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Stability of Traveling Waves to a Burgers Equation with 2nd-Order Nonlinear Diffusion

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ABSTRACT

We are interested in the study of asymptotic stability for Burgers equation with second order nonlinear diffusion. We first transform the original equation by the ansatz transformation to establish the existence of traveling wave. We further employ the energy estimate under small perturbation and arbitrary wave amplitude. This energy estimate is then used to establish the stability.

Keywords:
Stability; Arbitrary Wave Amplitude; Small Perturbation; Energy Estimate

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1. Introduction

Our goal is to establish stability of traveling waves to the following Burgers equation with nonlinear diffusion.

$$v_t + \frac{1}{2}(v^2)_x = \frac{\alpha}{2}(v^2)_{xx}, \qquad \alpha > 0, \tag{1}$$

where $v = v(x,t)$ and the initial data

$$v(x,0) = v_0(x) \to v_\pm \text{ as } x \to \pm\infty. \tag{2}$$

The equation (1) is the special case of the following equation,

$$v_t + (f(v))_x = \mu v_{xx}, \tag{3}$$

as studied by Il'in and Oleinik [1] and Sattinger [2] for the stability of shock profiles based on the maximum principle and spectral analysis respectively, where $\mu > 0$ and $f(v)$ is a smooth function.

Moreover, we refer to [3–5], for the existence of traveling waves to the system of Burgers equation and diffusionless Fisher,

$$u_t + a\sqrt{u}u_x = \mu u_{xx},$$

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$$v_t + b\sqrt{v}v_x = c\sqrt{v} - dv,$$

where a, b, c, d, μ are positive constant.

Other studies are in [6–9] for the existence of traveling waves to the Burgers equation containing square root term $\sqrt{v}$. It follows from (3), in this paper, we consider the special case $f(v) = \frac{v^2}{2}$ and nonlinear diffusion $\left(v^2\right)_{xx}$, where this paper is not only limited to study of the existence of traveling waves as the previous ones, but also for the stability of traveling waves to Burgers equation. We use the energy method to establish the a priori estimate under arbitrary wave amplitude and small perturbation, where this technique was also used to study the existence and stability of traveling waves to chemotaxis model as studied in [10, 11]. The elementary energy method can also be used to study the stability problem of traveling waves for coupled Burgers equation as studied in [11],

$$u_t + \frac{1}{2}\left(u^2 + v^2\right)_x = \mu u_{xx},$$
$$v_t + (uv)_x = \nu v_{xx}.$$

A similar problem with Hu [12] was studied by Li and Wang [10, 11], where the smallness of coefficients ε was considerable,

$$u_t - (uv)_x = Du_{xx},$$
$$v_t + \left(\varepsilon v^2 - u\right)_x = \nu v_{xx},$$

and this system was derived from Keller-Segel model, so-called chemotaxis model.

We organize this paper as follows. In Section 2, we present some steps to study the problem of this paper. In Section 3, we present the results and discussion of this paper which consist of the existence of traveling wave solution of V to the transformed Burgers equation (1), the energy estimate of transformed problem, and the stability of traveling waves to Burgers equation (1).

2. Methods

There are some steps to study the stability of traveling waves to Burgers equations with second-order nonlinear diffusion, including:

1. Establishing the existence of traveling waves of (1) by transforming the original equations through the ansatz traveling waves first.
2. Giving the appropriate perturbation to reformulate the traveling waves of (1) which is further used to establish the L^2-estimate, H^1-estimate, and H^2-estimate. Actually, this perturbation has the role as L^2-distance between v and traveling waves V.
3. Based on the energy estimates obtained before, we can prove the stability of traveling waves.

3. Results and Discussion

We first look for the traveling wave $V(x - st)$ of the Burgers equation with the nonlinear diffusion (1). Substituting the following transformation

$$v(x,t) = V(z), \; z = x - st, \tag{4}$$

into (1), where s is wave speed and z is the moving variable, we have

$$-sV_z = \frac{\alpha}{2}(V^2)_{zz} - \frac{1}{2}(V^2)_z \tag{5}$$

where the boundary conditions are imposed as

$$V(z) \to v_\pm \text{ as } z \to \pm\infty. \tag{6}$$

Now, integrating (5) in z and by the fact $V'(z) \to 0$ as $z \to \pm\infty$, we get

$$\frac{\alpha}{2}(V^2)_z - \frac{1}{2}V^2 + \frac{1}{2}v_\pm^2 = -sV + sv_\pm \tag{7}$$

