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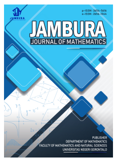

Modeling and Control of the Extreme Ideology Transmission Dynamics in a Society

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ABSTRACT

In this work, we propose a mathematical model to analyze the spread of extreme ideology in society. The so-called SERTA model divides the entire population into five compartments, namely susceptible, extremist, recruiter, treatment, and aware, to describe the state of the willingness of community members toward extreme ideology. We first present a model with constant control, i.e., a model without a dynamical control instrument, and provide the stability analysis of its equilibrium points based on the basic reproduction number. We then reformulate the model into an optimal control framework by introducing three control variables, namely prevention, disengagement, and deradicalization, to enable intervention of the dynamical process. The optimality conditions are obtained by employing Pontryagin's maximum principle, showing the optimal interdependence of state, co-state, and control variables. Numerical simulations based on the well-known Runge-Kutta algorithm and forward-backward sweep method are carried out to evaluate the effectiveness of control strategies under different scenarios. From the simulation results, it is found that by applying the three controls, the optimum solution is obtained. Besides that, in this study, disengagement contributes the most effect in suppressing extremist and recruiter populations, both by using single control and multiple controls.

Keywords:

Optimal Control Model; Pontryagin's Maximum Principle; Spread of Extreme Ideology; Stability Analysis

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1. Introduction

As social beings, humans cannot be separated from interactions with others. These interactions often have a negative effect, one of which is the influence of radicalism ideology. Radicalization is the process of someone holding extreme views or ideologies and acting toward violence [1]. People who are exposed to extreme ideologies will easily commit acts of violence, such as terrorism. These radical groups have very large developments in various parts of the world and spread extreme ideologies even when they are in prison.

People who have been exposed to extreme ideology, i.e., extremists, can easily carry out acts of terror. To prevent terror attacks, many countries establish special task forces or

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government bodies that focus on the issue of terrorism. In addition, the role of community leaders, religious figures, and even the community itself also have important contributions in eradicating the problem of terrorism. The war against extremists is often conducted with two strategies, namely hard and soft approaches. The hard approach aims to combat terrorism by prioritizing the security issue. In this approach, it is possible to prosecute terrorists, arrest, punish, or even kill them. However, this strategy will not be able to completely overcome acts of terror to their roots. At the same time, the soft approach is to seek disengagement and deradicalization initiatives. Based on this framework, we take a mathematical approach to find out the problem of radicalization by building a mathematical model of the spread of extreme ideologies.

The problem of spreading extreme ideology is modeled by dividing the compartmentalization of the radicalization process model into three classes, namely susceptible, extremists, and recruiters [2]. This point of view is useful for describing the process of individual radicalization and recruitment in terrorist organizations from a mathematical perspective. This model is simple as it only concentrates on the recruitment process. In the next study, Santoprete and Xu [3] developed the model for the deradicalization process, namely by adding the treatment compartment. It seeks to change radical extremist beliefs and violent behavior to reintegrate them into society. In addition, the spread of extreme ideologies can similarly be modeled as the spread of rumors. According to [4], the spread of rumors is a social phenomenon that spreads on a large scale in a short time through the communication chain. To analyze its spread and suppression, the spread of rumors is often modeled as a process of social contagion. The model consists of ignorant, spreaders, stiflers, and latent class. According to [5], the dynamic transmission of rumors divides the population into three compartments, namely ignorant, spreaders and stiflers, assuming that the process of spreading rumors is related to the psychological quality of individuals. By comparing the rumor spread model, this study assumes that the rate of rumor spread between ignorant and spreaders is nonlinear. Individuals leave ignorant for spreaders in a delayed time. Next, Jin, et al. [6] applied an epidemiological model to the dissemination of information through social media Twitter in relation to the spread of news and rumors. The mathematical model divides the population into four compartments that reflect the status of an individual, namely susceptible, infected, skeptic, and exposed. In the previous model for the spread of extreme ideology, it is necessary to consider the optimal control problem to find out the best strategy for overcoming the problem of spreading extreme ideology. This refers to the optimal control problem for the spread of rumors.

Based on the literatures above, we extend the existing susceptible - extremists - recruiters - treatment [3] model by introducing the so-called aware class [7]. The assumption behind this elaboration is that after the individual is treated, a new individual with extreme ideology awareness will emerge. This awareness can be seen as a form of success in the treatment process. Individuals who are successfully treated are grouped into a new awareness. In addition, by our model, the introduction of control actions is possible to intervene in the process of spreading an extreme ideology. We equip our model with three control variables, namely preventive, disengagement, and deradicalization. Preventive actions can be in the form of campaigns through digital media to disseminate counter-narrative efforts. We mean disengagement measures, social approaches to forcibly breaking ties of extremists from their groups.

Deradicalization is a psychological approach to restoring the original ideology of individuals under treatment. The primary objective of this current study is two-fold. Firstly, we consider the uncontrolled model and perform a stability analysis. Secondly, we consider the optimal control model and derive the necessary conditions for the optimal intervention based on Pontryagin's maximum principle. Simulations that show the effectiveness of control strategies are provided.

2. Model

2.1. Uncontrolled Model

In studies of the spread of extreme ideology or terrorism activities, certain variables, such as the properties of individual groups, evolve in time and space. To formulate the dynamics of extreme ideology transmission, we assume that the population in the society are divided into five classes, namely susceptible, extremist, recruiter, treatment, and aware. In this sense, we extend the bare-bones model of [2, 3] by adding new compartments. The susceptible class consists of individuals that have not adopted the extreme ideology but are at risk of radicalization. The number of individuals in this class at time t is denoted by $S(t)$. The extremist class comprises individuals that hold an extreme ideology and engage in terror activities. The number of extremists at time t is denoted by $E(t)$. Recruiters, denoted by $R(t)$, are defined as individuals that may not directly involve in extreme violence themselves but who radicalize, recruit and incite others to do so. This class includes propagandists and enablers. The treatment class $T(t)$ consists of extremists that prisoned in jails or deradicalized in rehabilitation centers [8]. The aware class $A(t)$ is the result of successful deradicalization processes and thus consists of individuals who are sensitized and dropped extreme ideology [7, 9]. We mean by an uncontrolled model, a model without dynamical control variables.

