laplacian matrix is

$$\mathfrak{L}(E\mathfrak{T}_{\mathbb{B}}(a_ia_j)) = \{\alpha_i\} = \{0, 0.3, 0.5\},$$
$$\mathfrak{L}(E\mathfrak{F}_{\mathbb{B}}(a_ia_j)) = \{\beta_i\} = \{0, 2.22679, 2.57321\},$$
$$(\mathfrak{L}E_{\mathfrak{T}}, \mathfrak{L}E_{\mathfrak{F}}) = \{(0,0), (0.3, 2.22679), (0.5, 2.57321)\}.$$

Now,

$$\rho_i = \alpha_i - \frac{\sum_{1\leq i<j\leq p}\mathfrak{T}_{\mathbb{B}}(a_ia_j)}{p},$$
$$\rho_1 = 0 - \frac{(0.1+0.1+0.2)}{3} = 0.13333,$$
$$\rho_2 = 0.3 - \frac{(0.1+0.1+0.2)}{3} = 0.16667,$$
$$\rho_3 = 0.5 - \frac{(0.1+0.1+0.2)}{3} = 0.36667,$$

and

$$\zeta_i = \beta_i - \frac{\sum_{1\leq i<j\leq p}\mathfrak{F}_{\mathbb{B}}(a_ia_j)}{p},$$
$$\zeta_1 = 0 - \frac{(0.7+0.9+0.8)}{3} = -0.8,$$
$$\zeta_2 = 2.22679 - \frac{(0.7+0.9+0.8)}{3} = 1.42679,$$
$$\zeta_3 = 2.57321 - \frac{(0.7+0.9+0.8)}{3} = 1.77321,$$
$$\sum_{i=1}^{p}|\rho_i| = 0.13333 + 0.16667 + 0.36667 = 0.66667,$$
$$\sum_{i=1}^{p}|\zeta_i| = 0.8 + 1.42679 + 1.77321 = 4.$$

The Laplacian Energy is

$$\mathfrak{LE}(G) = (\mathfrak{LE}((\mathfrak{T}_{\mathbb{B}}(a_ia_j)), \mathfrak{LE}(\mathfrak{F}_{\mathbb{B}}(a_ia_j))) = (\sum_{i=1}^{p}|\rho_i|, \sum_{i=1}^{p}|\zeta_i|) = (0.66667, 4)$$

Furthermore,

$$\sum_{\substack{i=1\\ \alpha_i\in\mathfrak{L}E_{\mathfrak{T}}}}^{p} \alpha_i = 0 + 0.3 + 0.5 = 0.8,$$
$$2\sum_{1\leq i<j\leq p}\mathfrak{T}_{\mathbb{B}}(a_ia_j) = 2(0.1+0.1+0.2) = 0.8,$$
$$\text{(i.e)} \sum_{\substack{i=1\\ \alpha_i\in\mathfrak{L}E_{\mathfrak{T}}}}^{p} \alpha_i = 2\sum_{1\leq i<j\leq p}\mathfrak{T}_{\mathbb{B}}(a_ia_j),$$
$$\sum_{\substack{i=1\\ \alpha_i\in\mathfrak{L}E_{\mathfrak{T}}}}^{p} \alpha_i^2 = 0 + 0.09 + 0.25 = 0.34,$$
$$2\sum_{1\leq i<j\leq p}(\mathfrak{T}_{\mathbb{B}}(a_ia_j))^2 + \sum_{i=1}^{p} d_o^2\mathfrak{T}_{\mathbb{A}}(a_i) = 2(0.01+0.01+0.04) + (0.04 + 0.09 + 0.09) = 0.34,$$
$$\text{(i.e)} \sum_{\substack{i=1\\ \alpha_i\in\mathfrak{L}E_{\mathfrak{T}}}}^{p} \alpha_i^2 = 2\sum_{1\leq i<j\leq p}(\mathfrak{T}_{\mathbb{B}}(a_ia_j))^2 + \sum_{i=1}^{p} d_o^2\mathfrak{T}_{\mathbb{A}}(a_i).$$