and the condition of Rankine-Hugoniot

$$s(v_- - v_+) = \frac{1}{2}\left(v_-^2 - v_+^2\right) \tag{8}$$

which gives

$$s = \frac{v_- + v_+}{2} > 0. \tag{9}$$

Then employing the classical phase-plane analysis, one can easily find the existence and uniqueness of traveling wave solutions to (1). It follows from (7) and (9), one has

$$V_z = \frac{V^2 - (v_+ + v_-)V + v_+v_-}{2\alpha V} := M(V).$$

We differentiate $M(V)$ with respect to V to get

$$\frac{dM(V)}{dV} = -\frac{1}{V^2}\left(\frac{V^2 - (v_+ + v_-)V + v_+v_-}{2\alpha}\right) + \frac{1}{V}\left(\frac{2V - (v_+ + v_-)}{2\alpha}\right).$$

By substituting the equilibrium (v_+, v_-) into $dM(V)/dV$ and applying the entropy condition $v_+ < v_-$, one has

$$\left.\frac{dM(V)}{dV}\right|_{V=v_+} = \frac{1}{v_+}\left(\frac{v_+ - v_-}{2\alpha}\right) < 0, \quad \left.\frac{dM(V)}{dV}\right|_{V=v_-} = \frac{1}{v_-}\left(\frac{v_- - v_+}{2\alpha}\right) > 0,$$

which implies that v_+ is stable point and v_- is unstable point.

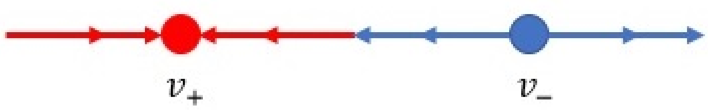

Figure 1. Phase portrait for V

The interpretation of the statement above can be represented as phase portrait in Figure 1. It can be seen that at point v_- all arrows move away from it and at point v_+ all arrows move toward it.

Lemma 1. *We assume that $v_\pm$ satisfy (8). Then there exists a monotone solution of traveling wave $V(x - st)$ to (5), which is unique up to a translation and satisfies $V_z < 0$. Moreover, as*

$z \to \pm\infty$, V decays exponentially with rates

$$V - v_{\pm} \sim e^{\sigma_{\pm} z}, \tag{10}$$

where

$$\sigma_{+} = \frac{1}{v_{+}}\left(\frac{v_{+} - v_{-}}{2\alpha}\right), \; \sigma_{-} = \frac{1}{v_{-}}\left(\frac{v_{-} - v_{+}}{2\alpha}\right).$$

For the transformed Burgers equation (1)*, we define*

$$\varphi_0(z) = \int_{-\infty}^{z} (v_0 - V)(y)dy,$$

which is the zero-mass perturbation (see [13, 14]). Then we have the following stability result.

Theorem 1. *Let one has the traveling wave $V(x - st)$ obtained in Lemma 1. Then we have a constant $\varepsilon_0 > 0$ such that if $\| v_0 - V\|_1 + \| \varphi_0 \| \leq \varepsilon_0$, then the Cauchy problem* (1)-(2) *has a unique global solution $u(x,t)$ satisfying*

$$v - V \in C([0,\infty); H^1) \cap L^2([0,\infty); H^1),$$

and

$$\sup_{x\in\mathcal{R}} |v(x,t) - V(x - st)| \to 0 \text{ as } t \to +\infty.$$

By changing the variables $(x,t) \to (z = x - st, t)$, the Burgers equation (1) becomes

$$v_t - sv_z = \frac{\alpha}{2}(v^2)_{zz} - \frac{1}{2}(v^2)_z \tag{11}$$

We decompose the solution v of (11) as

$$v\,(z,t) = V\,(z) + \varphi_z\,(z,t)\,. \tag{12}$$

Then

$$\varphi(z,t) = \int_{-\infty}^{z} (v(y,t) - V(y))dy \tag{13}$$

Substituting (12) into (11) and the results are integrated in z, one has

$$\varphi_t = (s - V)\varphi_z + \alpha(V\varphi_z)_z - \frac{V^2 + \varphi_z^2}{2} + F, \tag{14}$$

where $F = \alpha((V + \varphi_z)^2 - V^2 - V\varphi_z)_z$. The initial condition of φ is given by

$$\varphi(z,0) = \varphi_0(z) = \int_{-\infty}^{z} (v_0 - V)dy, \tag{15}$$

with $\varphi_0(\pm\infty) = 0$. We find the solution of equation (14)-(15) in the region