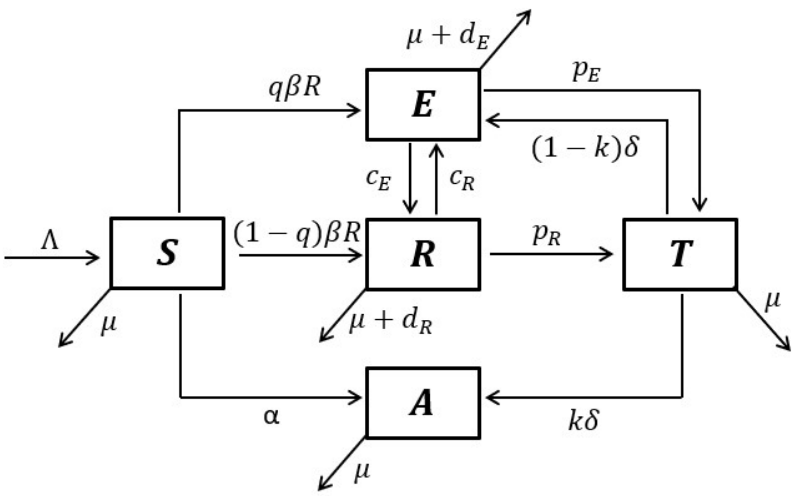

Figure 1. Uncontrolled model of extreme ideology transmission

The compartmental diagram of the model is depicted in Figure 1 and is constructed by assuming that individuals who are successfully treated have two possibilities, namely, returning to being extremists or aware. We also assume that an aware individual will not return to being susceptible because they already know extreme ideologies.

The dynamics of the model are then formulated as a system of nonlinear ordinary differential equations as follows:

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda - \beta RS - (\mu + \alpha) S \\
\frac{dE}{dt} &= q\beta RS - (\mu + d_E + c_E + p_E) E + c_R R + (1-k)\delta T \\
\frac{dR}{dt} &= (1-q)\beta RS + c_E E - (\mu + d_R + c_R + p_R) R \\
\frac{dT}{dt} &= p_E E + p_R R - (\mu + \delta) T \\
\frac{dA}{dt} &= \alpha S + k\delta T - \mu A.
\end{aligned} \tag{1}$$

For further analysis, we introduce the following simplified expressions:

$$\begin{aligned}
w_E &= \mu + d_E + c_E + p_E, \\
w_R &= \mu + d_R + c_R + p_R.
\end{aligned} \tag{2}$$

All of the parameters used in the model (1) are non-negative. The description of the parameters is given in Table 1.

Table 1. Description of parameters

Parameter	Description
Λ	Recruitment rate into the population
μ	Natural death rate
α	The rate for susceptible switch to aware
β	The rate for susceptible switch to recruiters
q	Fraction entering the extremist class
c_R	The rate for recruiters to switch to extremists
c_E	The rate for extremists switch to recruiters
d_R	The death rate of recruiters due to being killed
d_E	The death rate of extremists due to being killed
p_R	Treatment rate for recruiters
p_E	Treatment rate for extremists
k	The fraction of treated individuals is removed
δ	The rate of individuals leaving treatment

2.2. Optimal Control Model

In this section, we expand the uncontrolled model into one where we are able to intervene in the dynamics of radical ideology transmission by means of control variables. Instead of analyzing the effects of disengagement p_E, p_R, deradicalization k, and prevention α as constant parameters of the model, we represent such efforts as dynamic variables whose values are to be optimally selected. More specifically, we denote by u_1 the prevention program aims to terminate the spread of extreme ideology activity before it starts, by u_2 the disengagement programs striving to obstruct the spread of extreme ideology while it is taking place, and by u_3 the deradicalization programs intending to rehabilitate convicted extremists with the fundamental goal of social reintegration. These three programs are known as countering violent extremism

(CVE) initiatives [2, 3, 10–12].

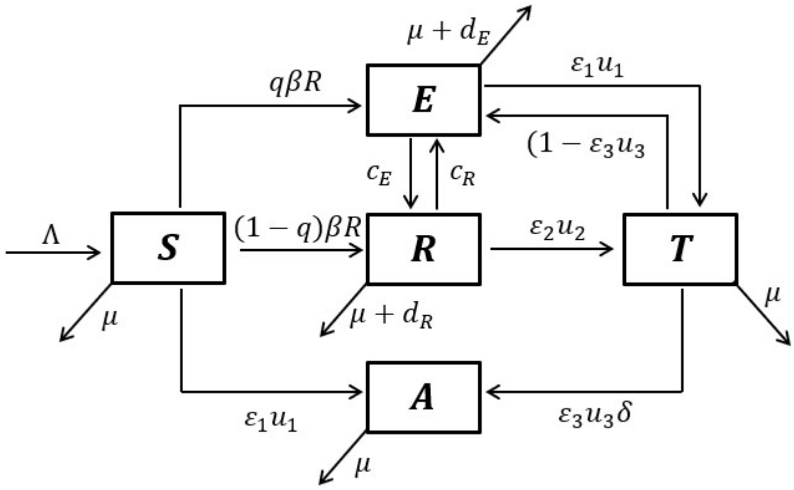

Figure 2. Optimal control model of extreme ideology transmission

A successful prevention program will transfer individuals in the susceptible class into an aware class. Disengagement programs move extremists and recruiters into treatment classes. An effective deradicalization program will permanently make rehabilitated individuals aware. At the same time, an unsuccessful program will send back treated individuals into the extremist class. The model diagram is depicted in Figure 2, and the equations of motion is formulated as follows:

$$\begin{aligned}\frac{dS}{dt} &= \Lambda - \beta RS - (\mu + \varepsilon_1 u_1) S \\ \frac{dE}{dt} &= q\beta RS - (\mu + d_E + c_E + \varepsilon_2 u_2) E + c_R R + (1 - \varepsilon_3 u_3) \delta T \\ \frac{dR}{dt} &= (1-q)\beta RS + c_E E - (\mu + d_R + c_R + \varepsilon_2 u_2) R \\ \frac{dT}{dt} &= \varepsilon_2 u_2 (E + R) - (\mu + \delta) T \\ \frac{dA}{dt} &= \varepsilon_3 u_3 \delta T - \mu A + \varepsilon_1 u_1 S.\end{aligned} \tag{3}$$

We denote by $u_1(t)$, $u_3(t)$, and $u_3(t)$, respectively the number of population (in percent) in the respecting classes involved in the prevention, disengagement, and deradicalization programs at time t. We assume that the effectiveness of the prevention, disengagement and deradicalization programs are denoted by ε_1, ε_2, and ε_3, respectively.