Also,

$$\sum_{\substack{i=1\\ \beta_i\in\mathfrak{L}E_{\mathfrak{F}}}}^{p} \beta_i = 0 + 2.22679 + 2.57321 = 4.8,$$
$$2\sum_{1\leq i<j\leq p}\mathfrak{F}_{\mathbb{B}}(a_ia_j) = 2(0.7+0.9+0.8) = 4.8,$$
$$\text{(i.e)} \sum_{\substack{i=1\\ \beta_i\in\mathfrak{L}E_{\mathfrak{F}}}}^{p} \beta_i = 2\sum_{1\leq i<j\leq p}\mathfrak{F}_{\mathbb{B}}(a_ia_j),$$
$$\sum_{\substack{i=1\\ \beta_i\in\mathfrak{L}E_{\mathfrak{F}}}}^{p} \beta_i^2 = 0 + 4.9586 + 6.6214 = 11.58,$$
$$2\sum_{1\leq i<j\leq p}(\mathfrak{F}_{\mathbb{B}}(a_ia_j))^2 + \sum_{i=1}^{p} d_o^2\mathfrak{F}_{\mathbb{A}}(a_i) = 2(0.49+0.64+0.81) + (2.56 + 2.89 + 2.25) = 11.58,$$
$$\text{(i.e)} \sum_{\substack{i=1\\ \beta_i\in\mathfrak{L}E_{\mathfrak{F}}}}^{p} \beta_i^2 = 2\sum_{1\leq i<j\leq p}(\mathfrak{F}_{\mathbb{B}}(a_ia_j))^2$$

$$+\sum_{i=1}^{p} d_o^2 \mathfrak{F}_{\mathbb{A}}(a_i).$$

Observation

Table 1. Comparison Table: Energy vs Laplacian Energy in PIFG

Feature	Energy	Laplacian Energy
Based on	Connection Matrix	Laplacian Matrix
Eigen values from	Connection matrix	Laplacian matrix
Reflects	Total present or intensity of relations	Imbalance or variation of influence
Degree usage	Not used	Used
Interpretation	Strength of association	Structural irregularity

In PIFG, energy quantifies the magnitude ("How much?"), whereas Laplacian energy measures the uniformity ("How uniformly?").

4. Application in Medical Diagnosis System

In medical diagnosis, patient data often contains uncertainty, vagueness and hesitation due to symptoms being subjective or overlapping across multiple conditions. Traditional crisp or even intuitionistic fuzzy graph models may fall short in fully capturing such ambiguity. Pythagorean intuitionistic fuzzy graphs (PIFGs) offer a richer framework by accommodating more flexible truth membership and falsity membership degrees, making them particularly suitable for modelling diagnostic networks.

In the context of medical diagnosis networks, vertices signify key medical parameters, encompassing symptoms, diagnostic tests and diseases. The edges connecting these vertices quantify the strength and confidence of relationships between parameters through Pythagorean intuitionistic fuzzy values, thereby capturing inherent uncertainties. Utilizing a specifically defined connection matrix facilitates the numerical representation of these complex relationships. Furthermore, energy and Laplacian energy serve as metrics to assess the overall disorder or instability within the network, offering valuable insights into the structural dynamics of medical diagnoses.

PIFG for heart condition diagnosis

1. Identify vertices: symptoms (chest pain, shortness of breath), test results (ECG, BP, Cholesterol), diagnoses (Angina, myocardial infarction).
2. Use Pythagorean intuitionistic fuzzy values for edges
3. Calculate energy and Laplacian energy for each patient's graph.

Insights: Higher energy – unstable diagnosis, further investigations needed.
Lower Energy: More confident diagnosis.
Benefits: Improved diagnostic accuracy and personalized treatment plans.

This approach enhances heart condition diagnosis by quantifying uncertainty.