$$X\,(0,T) := \left\{\varphi(z,t) \in C([0,T), H^2) : \varphi_z \in L^2((0,T); H^2))\right\},$$

for $0 < T \leq +\infty$. Let

$$N(t) := \sup_{0\le\tau\le t}\{\|\varphi(.,\tau)\|_2\}.$$

It follows from the Sobolev's inequality $\| f\|_{L^\infty} \le \sqrt{2}\, \| f\|_{L^2}^{\frac{1}{2}}\, \| f_x\|_{L^2}^{\frac{1}{2}}$, one has

$$\sup_{\tau\in[0,t]}\{\|\varphi(\cdot,\tau)\|_{L^\infty}, \|\varphi_z(\cdot,\tau)\|_{L^\infty}\} \le N(t).$$

For (14)-(15), we have the following global well-posedness.

Theorem 2. *By the assumptions of Theorem 1, one has a constant $\delta_1 > 0$ such that if $N(0) \le \delta_1$, then the Cauchy problem* (14)-(15) *has unique global solution $\varphi \in X(0, +\infty)$ such that*

$$\|\varphi(.,t)\|_2^2 + \int_0^t \|\varphi_z(.,\tau)\|_2^2\, d\tau \le C\, \|\varphi_0\|_2^2 \tag{16}$$

Moreover, it holds that

$$\sup_{z\in R}|\varphi_z(z,t)| \to 0 \text{ as } t \to +\infty \tag{17}$$

Since the local existence can be established in a standard way (see [15]), we need to prove a priori estimate.

Proposition 1. *We consider that $\varphi \in X(0, T)$ be a solution of* (14)-(15) *for some time $T > 0$. Then one has a constant $\varepsilon_1 > 0$, such that if $N(T) < \varepsilon_1$, then φ satisfies* (16) *for any $0 \le t \le T$.*

We further present the energy estimates for solution φ of (14)-(15), and hence prove Proposition 1. We first derive the basic L^2 estimate.

Lemma 2. *Under the same assumptions of Proposition 1, if $N(t) \ll 1$, then*

$$\|\varphi(.,t)\|^2 + \int_0^t \|\varphi_z(.,\tau)\|^2 d\tau \le C\, \|\varphi_0\|^2 + C\alpha N(t)\int_0^t\int \varphi_{zz}^2 \tag{18}$$

Proof. Multiplying (14) by $\frac{\varphi}{V}$, integrating the resulting equations, we have

$$\frac{1}{2}\frac{d}{dt}\int\frac{\varphi^2}{V} + \left(\alpha+\frac{1}{2V}\right)\int\varphi_z^2 = \int\frac{\varphi^2}{2}\left(\left(\frac{s-V}{V}\right)_z - \alpha\left(V\left(\frac{1}{V}\right)_z\right)_z\right) + \int\left(\frac{F\varphi}{V}\right) - \frac{V\varphi}{2} \tag{19}$$

It follows from (17), then (19) becomes

$$\begin{aligned}\left(\frac{s-V}{V}\right)_z - \alpha\left(V\left(\frac{1}{V}\right)_z\right)_z &= \left(\frac{s-V}{V} - \alpha V\left(\frac{1}{V}\right)_z\right)_z \\ &= \left(\frac{sv_+ + \frac{1}{2}v_+^2}{V^2}\right)_z \\ &= -\frac{v_+(2s+v_+)V_z}{2V^3} > 0\end{aligned} \tag{20}$$

It follows from $V \ge v_+ > 0$ and $\varphi_{z\,L^\infty} \le N(t) \ll 1$, then we have

$$|F| \le C\alpha\left(|\varphi_{zz}|\,|\varphi_z| + |\varphi_z|^2\right). \tag{21}$$

Then by Young's inequality and $\varphi_{L^\infty} \leq N(t)$, obtained

$$\left| \int \frac{F\varphi}{V} \right| \leq C\alpha N(t) \int \left(|\varphi_z|^2 + |\varphi_{zz}|^2 \right). \tag{22}$$

Substituting (20), (22) into (19), and using the fact that $V \geq v_+ > 0$, then the proof of Lemma 3.2 is finished. ■

The following lemma gives the H^1 estimate of φ.