The optimal control problem aims to control the spread of extreme ideology by minimizing the number of extremists and recruiters jointly with the costs of implementing controls $u_1(t)$, $u_2(t)$, and $u_3(t)$. The control performance is represented by the following objective functional:

$$J(u_1, u_2, u_3) = \int_0^{t_f} [B_1 E(t) + B_2 R(t) + C_1 {u_1}^2(t) + C_2 {u_2}^2(t) + C_3 {u_3}^2(t)] dt, \tag{4}$$

where B_1, B_2 are the weighting constants for extremists and recruiters, respectively. C_1, C_2, C_3 are the cost weights for control $u_1(t)$, $u_2(t)$, and $u_3(t)$, respectively. As the control efforts are limited, then we consider bounded control variables:

$$0 \leq u_i(t) \leq \overline{u}_i, \tag{5}$$

for $i = 1, 2, 3$ and $t \in [0, t_f]$, where t_f is the control period. In equation (5), $\overline{u}_i$ are the upper bound of the control application.

3. Results and Discussions

3.1. Positivity and Boundedness

Let N be denoted as the total population:

$$N = S + E + R + T + A. \tag{6}$$

From which we then have

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dE}{dt} + \frac{dR}{dt} + \frac{dT}{dt} + \frac{dA}{dt}. \tag{7}$$

By substituting equation (1) into equation (7) we have

$$\frac{dN}{dt} = \Lambda - \mu(S + E + R + T + A) - d_E E - d_R R. \tag{8}$$

Since $d_E E$ and $d_R R$ are both non-negative. We may drop them from the equation to get the following inequality

$$\frac{dN}{dt} \leq \Lambda - \mu N, \tag{9}$$

from which we obtain

$$0 \leq N(t) \leq \frac{\Lambda}{\mu} + \left(N_0 - \frac{\Lambda}{\mu}\right) e^{-\mu t}. \tag{10}$$

where $N_0 = N(0)$ is the initial value. Thus, the positivity and boundedness properties of the system (5) can be stated in the invariant region $\mathbb{S}$ as follows:

$$\mathbb{S} = (S, E, R, T, A)^T \in \mathbb{R}^5_+ | 0 \leq S + E + R + T + A \leq \frac{\Lambda}{\mu}. \tag{11}$$

3.2. Extremist-free Equilibrium and Basic Reproduction Number

Equilibrium points of a nonlinear system (1) can be found by solving the following homogeneous equations system:

$$\begin{aligned} \Lambda - \beta RS - (\mu + \alpha) S &= 0 \\ q\beta RS - (\mu + d_E + c_E + p_E) E + c_R R + (1-k)\delta T &= 0 \\ (1-q)\beta RS + c_E E - (\mu + d_R + c_R + p_R) R &= 0 \\ p_E E + p_R R - (\mu + \delta) T &= 0 \\ \alpha S + k\delta T - \mu A &= 0. \end{aligned} \tag{12}$$

Extremism-free equilibrium P^0 can be found by setting $R = 0$, from which we then get

$$P^0 = \left(S^0,\ 0,\ 0, 0,\ A^0\right), \tag{13}$$

where

$$\begin{aligned} S^0 &= \frac{\Lambda}{\mu + \alpha}, \\ A^0 &= \frac{\alpha}{\mu} S^0. \end{aligned}$$

The Jacobian matrix of nonlinear system (1) is given by

$$J = \begin{bmatrix} -\beta R - (\mu + \alpha) & 0 & -\beta S & 0 & 0 \\ q\beta R & -w_E & q\beta S + c_R & (1-k)\delta & 0 \\ (1-q)\beta R & c_E & (1-q)\beta S - w_R & 0 & 0 \\ 0 & p_E & p_R & -(\mu+\delta) & 0 \\ \alpha & 0 & 0 & k\delta & -\mu \end{bmatrix}. \tag{14}$$

From Jacobian matrix (14), we can identify the so-called infection subsystem which determined by classes E, R, and T as follows (in matrix form):

$$\begin{bmatrix} \frac{dE}{dt} \\ \frac{dR}{dt} \\ \frac{dT}{dt} \end{bmatrix} = \begin{bmatrix} -w_E & q\beta S + c_R & (1-k)\delta \\ c_E & (1-k)\beta S - w_R & 0 \\ p_E & p_R & -(\mu+\delta) \end{bmatrix} \begin{bmatrix} E \\ R \\ T \end{bmatrix}. \tag{15}$$

Suppose $X = (E,\ R,\ T)^{\mathrm{T}}$ be a vector of infected classes and $Y = (S,\ A)^{\mathrm{T}}$ be a vector of uninfected classes. Vector of flows from X to Y, denoted by $\mathcal{F}(X,\ Y)$, and that of other flows denoted by $\mathcal{V}(X,\ Y)$, are respectively given by

$$\mathcal{F}(X,Y) = \begin{bmatrix} q\beta SR \\ (1-q)\beta SR \\ 0 \end{bmatrix}, \mathcal{V}(X,Y) = \begin{bmatrix} w_E E - c_R R - (1-k)\delta R \\ -c_E E + w_R R \\ -p_E E - p_R R + (\mu+\delta) T \end{bmatrix}. \tag{16}$$

and thus

$$\frac{dX}{dt} = \mathcal{F}(X,Y) - \mathcal{V}(X,Y) \tag{17}$$

If we denote by $\mathcal{F}$ the Jacobian matrix of $\mathcal{F}$ and by $\mathcal{V}$ the Jacobian matrix of $\mathcal{V}$, both evaluated at equilibrium point P^0, then we have

$$F = \beta S^0 \begin{bmatrix} 0 & q & 0 \\ 0 & 1-q & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{18}$$

$$V = \begin{bmatrix} w_E & -c_R & -(1-k)\delta \\ -c_E & w_R & 0 \\ -p_E & -p_R & \mu+\delta \end{bmatrix}. \tag{19}$$