Example 6. $G = (\mathbb{A}, \mathbb{B})$ is a Pythagorean intuitionistic fuzzy graph (PIFG), where vertex set $\mathbb{A} = \{a_1, a_2, a_3, a_4, a_5\}$. Assign truth membership to which a symptom or test confirms the presence of disease, falsity membership to which it denies or contradicts it and Pythagorean intuitionistic hesitation to uncertainty or lack of information. The values in Table 2 indicate the vertex values that is the relevance of each symptom/test to heart disease diagnosis. The values in Table 3 indicate the edge values that represent the strength of influence between symptom/tests and the diagnosis or another symptom (target).
The Connection matrix for the above PIFG is

$$\mathcal{A}(G) = [a_{ij}], \quad a_{ij} = ((\mathfrak{T}_{\mathbb{B}}(a_i a_j)), (\mathfrak{F}_{\mathbb{B}}(a_i a_j))),$$

where

$$\mathcal{A}\mathfrak{T}_{\mathbb{B}}(a_i a_j) = \begin{pmatrix} 0 & 0.45 & 0 & 0 & 0.54 \\ 0.45 & 0 & 0.36 & 0 & 0.45 \\ 0 & 0.36 & 0 & 0 & 0.36 \\ 0 & 0 & 0 & 0 & 0.54 \\ 0.54 & 0.45 & 0.36 & 0.54 & 0 \end{pmatrix},$$

$$\mathcal{A}\mathfrak{F}_{\mathbb{B}}(a_i a_j) = \begin{pmatrix} 0 & 0.4 & 0 & 0 & 0.3 \\ 0.4 & 0 & 0.5 & 0 & 0.4 \\ 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0.35 \\ 0.3 & 0.4 & 0.5 & 0.35 & 0 \end{pmatrix},$$

$$\mathcal{A}(G) = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} \\ m_{31} & m_{32} & m_{33} & m_{34} & m_{35} \\ m_{41} & m_{42} & m_{43} & m_{44} & m_{45} \\ m_{51} & m_{52} & m_{53} & m_{54} & m_{55} \end{pmatrix},$$

$$\begin{aligned}
m_{11} &= <0,0>, \quad m_{12} =<0.45,0.4>,\\
m_{13} &= <0,0>, \quad m_{14} =<0,0>,\\
m_{15} &= <0.54,0.3>, \quad m_{21} =<0.45,0.4>,\\
m_{22} &= <0,0>, \quad m_{23} =<0.36,0.5>,\\
m_{24} &= <0,0>, \quad m_{25} =<0.45,0.4>,\\
m_{31} &= <0,0>, \quad m_{32} =<0.36,0.5>,\\
m_{33} &= <0,0>, \quad m_{34} =<0,0>,\\
m_{35} &= <0.36,0.5>, \quad m_{41} =<0,0>,\\
m_{42} &= <0,0>, \quad m_{43} =<0,0>,\\
m_{44} &= <0,0>, \quad m_{45} =<0.54,0.35>,\\
m_{51} &= <0.54,0.3>, \quad m_{52} =<0.45,0.4>,\\
m_{53} &= <0.36,0.5>, \quad m_{54} =<0.54,0.35>,\\
m_{55} &= <0,0>.
\end{aligned}$$

The spectrum of the above PIFG is $(E_{\mathfrak{T}}, E_{\mathfrak{F}})$, where

$$E_{\mathfrak{T}}(\mathfrak{T}_{\mathbb{B}}(a_i a_j)) = \{1.2075, 0.8444, -0.0000, -0.5407, 0.1776\},$$
$$E_{\mathfrak{F}}(\mathfrak{F}_{\mathbb{B}}(a_i a_j)) = \{1.1201, 0.7222, 0.0000, -0.5227, 0.1247\}.$$

Table 2. Vertex attributes with Truth, Falsity, and Hesitancy values in $\mathbb{A}$

Vertex	Description	Truth $\mathfrak{T}_{\mathbb{A}}(a_i)$	Falsity $\mathfrak{F}_{\mathbb{A}}(a_i)$	Hesitancy $\pi_{\mathbb{A}}(a_i)$
a_1: Chest Pain	Common symptom	0.6	0.3	0.1
a_2: High Blood Pressure	Key risk factor	0.5	0.4	0.1
a_3: High Cholesterol	Indirect indicator	0.4	0.5	0.1
a_4: Shortness of Breath (SOB)	Related symptom	0.6	0.35	0.05
a_5: Heart Disease (Target)	Diagnosis class	1.0	0.0	0.0