Lemma 3. *Under the same assumptions of Proposition 1, if $N(t) \ll 1$, it holds that*

$$\| \varphi(.,t) \|_1^2 + \int_0^t \|\varphi_z(.,\tau)\|_1^2 d\tau \leq C \| \varphi_0\|_1^2. \tag{23}$$

Proof. Differentiating (14) in z gives

$$\varphi_{zt} = (s-V)\varphi_{zz} + \alpha(V\varphi_z)_{zz} - (V_z\varphi_z + VV_z + \varphi_z\varphi_{zz}) + F_z. \tag{24}$$

Multiplying (24) by $\frac{\varphi_z}{V}$, then integrating the resulting equation, we have

$$\frac{1}{2}\frac{d}{dt}\int \frac{\varphi_z^2}{V} + \alpha \int \varphi_{zz}^2 = \int \frac{\varphi_z^2}{2}\left(\frac{s-V}{V}\right)_z - 2\alpha \int \frac{\varphi_z^2}{2}\left(V\left(\frac{1}{V}\right)_z\right)_z + \int \varphi_z^2 \left(V\left(\frac{1}{V}\right)_z\right)^2 \tag{25}$$

$$+ \int F\left(\frac{\varphi_{zz}}{V} + \varphi_z\left(\frac{1}{V}\right)_z\right) - \int \left(\frac{V_z\varphi_z^2}{V} + \varphi_z V_z + \frac{\varphi_z^2\varphi_{zz}}{V}\right).$$

It follows from $V \geq v_+ > 0$ and $V_z \leq C$, it holds that

$$\left| \int F\left(\frac{\varphi_{zz}}{V} + \varphi_z\left(\frac{1}{V}\right)_z\right)\right| \leq C\int \left(\varphi_{zz}^2 + \varphi_z^2\right), \tag{26}$$

$$\left| \int \left(\frac{V_z\varphi_z^2}{V} + \frac{\varphi_z^2\varphi_{zz}}{V}\right)\right| \leq C\int \left(\varphi_{zz}^2 + \varphi_z^2\right).$$

Moreover, it follows from (7), then two terms in right hand side, one has

$$\frac{\varphi_z^2}{2}\left(\frac{s-V}{V} - 2\alpha V\left(\frac{1}{V}\right)_z\right)_z = \frac{\varphi_z^2}{2}\left(\frac{sv_+ + v_+^2}{V^2}\right)_z = -\frac{\varphi_z^2}{2}\left(\frac{v_+(s+v_+)V_z}{2V^3}\right) > 0. \tag{27}$$

Substituting (26), (27) into (25), and combining the results with (18), one has

$$\int \tfrac{\varphi_z^2}{V} + \alpha(1 - CN(t)) \int_0^t \int \varphi_{zz}^2 \leq C \int \varphi_{0z} \ .$$

We use the fact that $N(t) \ll 1$ and $V \geq v_+ > 0$ to above inequality, then the proof (23) in this Lemma 3 is completed. ■

Next, we present H^2 estimate of φ.

Lemma 4. *Under the same assumptions of Proposition 1, if $N(t) \ll 1$, it holds that*

$$\| \varphi(.,t) \|_2^2 + \int_0^t \|\varphi_z(.,\tau)\|_2^2 \leq C \| \varphi_0\|_2^2. \tag{28}$$

Proof. Differentiating (24) with respect to z gives

$$\varphi_{zzt} = (s-V)\varphi_{zzz} + \alpha(V_z\varphi_{zz} + V\varphi_{zzz})_z + \alpha(V_{zz}\varphi_z + V_z\varphi_{zz})_z + F_{zz} \quad (29)$$
$$-\left(V_{zz}\varphi_z + V_z\varphi_{zz} + V_z^2 + VV_{zz}\varphi_{zz}^2 + \varphi_z\varphi_{zzz}\right)$$

Multiplying (29) by $\frac{\varphi_{zz}}{V}$, we have

$$\frac{1}{2}\frac{d}{dt}\int\frac{\varphi_{zz}^2}{V} + \alpha\int\varphi_{zzz}^2 = \int\frac{\varphi_{zz}^2}{2}\left(\frac{s-V}{V} - 2\alpha V\left(\frac{1}{V}\right)_z\right)_z + \int F_z\left(\frac{\varphi_{zzz}}{V} + \varphi_{zz}\left(\frac{1}{V}\right)_z\right) \quad (30)$$
$$-\int\left(\frac{V_{zz}\varphi_z\varphi_{zz}}{V} + \frac{V_z\varphi_{zz}^2}{V} + \frac{V_z^2\varphi_{zz}}{V} + V_{zz}\varphi_{zz}^3 + \frac{\varphi_z\varphi_{zz}\varphi_{zzz}}{V}\right)$$