The determinant and inverse of V in (19) are respectively given by

$$|V| = g_2 w_R - g_1 c_R - g_3 p_R, \tag{20}$$

$$V^{-1} = \frac{1}{|V|} \begin{bmatrix} w_R(\mu+\delta) & (\mu+\delta)c_R + (1-k)\delta w_R & (1-k)\delta w_R \\ g_1 & g_2 & g_3 \\ p_R c_E + w_R p_E & c_R p_E + p_R w_E & w_R w_E + c_R c_E \end{bmatrix}, \tag{21}$$

where

$$\begin{aligned} g_1 &= c_E(\mu+\delta) \\ g_2 &= w_E(\mu+\delta) - (1-k)\delta p_E \\ g_3 &= (1-k)\delta c_E \end{aligned}$$

The next generations matrix [13] G is then can be calculated as

$$G = FV^{-1} = \frac{\beta S^0}{|V|} \begin{bmatrix} qg_1 & qg_1 & qg_3 \\ (1-q)g_1 & (1-q)g_2 & (1-q)g_3 \\ 0 & 0 & 0 \end{bmatrix}. \tag{22}$$

Let define $\hat{G} = \frac{|V|}{\beta S^0} G$. The characteristic polynomial of $\hat{G}$ is expressed as follows:

$$\left|\hat{G} - \widehat{\lambda} I_3\right| = \widehat{\lambda}^2 \left(qg_1 + (1-q)g_2 - \widehat{\lambda}\right)$$

Since two of eigenvalues are zero, then the dominant eigenvalues of $\hat{G}$ is provided by

$$\widehat{\lambda} = qg_1 + (1-q)g_2,$$

and thus the dominant eigenvalue of G is given by

$$\lambda = \frac{\beta S^0 \widehat{\lambda}}{|V|}. \tag{23}$$

Since the basic reproduction number $\mathcal{R}_0$ is defined as the dominant eigenvalue of the next generation matrix G, then we have

$$\mathcal{R}_0 = \frac{\beta S^0 \left(q g_1 + (1-q)\, g_2\right)}{g_2 w_R - g_1 c_R - g_3 p_R}. \tag{24}$$

3.3. Stability of Extremist-free Equilibrium Point

Theorem 1. *Suppose that* $(\mu + \delta + w_E)\, m_2 > c_E m_1$ *and* $\mathcal{R}_0 < 1$. *Then,* P^0 *is globally asymptotically stable.*

Proof. Let J^0 is the Jacobian matrix (14) evaluated at an equilibrium point P^0. Then J^0 is the coefficient matrix of the linearized system around P^0 and given by

$$J^0 = \begin{bmatrix} -(\mu+\alpha) & 0 & -\beta S & 0 & 0 \\ 0 & -w_E & m_1 & (1-k)\,\delta & 0 \\ 0 & c_E & -m_2 & 0 & 0 \\ 0 & p_E & p_R & -\mu-\delta & 0 \\ \alpha & 0 & 0 & k\delta & -\mu \end{bmatrix}, \tag{25}$$

where m_1 and m_2 in equation (25) are defined as

$$m_1 = q\beta S^0 + c_R, m_2 = w_R - (1-q)\,\beta S^0. \tag{26}$$

The characteristic polynomial of J^0 is

$$\left|J^0 - \lambda I_5\right| = -(\lambda+\mu+\alpha)(\lambda+\mu)\left(\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0\right), \tag{27}$$

where

$$\begin{aligned} a_2 &= \mu + \delta + w_E + m_2, \\ a_1 &= (\mu+\delta+w_E)\, m_2 + g_2 - m_1 c_E, \\ a_0 &= g_2 m_2 - g_1 m_1 - g_3 p_R. \end{aligned} \tag{28}$$

From (28), we can see that J^0 has at most five eigenvalues with two of them are as follows:

$$\lambda_1 = -(\mu+\alpha) < 0, \tag{29}$$

$$\lambda_2 = -\mu < 0. \tag{30}$$

The other three eigenvalues are obtained by solving the following equation:

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0. \tag{31}$$

The stability of the fixed point without the spread of extreme ideologies P^0 is based on equation (31) according to the Routh-Hurwitz criteria [14], P^0 is stable if it meets the following stability conditions:

$$a_2 > 0,\ a_0 > 0,\ a_2 a_1 > a_0. \tag{32}$$

It will be proven that the stability requirements of the Routh-Hurwitz Criteria in equation (32) are met by using the $\mathcal{R}_0$ condition. Suppose $\mathcal{R}_0 < 1$, then based on equation (24) it is obtained:

$$\beta S^0 \left(q g_1 + (1-q) g_2\right) < g_2 w_R - g_1 c_R - g_3 p_R. \tag{33}$$

By considering equations (26) it is obtained

$$m_2 g_2 > m_1 g_1 + g_3 p_R. \tag{34}$$

Based on the inequality (34), we get $a_0 > 0$. In equation (28), because each parameter is positive, we then show that $a_2 > 0$. Next, it will be shown that $a_2 a_1 > a_0$. By substituting the values a_2, a_1, and a_0 given in (28), we may obtain

$$a_2 a_1 - a_0 = (C + m_2)(C m_2 - m_1 c_E) + C g_2 + m_1 g_1 + g_3 p_R, \tag{35}$$

where $C = \mu + \delta + w_E$. The sign of $a_2 a_1 - a_0$ depends on the sign of $C m_2 - m_1 c_E$. Thus if we assume that $C m_2 - m_1 c_E > 0$, then $a_2 a_1 - a_0 > 0$ and the Routh-Hurwitz criteria are all satisfied when $\mathcal{R}_0 < 1$, so P^0 is globally asymptotically stable. □

3.4. Endemic Equilibrium

An endemic equilibrium can be identified by solving equations system (12) for which at least one of the populations E, R, T, and A is nonzero. Preliminary analysis shows that an endemic equilibrium is obtained by setting $E \neq 0$, $R \neq 0$, and $T \neq 0$. Let denote by P^* the endemic equilibrium:

$$P^* = \left(S^*, E^*, R^*, T^*, A^*\right). \tag{36}$$

Suppose that $R^* \neq 0$ is given. Successively from (12) we obtain the followings:

$$S^* = \frac{\Lambda}{\mu + \alpha + \beta R^*}, \tag{37}$$

$$E^* = \frac{1}{c_E}\left(w_R - (1-q)\beta S^*\right) R^*, \tag{38}$$

$$T^* = \frac{1}{\mu + \delta}\left(\frac{p_E}{c_E}\left(w_R - (1-q)\beta S^*\right) + p_R\right) R^*, \tag{39}$$

$$A^* = \frac{k\delta}{\mu(\mu + \delta)}\left(\frac{p_E}{c_E}\left(w_R - (1-q)\beta S^*\right) + p_R\right) R^* + \frac{\alpha}{\mu} S^*. \tag{40}$$