Table 3. Edge attributes with Truth, Falsity, and Hesitancy values in $\mathbb{B}$

Edge From → To	Description	Truth $\mathfrak{T}_{\mathbb{B}}(a_i, a_j)$	Falsity $\mathfrak{F}_{\mathbb{B}}(a_i, a_j)$	Hesitancy $\pi_{\mathbb{B}}(a_i, a_j)$
$a_1 \to a_5$	Chest pain indicates heart disease	0.54	0.3	0.6184
$a_2 \to a_5$	High BP suggests heart risk	0.45	0.4	0.6375
$a_3 \to a_5$	High cholesterol contributes to heart risk	0.36	0.5	0.6204
$a_4 \to a_5$	SOB indicates worsening cardiac condition	0.54	0.35	0.5859
$a_1 \to a_2$	Chest pain may be caused by high BP	0.45	0.4	0.6375
$a_3 \to a_2$	Cholesterol indirectly relates to BP	0.36	0.5	0.6204

The Energy of the above PIFG is

$$\mathfrak{E}(G) = (\mathfrak{E}((\mathfrak{T}_{\mathbb{B}}(a_i, a_j)), \mathfrak{E}(\mathfrak{F}_{\mathbb{B}}(a_i, a_j))) = (2.7702, 2.4897)$$

with

$$\begin{aligned}\mathfrak{E}(\mathfrak{T}_{\mathbb{B}}(a_i a_j)) &= 1.2075 + 0.8444 + 0.0000 + 0.5407 + 0.1776\\ &= 2.7702,\end{aligned}$$

and

$$\begin{aligned}\mathfrak{E}(\mathfrak{F}_{\mathbb{B}}(a_i a_j)) &= 1.1201 + 0.7222 + 0.0000 + 0.5227 + 0.1247\\ &= 2.4897,\end{aligned}$$

$$\begin{aligned}\sum_{\substack{i=1\\ \lambda_i \in E_{\mathfrak{T}}}}^{5} \lambda_i &= 1.2075 - 0.8444 + 0.0000 - 0.5407 + 0.1776\\ &= 0,\end{aligned}$$

$$\begin{aligned}\sum_{\substack{i=1\\ \delta_i \in E_{\mathfrak{F}}}}^{5} \delta_i &= 1.1201 - 0.7222 + 0.0000 - 0.5227 + 0.1247\\ &= 0,\end{aligned}$$

$$\begin{aligned}\sum_{\substack{i=1\\ \lambda_i \in E_{\mathfrak{T}}}}^{5} \lambda_i^2 &= (1.2075)^2 + (-0.8444)^2 + (-0.5407)^2\\ &\quad + (0.1776)^2\\ &= 2.4950\\ &= 2(1.2474),\\ &= 2((0.45)^2 + (0.54)^2 + (0.36)^2 + (0.45)^2\\ &\quad + (0.36)^2 + (0.54)^2),\\ &= 2(\mathfrak{T}_{\mathbb{B}}(a_1, a_2)^2 + \mathfrak{T}_{\mathbb{B}}(a_1, a_5)^2 + \mathfrak{T}_{\mathbb{B}}(a_3, a_2)^2\\ &\quad + \mathfrak{T}_{\mathbb{B}}(a_2, a_5)^2 + \mathfrak{T}_{\mathbb{B}}(a_3, a_5)^2 + \mathfrak{T}_{\mathbb{B}}(a_4, a_5)^2),\\ &= 2 \sum_{1 \le i < j \le 5} \mathfrak{T}_{\mathbb{B}}(a_i, a_j)^2,\end{aligned}$$