Notice that

$$F_z = \alpha((V+\varphi_z)^2 - V^2 - V\varphi_z)_{zz} = \alpha\left(2V_z\varphi_{zz} + 2\varphi_{zz}^2 + V\varphi_{zzz} + \varphi_z V_{zz} + \varphi_z\varphi_{zzz}\right) \quad (31)$$

Then, we use the fact that $V \geq v_+ > 0,\ V_z \leq C,\ V_{zz} \leq C$, and $\varphi_{z\,L^\infty} \leq N(t)$, we have

$$\left|\int F_z\left(\frac{\varphi_{zzz}}{V} + \varphi_{zz}\left(\frac{1}{V}\right)_z\right)\right| \leq CN(t)\int\left(|\varphi_{zz}|^2 + |\varphi_{zzz}|^2\right), \quad (32)$$
$$\left|\int\frac{\varphi_z\varphi_{zz}\varphi_{zzz}}{V}\right| \leq CN(t)\int\left(|\varphi_{zz}|^2 + |\varphi_{zzz}|^2)\right).$$

Moreover, the first term of right-hand side is obtained

$$\frac{\varphi_{zz}^2}{2}\left(\frac{s-V}{V} - 2\alpha V\left(\frac{1}{V}\right)_z\right)_z = \frac{\varphi_{zz}^2}{2}\left(\frac{sv_+ + v_+^2}{V^2}\right)_z = -\frac{\varphi_{zz}^2}{2}\left(\frac{v_+(s+v_+)V_z}{2V^3}\right) > 0. \quad (33)$$

By substituting (32), (33) into (30), and $V \geq v_+ > 0$, then Lemma 4 is proved. ■

Based on the energy estimates, we are now ready to prove the main results in this paper. Owing to the transformation (12), Theorem 1 is a consequence of Theorem 2.

Proof, **Theorem 2**. The energy estimate (16) guarantees that $N(t)$ is small if $N(0)$ is small enough. Thus, applying the standard extension procedure, we get the global well-posedness of (14)-(15) in $X(0,+\infty)$.

Next, we prove the convergence (17). Owing to the global estimate (16), we get

$$\int_0^t\int_{-\infty}^{\infty}\varphi_z^2(z,\tau)\,dzd\tau \leq C\parallel\varphi_0\|_2^2 < \infty. \quad (34)$$

In view of the first equation of (14), by Young's inequality,

$$\frac{d}{dt}\int_{-\infty}^{\infty}\varphi_z^2(z,t)dz = -2\int_{-\infty}^{\infty}\varphi_t\varphi_{zz}dz$$

$$= -2\int_{-\infty}^{\infty}\varphi_{zz}\left((s-V)\varphi_z + \alpha(V\varphi_z)_z + \alpha((V+\varphi_z)^2 - V^2 - V\varphi_z)_z - \frac{V^2+\varphi_z^2}{2}\right) \leq C\int_{-\infty}^{\infty}(\varphi_{zz}^2 + \varphi_z^2).$$

It then follows that

$$\int_0^\infty \left| \frac{d}{dt} \int_{-\infty}^\infty \varphi_z^2(z,t)dz \right| \le C \int_0^\infty \int_{-\infty}^\infty (\varphi_{zz}^2 + \varphi_z^2) \le C \parallel \varphi_0 \|_2^2 < \infty. \tag{35}$$

By (34) and (35), we get

$$\int_{-\infty}^\infty \varphi_z^2(z,t)dz \to 0 \text{ as } t \to +\infty.$$

By Cauchy-Schwarz inequality, we further have

$$\varphi_z^2(z,t) = 2\int_{-\infty}^z \varphi_z\varphi_{zz}(y,t)dy \le 2\left(\int_{-\infty}^{+\infty} \varphi_z^2\,(y,t)dy\right)^{\frac{1}{2}} \left(\int_{-\infty}^{+\infty} \varphi_{zz}^2\,(y,t)dy\right)^{\frac{1}{2}} \to 0 \text{ as } t \to +\infty.$$

Hence (17) is proved. ■

4. Conclusion

Based on the results and discussions, we can conclude that the conditions of large wave amplitude and small perturbation are applied to establish the stability of traveling waves to Burgers equation with second-order nonlinear diffusion. The transformation by ansatz traveling waves is employed to the original equation as the first step to establish the existence of traveling waves. Then we present the energy estimates under the appropriate perturbation which is then used to prove the stability.

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