Substitution of equation (38) and (39) into equation (12)(b) provides

$$q\beta R^* S^* + c_R R^* = \frac{w_E}{c_E}\left(w_R - (1-q)\beta S^*\right) R^* - \frac{(1-k)\delta}{\mu + \delta}\left(\frac{p_E}{c_E}\left(w_R - (1-q)\beta S^*\right) + p_R\right) R^*.$$

Canceling R^* and collecting βS^* terms into left-hand side yield

$$\left(q+\frac{w_E\left(\mu+\delta\right)-\left(1-k\right)\delta p_E}{c_E\left(\mu+\delta\right)}\left(1-q\right)\right)\beta S^*=\frac{w_E w_R}{c_E}-\frac{\left(1-k\right)\delta}{\mu+\delta}\left(\frac{p_E w_R}{c_E}+p_R\right)-c_R.$$

Because of equation (20), and equation (21), we have

$$\left(q+\frac{g_2}{g_1}\left(1-q\right)\right)\beta S^*=\frac{w_E w_R}{c_E}-\frac{\left(1-k\right)\delta}{\mu+\delta}\left(\frac{p_E w_R}{c_E}+p_R\right)-c_R$$

$$\left(g_1 q+g_2\left(1-q\right)\right)\beta S^*=\frac{g_1 w_E w_R}{c_E}-\frac{g_1\left(1-k\right)\delta}{\mu+\delta}\left(\frac{p_E w_R}{c_E}+p_R\right)-g_1 c_R$$

$$\left(g_1 q+g_2\left(1-q\right)\right)\beta S^*=g_2 w_R-g_3 p_R-g_1 c_R$$

$$S^*=\frac{|V|}{\beta\left(g_1 q+g_2\left(1-q\right)\right)},$$

and then by equation (24), we obtain

$$S^*=\frac{S^0}{\mathcal{R}_0}. \tag{41}$$

Substituting equation (41) into equation (37) gives

$$R^*=\frac{\Lambda}{\beta S^0}\left(\mathcal{R}_0-1\right). \tag{42}$$

Finally, by inserting equation (42) into equation (38), (39), and (40), we have

$$E^*=\frac{\mu+\alpha}{c_E}\left(w_R-\frac{\left(1-q\right)\beta S^0}{\mathcal{R}_0}\right)\left(\mathcal{R}_0-1\right), \tag{43}$$

$$T^*=\frac{\mu+\alpha}{\mu+\delta}\left(\frac{p_E}{c_E}\left(w_R-\frac{\left(1-q\right)\beta S^0}{\mathcal{R}_0}+p_R\right)\right)\left(\mathcal{R}_0-1\right), \tag{44}$$

$$A^*=\frac{k\delta}{\mu}\frac{\mu+\alpha}{\mu+\delta}\left(\frac{p_E}{c_E}\left(w_R-\frac{\left(1-q\right)\beta S^0}{\mathcal{R}_0}+p_R\right)\right)\left(\mathcal{R}_0-1\right)+\frac{A^0}{\mathcal{R}^0}. \tag{45}$$

Stability analysis of the endemic equilibrium point P^* can be undertaken by following the approach in [3] and [12].

3.5. Optimality Conditions

Our control objective is to determine the control variables $u_1(t)$, $u_2(t)$, and $u_3(t)$ such that minimize the objective functional given in equation (4) subject to system (3) with initial condition $S(0)=S_0$, $E\left(0\right)=E_0$, $R(0)=R_0$, $T(0)=T_0$, and $A(0)=A_0$ and free terminal times. We also consider bounded control variables in equation (5). Deriving the optimality conditions using Pontryagin's maximum principle requires the following Hamiltonian:

$$\begin{aligned} H = {} & B_1E + B_2R + C_1{u_1}^2 + C_2{u_2}^2 + C_3{u_3}^2 + \; p_1\left[\Lambda - \beta SR - (\mu + \varepsilon_1 u_1)\, S \;\right] \\ & + p_2\left[q\beta SR - (\mu + d_E + c_E + \varepsilon_2 u_E)\, E + c_R R + (1 - \varepsilon_3 u_3)\delta T\right] \\ & + p_3\left[(1-q)\,\beta SR + c_E E - (\mu + d_R + c_R + \varepsilon_2 u_2)\, R\right] \\ & + p_4\left[\varepsilon_2 u_2\,(E+R) - (\mu+\delta)\, T\right] + p_5\left[\varepsilon_3 u_3 \delta T - \mu A + \varepsilon_1 u_1 S\right] \end{aligned} \tag{46}$$

with $p_i\ (i = 1,2,3,4,5)$ are adjoin functions that must be optimally determined through the optimization process.

The necessary conditions for optimality according to Pontryagin's maximum principle are given by:

1. Minimize the Hamiltonian H with respect to u_i, given by the following stationary conditions:

$$\frac{\partial H}{\partial u_i} = 0, \quad i = 1,\ 2, 3. \tag{47}$$

Solving equation (47) and by considering bounded control equation (5) produces the following optimal controls:

$${u_1}^* = \min\{\max\{0, \frac{(p_1 - p_5)\,\varepsilon_1 S}{2C_1}\}, \overline{u}_1\}, \tag{48}$$

$${u_2}^* = \min\{\max\{0, \frac{(p_2 - p_4)\,E + (p_3 - p_4)\,R}{2C_2}\}, \overline{u}_2\}, \tag{49}$$

$${u_3}^* = \min\{\max\{0, \frac{(p_2 - p_5)\,\varepsilon_3 \delta T}{2C_3}\}, \overline{u}_3\}. \tag{50}$$

2. State variable $x_i \in (S, E, R, T, A)$ satisfy the differential equations system

$$\dot{x}_i\,(t) = \frac{\partial H}{\partial p_i}, \quad i = 1,2,3,4,5, \quad x_i \in (S, E, R, T, A)\,. \tag{51}$$

Solving equation (51) will provide the state system (3) together with initial conditions.