$$\begin{aligned}\sum_{\substack{i=1\\ \delta_i \in E_{\mathfrak{F}}}}^{5} \delta_i^2 &= (1.1201)^2 + (-0.7222)^2 + (-0.5227)^2\\ &\quad + (0.1247)^2\\ &= 2.0649\\ &= 2(1.0325),\\ &= 2((0.4)^2 + (0.3)^2 + (0.5)^2 + (0.4)^2 + (0.5)^2 + (0.35)^2),\\ &= 2(\mathfrak{F}_{\mathbb{B}}(a_1 a_2)^2 + \mathfrak{F}_{\mathbb{B}}(a_1 a_5)^2 + \mathfrak{F}_{\mathbb{B}}(a_3 a_2)^2 + \mathfrak{F}_{\mathbb{B}}(a_2 a_5)^2\\ &\quad + \mathfrak{F}_{\mathbb{B}}(a_3 a_5)^2 + \mathfrak{F}_{\mathbb{B}}(a_4 a_5)^2),\\ &= 2 \sum_{1 \le i < j \le 5} \mathfrak{F}_{\mathbb{B}}(a_i a_j)^2.\end{aligned}$$

Theorem 1 is proved for this data.
To Find Laplacian Energy:

$$\mathfrak{D}(G) = \begin{pmatrix} n_{11} & n_{12} & n_{13} & n_{14} & n_{15}\\ n_{21} & n_{22} & n_{23} & n_{24} & n_{25}\\ n_{31} & n_{32} & n_{33} & n_{34} & n_{35}\\ n_{41} & n_{42} & n_{43} & n_{44} & n_{45}\\ n_{51} & n_{52} & n_{53} & n_{54} & n_{55} \end{pmatrix},$$

$$\begin{aligned}n_{11} &= < 0.99, 0.7 >,\ n_{12} =< 0, 0 >,\ n_{13} =< 0, 0 >,\\ n_{14} &= < 0, 0 >,\ n_{15} =< 0, 0 >,\ n_{21} =< 0, 0 >,\\ n_{22} &= < 1.26, 1.3 >,\ n_{23} =< 0, 0 >,\ n_{24} =< 0, 0 >,\\ n_{25} &= < 0, 0 >,\ n_{31} =< 0, 0 >,\ n_{32} =< 0, 0 >,\\ n_{33} &= < 0.72, 1 >,\ n_{34} =< 0, 0 >,\ n_{35} =< 0, 0 >,\\ n_{41} &= < 0, 0 >,\ n_{42} =< 0, 0 >,\ n_{43} =< 0, 0 >,\\ n_{44} &= < 0.54, 0.35 >,\ n_{45} =< 0, 0 >,\ n_{51} =< 0, 0 >,\\ n_{52} &= < 0, 0 >,\ n_{53} =< 0, 0 >,\ n_{54} =< 0, 0 >,\\ n_{55} &= < 1.89, 1.55 >,\end{aligned}$$

$$\mathcal{A}(G) = \begin{pmatrix} o_{11} & o_{12} & o_{13} & o_{14} & o_{15}\\ o_{21} & o_{22} & o_{23} & o_{24} & o_{25}\\ o_{31} & o_{32} & o_{33} & o_{34} & o_{35}\\ o_{41} & o_{42} & o_{43} & o_{44} & o_{45}\\ o_{51} & o_{52} & o_{53} & o_{54} & o_{55} \end{pmatrix},$$

$$\begin{aligned}o_{11} &= < 0, 0 >,\ o_{12} =< 0.45, 0.4 >,\ o_{13} =< 0, 0 >,\\ o_{14} &= < 0, 0 >,\ o_{15} =< 0.54, 0.3 >,\\ o_{21} &= < 0.45, 0.4 >, o_{22} =< 0, 0 >,\\ o_{23} &= < 0.36, 0.5 >,\ o_{24} =< 0, 0 >,\end{aligned}$$