3. Adjoin variables $p_i(t)$ for $i = 1,2,3,4,5$, satisfy the differential equations system

$$\dot{p}_i\,(t) = -\frac{\partial H}{\partial x}, \quad i = 1,2,3,4,5, \quad x_i \in (S, E, R, T, A)\,. \tag{52}$$

Solving equation (52) will give the following costate system:

$$\begin{aligned} \dot{p}_1 &= p_1\,(\beta R + \mu + \varepsilon_1 u_1) - (p_2 q\beta R + p_3\,(1-q)\,\beta R + p_5\varepsilon_1 u_1) \\ \dot{p}_2 &= p_2\,(\mu + d_E + c_E + \varepsilon_2 u_2) - (p_3 c_E + p_4 \varepsilon_2 u_2 + B_1) \\ \dot{p}_3 &= p_1\beta S - p_2\,(q\beta S + c_R) - p_3\,((1-q)\,\beta S - (\mu + d_R + c_R + \varepsilon_2 u_2)) - p_4\varepsilon_2 u_2 - B_2 \\ \dot{p}_4 &= p_4\,(\mu+\delta) - (p_2\,(1 - \varepsilon_3 u_3)\,\delta + p_5 \varepsilon_3 u_3 \delta \\ \dot{p}_5 &= p_5\mu. \end{aligned} \tag{53}$$

Since at terminal times all states are free, then transversality conditions $p_i\left(t_f\right) = 0$

for $i = 1, 2, 3, 4, 5$ must be satisfied [15].
Thus, the optimal control variables (48)-(50) can be obtained by numerically solving the dynamical system (3) simultaneously with costate system (53) under initial conditions and transversality conditions.

3.6. Numerical Simulations

In this section we provide an example for verifying the stability properties as well as assessing the effectiveness of the control strategy. We consider a small and closed society with population of 17 million. The initial values of state variables are $S_0 = 10$ million, $E_0 = 50$, $R_0 = 10$, $T_0 = 5$, and $A_0 = 7$ million.

3.6.1. Uncontrolled Model

Numerical simulations were carried out to show population dynamics without the spread of extreme ideology when $\mathcal{R}_0 < 1$ and with the spread of extreme ideological when $\mathcal{R}_0 > 1$. The following values of parameters are used for both cases: $\Lambda = 600$, $\mu = 0.000034247$, $\alpha = 0.75$, $\beta = 0.000055$, $q = 0.86$, $c_R = 0.0008$, $c_E = 0.0006$, $d_R = d_E = 0.00083$, $k = 0.56$, and $\delta = 0.0016$. For the case of $\mathcal{R}_0 < 1$ we set $p_R = p_E = 0.072$ and for the case of $\mathcal{R}_0 < 1$ we assign $p_R = p_E = 0.008$. Under these choices of parameter values, we obtain $\mathcal{R}_0 = 0.0914376 < 1$ and $\mathcal{R}_0 = 1.07481 > 1$.

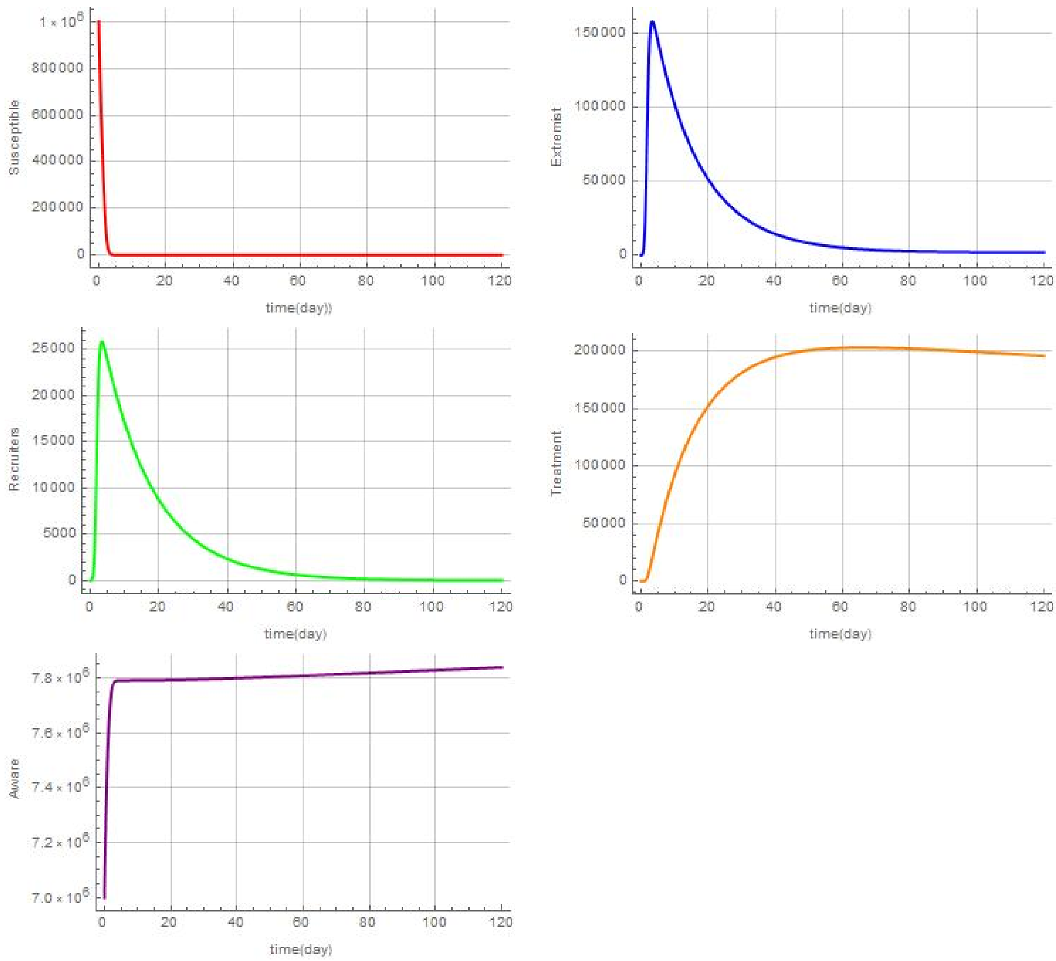

Figure 3. Population dynamics for $\mathcal{R}_0 < 1$

The population dynamics for $\mathcal{R}_0 < 1$ are shown in Figure 3. It can be seen that the population of susceptible individuals has decreased very significantly in less than 20 days. The number of extremists and recruiters initially increased but then decreased continuously until it reached 0 in less than 80 days. The number of individuals under treatment increased and then decreased slowly after 60 days. Furthermore, the number of aware individuals increased significantly in less than 20 days, and continues to increase slowly after that.

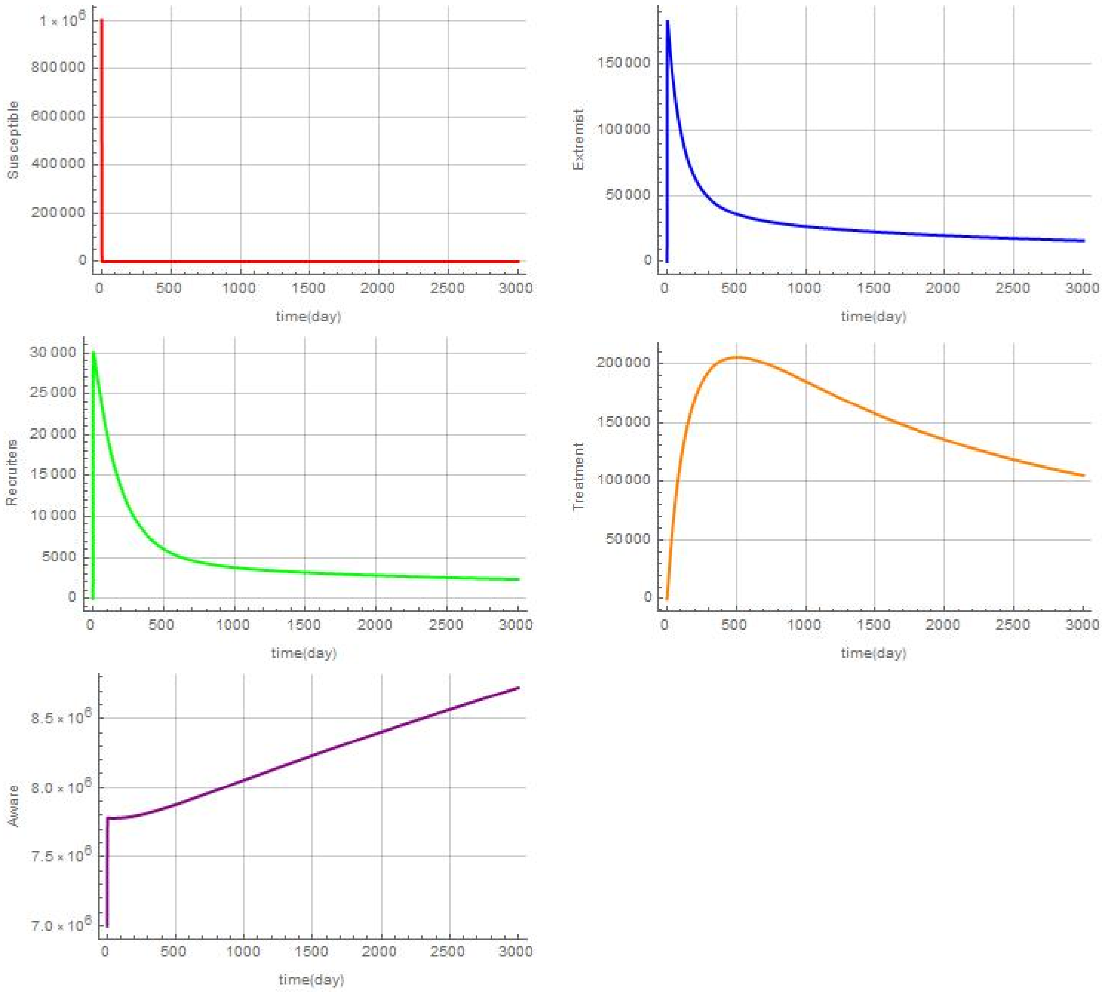

Figure 4. Population dynamics for $\mathcal{R}_0 > 1$

The population dynamics for $\mathcal{R}_0 > 1$ are shown in Figure 4. We can see that the population of susceptible has decreased very significantly within 20 days. The total populations of extremists and recruiters have increased continuously, then decreased in less than 500 days, but the total population did not reach 0 within 3000 days. This means that extremists and recruiters will always be around, and the extreme ideology will be transmitted among susceptible individuals. The number of individuals under treatment experienced a very significant increase in less than 500 days and then decreased slowly. Furthermore, the number of aware individuals continues to increase.

3.6.2. Optimal Control Model

For model with control variables (3), we use the following values of parameter: $\Lambda = 18000$, $\mu = 0.00102741$, $\beta = 1.68 \times 10^{-8}$, $q = 0.86$, $c_R = 0.0024$, $c_E = 0.0018$, $d_R = d_E = 0.0249$, and $\delta = 0.0048$, all are in per month unit. To evaluate the effectiveness of control combinations, we develop two control scenarios relating to the number of control instruments as depicted in Table 2. Particularly, we aim to compare the effectiveness of single control and multiple controls with constant control. In constant control, we set all control variable constant all the time, i.e., $u_1 = u_2 = 0.05$ and $u_3 = 0.10$. In single control, we optimize one control variable and set other two control variables constant. In multiple controls, we optimize two or three control variables. From Table 2 we can see that the control upper bounds are $\overline{u}_1 = 0.10$, $\overline{u}_2 = 0.15$, and $\overline{u}_3 = 0.20$. In the simulation we also assume that all control actions have certain effectivity levels, i.e., $\varepsilon_1 = 0.80$, $\varepsilon_2 = 0.70$, and $\varepsilon_3 = 0.40$.

Table 2. Control scenarios

Scenario	Prevention u_1	Disengagement u_2	Deradicalization u_3
Constant Control	0.05	0.05	0.10
Single Control	$0 - 0.10$	0.05	0.10
	0.05	$0 - 0.15$	0.10
	0.05	0.05	$0 - 0.20$
Multiple Controls	$0 - 0.10$	$0 - 0.15$	0.10
	$0 - 0.10$	0.05	$0 - 0.20$
	0.05	$0 - 0.15$	$0 - 0.20$
	$0 - 0.10$	$0 - 0.15$	$0 - 0.20$

The well-known forward-backward sweep method was applied to numerically solve the three blocks of differential equations system characterized by Pontryagin's maximum principle. The state system (3), which has initial conditions, was initially solved in forward by implementation of the fourth order Runge-Kutta algorithm. Then, the costate system (53), which has terminal conditions, was solved backwards in time with the same algorithm, following by controls updating according to equation (48)-(50). This step generates new approximations of the state, costate, and control variables. The process feedbacks by utilizing these new updates and generating new approximations of Runge-Kutta and control updates with the objective of reaching fixed variables. The sweep method is terminated when a sufficiently small level of tolerance reached.