$$o_{25} = \; < 0.45, 0.4 >, \; o_{31} =< 0, 0 >,$$
$$o_{32} = \; < 0.36, 0.5 >, \; o_{33} =< 0, 0 >,$$
$$o_{34} = \; < 0, 0 >, \; o_{35} =< 0.36, 0.5 >,$$
$$o_{41} = \; < 0, 0 >, \; o_{42} =< 0, 0 >, \; o_{43} =< 0, 0 >,$$
$$o_{44} = \; < 0, 0 >, \; o_{45} =< 0.54, 0.35 >,$$
$$o_{51} = \; < 0.54, 0.3 >, \; o_{52} =< 0.45, 0.4 >,$$
$$o_{53} = \; < 0.36, 0.5 >, \; o_{54} =< 0.54, 0.35 >,$$
$$o_{55} = \; < 0, 0 >,$$
$$\mathfrak{L}(G) = \begin{pmatrix} r_{11} & r_{12} & r_{13} & r_{14} & r_{15} \\ r_{21} & r_{22} & r_{23} & r_{24} & r_{25} \\ r_{31} & r_{32} & r_{33} & r_{34} & r_{35} \\ r_{41} & r_{42} & r_{43} & r_{44} & r_{45} \\ r_{51} & r_{52} & r_{53} & r_{54} & r_{55} \end{pmatrix},$$
$$r_{11} = \; < 0.99, 0.7 >, \; r_{12} =< -0.45, -0.4 >,$$
$$r_{13} = \; < 0, 0 >, \; r_{14} =< 0, 0 >,$$
$$r_{15} = \; < -0.54, -0.3 >, \; r_{21} =< -0.45, -0.4 >,$$
$$r_{22} = \; < 1.26, 1.3 >, \; r_{23} =< -0.36, -0.5 >,$$
$$r_{24} = \; < 0, 0 >, \; r_{25} =< -0.45, -0.4 >,$$
$$r_{31} = \; < 0, 0 >, \; r_{32} =< 0, 0 >, \; r_{33} =< 0.72, 1 >,$$
$$r_{34} = \; < 0, 0 >, \; r_{35} =< -0.36, -0.5 >,$$
$$r_{41} = \; < 0, 0 >, \; r_{42} =< 0, 0 >,$$
$$r_{43} = \; < 0, 0 >, \; r_{44} =< 0.54, 0.35 >,$$
$$r_{45} = \; < -0.54, -0.35 >, \; r_{51} =< -0.54, -0.3 >,$$
$$r_{52} = \; < -0.45, -0.4 >, \; r_{53} =< -0.36, -0.5 >,$$
$$r_{54} = \; < -0.54, -0.35 >, \; r_{55} =< 1.89, 1.55 >,$$

The spectrum of the above Laplacian matrix is

$$\begin{aligned} \mathfrak{L}E_{\mathfrak{T}} &= \mathfrak{L}(E\mathfrak{T}_{\mathbb{B}}(a_i a_j)) \\ &= \{a_i\} \\ &= \{2.3894, 1.5974, 0.0723, 0.5702, 0.7705\}, \\ \mathfrak{L}E_{\mathfrak{F}} &= \mathfrak{L}(E\mathfrak{F}_{\mathbb{B}}(a_i a_j)) \\ &= \{\beta_i\} \\ &= \{0.0936, 0.8633, 1.5611, 1.9790, 0.4029\}, \\ (\mathfrak{L}E_{\mathfrak{T}}, \mathfrak{L}E_{\mathfrak{F}}) &= \{(2.3894, 0.0936), (1.5974, 0.8633)(0.0723, \\ & \quad 1.5611), (0.5702, 1.9790), (0.7705, 0.4029)\}. \end{aligned}$$

Now,

$$\rho_i = a_i - \frac{\sum_{1 \le i < j \le p} \mathfrak{T}_{\mathbb{B}}(a_i a_j)}{p},$$
$$\rho_1 = 2.3894 - \frac{(0.6 + 0.5 + 0.4 + 0.6 + 1.0)}{5} = 1.7694,$$
$$\rho_2 = 1.5974 - \frac{(0.6 + 0.5 + 0.4 + 0.6 + 1.0)}{5} = 0.9774,$$
$$\rho_3 = 0.0723 - \frac{(0.6 + 0.5 + 0.4 + 0.6 + 1.0)}{5} = -0.5477,$$
$$\rho_4 = 0.5702 - \frac{(0.6 + 0.5 + 0.4 + 0.6 + 1.0)}{5} = -0.0498,$$
$$\rho_5 = 0.7705 - \frac{(0.6 + 0.5 + 0.4 + 0.6 + 1.0)}{5} = 0.1505,$$
$$\zeta_i = \beta_i - \frac{\sum_{1 \le i < j \le p} \mathfrak{F}_{\mathbb{B}}(a_i a_j)}{p},$$
$$\zeta_1 = 0.0936 - \frac{(0.3 + 0.4 + 0.5 + 0.35 + 0.0)}{5} = -0.2164,$$
$$\zeta_2 = 0.8633 - \frac{(0.3 + 0.4 + 0.5 + 0.35 + 0.0)}{5} = 0.5533,$$
$$\zeta_3 = 1.5611 - \frac{(0.3 + 0.4 + 0.5 + 0.35 + 0.0)}{5} = 1.2511,$$
$$\zeta_4 = 1.9790 - \frac{(0.3 + 0.4 + 0.5 + 0.35 + 0.0)}{5} = 1.669,$$
$$\zeta_5 = 0.4029 - \frac{(0.3 + 0.4 + 0.5 + 0.35 + 0.0)}{5} = 0.0929,$$
$$\sum_{i=1}^{p} |\rho_i| = 1.7694 + 0.9774 + 0.5477 + 0.0498 + 0.1505 = 3.4948,$$
$$\sum_{i=1}^{p} |\zeta_i| = 0.2164 + 0.5533 + 1.2511 + 1.669 + 0.0929 = 3.7827.$$

The Laplacian Energy is

$$\begin{aligned} \mathfrak{LE}(G) &= (\mathfrak{LE}((\mathfrak{T}_{\mathbb{B}}(a_i a_j)), \mathfrak{LE}(\mathfrak{F}_{\mathbb{B}}(a_i a_j))) \\ &= (\sum_{i=1}^{p} |\rho_i|, \sum_{i=1}^{p} |\zeta_i|), \\ &= (3.4948, 3.7827). \end{aligned}$$

5. Interpretation for Medical Diagnosis:

A Pythagorean intuitionistic fuzzy graph is useful in modelling uncertain, imprecise or conflicting data like medical symptoms, lab test results or patient responses.

1. Energy values — indicate interaction strength
   In this problem, $\mathfrak{E}(\mathfrak{T}_{\mathbb{B}}(a_i a_j)) = 2.7702$ represents the total positive interaction strength among symptoms, test results, or patient parameters. A moderate value suggests meaningful connections exist between medical features.
   Also, $\mathfrak{E}(\mathfrak{F}_{\mathbb{B}}(a_i a_j)) = 2.4897$ measures the overall negative or uncertain interaction strength. The slightly lower value compared to the truth part means the degree of uncertainty or contradiction is present but slightly lower.

Table 4. Diagnostic energy interpretation table

Laplacian Energy Range	Mathematical Interpretation	Physical Interpretation	Recommended Action
0-1.0	Very low energy	Clear diagnosis, minimal conflict or uncertainty. One or two key symptoms strongly match diagnosis	Direct diagnosis may be reliable
1.0-2.0	Low energy	Mild complexity, mostly consistent symptoms. Most symptoms align; minor contradictions	Proceed with diagnosis; low risk
2.0-3.0	Moderate energy	Some uncertainty or overlapping symptom patterns. Possible overlap with other diseases; symptoms moderately spread	Recommend more tests or observation
3.0-4.0	High energy	Complex or conflicting diagnostic evidence. Conflicting signs; fuzzy membership in multiple diagnostic classes	Consider further investigation or referral
4.0-5.0	Very high energy	Highly uncertain or inconsistent symptom graph. Poorly structured or conflicting data	Reevaluate inputs; check for data errors

Table 5. Comparison of IFG, PFG, and PIFG

Feature	IFG	PFG	PIFG
Constraint	$T + F \leq 1$	$T^2 + F^2 \leq 1$	Vertex and Edge: $0 \leq T_v + F_v \leq 1$; Edge: $0 \leq T_e^2 + F_e^2 \leq 1$
Uncertainty space	Limited	Larger	Largest
Vertex hesitation	Yes	No	Yes
Edge uncertainty modeling	Limited	High	Highest (flexible hybrid)
Suitability for complex systems	Moderate	Moderate	High

2. Laplacian energy — measures deviation from uniformity
   $\mathfrak{LE}(\mathfrak{T}_{\mathbb{B}}(a_i a_j)) = 3.4948$: A high Laplacian energy in the truth-membership domain indicates variation or irregularity in how strongly symptoms/test results are associated. Certain symptoms or tests are strongly connected (critical) while others are not.
   $\mathfrak{LE}(\mathfrak{F}_{\mathbb{B}}(a_i a_j)) = 3.7827$: even higher in the falsity domain, suggesting more variability in contradictory or inconsistent information (e.g., atypical symptoms, conflicting reports).

Medical Insights:

The PIFG energy and Laplacian energy values suggest a moderately coherent medical diagnostic pattern, with stronger support for true associations among symptoms and parameters. However, the relatively high Laplacian energies indicate the presence of irregularities or conflicts, potentially highlighting rare cases, misdiagnoses or the need for deeper investigation into patient data. For that, let us present below the five point range Laplacian energy prediction to physical interpretation of medical diagnosis in the Table 4. From the Table 4, it is clear that

- If energy is low, diagnosis is easy.
- If energy is high , it's unclear – more tests needed.

6. Conclusion

This study successfully extends the concept of energy to PIFG, offering a deeper insight into uncertainty and complexity in fuzzy network structures. In this work, the concept of connection matrix were introduced also defined the energy and Laplacian energy specific of PIFGs and established corresponding upper and lower bounds. These measures provide a useful framework for analysing decision-making systems, particularly in diagnostic scenarios where relationship between symptoms and outcomes are not crisp. The interpretation of energy levels helps in identifying the clarity or ambiguity of diagnosis. In future work, this approach can be further generalized to Pythagorean Interval-Valued intuitionistic fuzzy graphs, allowing for even finer representation of uncertainty in complex systems.

Advantages of PIFGs over IFGs and PFGs

1. Advantage over intuitionistic fuzzy graphs (IFGs)
   In intuitionistic fuzzy graphs, the membership and non-membership degrees are constrained by the linear condition. This restriction significantly limits the admissible uncertainty space, making IFGs less effective in modeling systems where hesitation and ambiguity are high. In contrast, Pythagorean intuitionistic fuzzy graphs adopt the squared constraint, which allows both membership and non-membership degrees to take relatively higher values simultaneously. As a result, PIFGs can represent higher degrees of uncertainty and hesitation, particularly in complex relational structures. This increased flexibility leads to better sensitivity in edge modeling and energy-based analysis, which is crucial in applications such as medical diagnosis and decision support systems.
2. Advantage over Pythagorean fuzzy graphs (PFGs)
   Pythagorean fuzzy graphs consider only membership and non-membership degrees but do not explicitly incorporate intuitionistic hesitation at the vertex level. Consequently, PFGs may not adequately distinguish between object-level uncertainty and relation-level uncertainty. PIFGs overcome this limitation by integrating intuitionistic fuzzy modeling for vertices and Pythagorean fuzzy modeling for edges. This hybrid structure enables PIFGs to preserve intuitionistic interpretability for individual entities (vertices), Capture higher relational uncertainty through Pythagorean constraints on edges. Thus, PIFGs provide a more expressive and structured framework for modeling uncertainty in graph-based systems. Unified and enhanced modeling ca-

pability. By combining the strengths of both IFGs and PFGs, PIFGs offer greater admissible uncertainty space, improved discrimination capability in spectral and energy measures, enhanced modeling of nonlinear and interaction-based uncertainty. This makes PIFGs particularly suitable for real-world applications involving noisy, imprecise, or incomplete relational data. We present below a comparison of IFG, PFG, and PIFG in Table 5

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