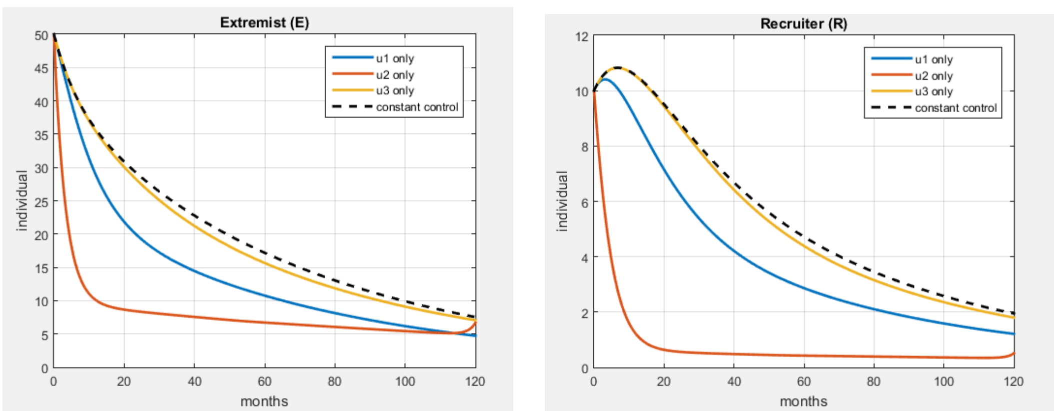

Figure 5. Number of extremists and recruiters under single control

Figure 5 depicts the dynamics of the number of extremists and recruiters under single control. We can see that all single control strategies successfully suppress extremist and recruiter populations. Disengagement (u_2) is the most effective strategy in reducing both populations, followed by preventive actions (u_1), and deradicalization (u_3). Figure 6 shows the level of extremist and recruiter populations under application of two and three control instruments. These control combinations include the application of preventive control u_1 and disengagement control u_2, preventive control u_1 and deradicalization control u_3, disengagement control u_2 and u_3 deradicalization control, as well as the implementation of three control variables. It is shown that the control combinations (u_1, u_2), (u_2, u_3), and (u_1, u_2, u_3) similarly perform the most contributions in reducing the number of extremists and recruiters. In other words, any control strategy without implementation of disengagement (u_2) will contribute less effects in suppressing extremist and recruiter populations.

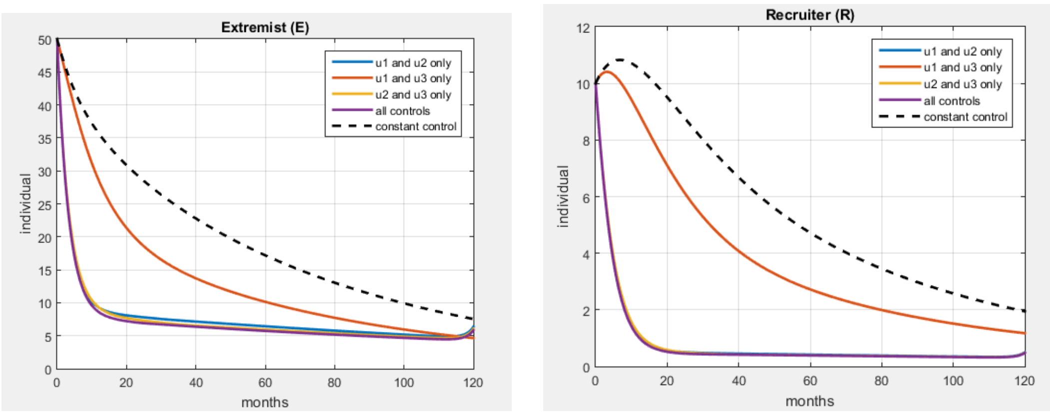

Figure 6. Number of extremists and recruiters under multiple controls

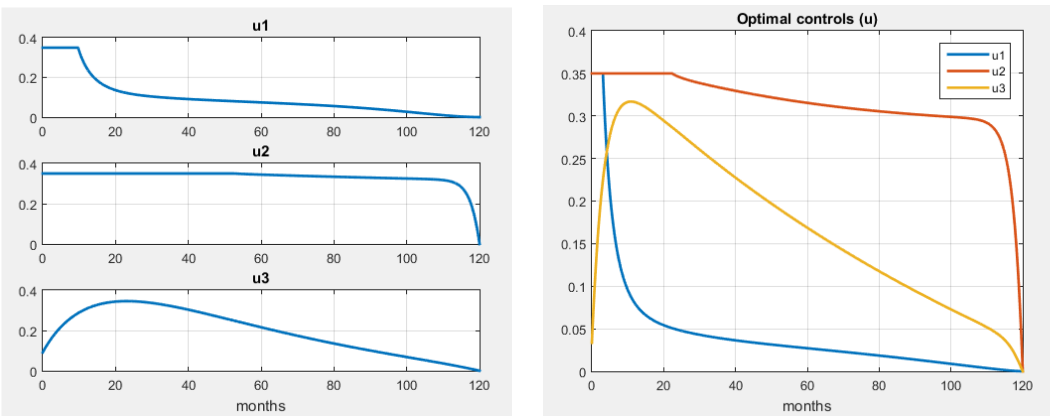

Figure 7. Single and multiple optimal controls

Figure 7 presents the optimal controls, analytically expressed in equation (48)-(50), when they are implemented in a single and multiple manners. When u_2 and u_3 are set constant, the optimal setting of u_1 is applied at maximum intensity in the first ten days and suddenly reduced until the end of the control period. When u_1 and u_3 are set constant, the control u_2 must be implemented at the maximum level almost throughout the period of control. Application of single optimal control u_3 suggests the gradual implementation of this control variable. In the case of multiple controls strategy

(u_1, u_2, u_3), it is required the implementation of disengagement (u_2) in higher intensity than other control variables.

4. Conclusion

We have proposed a mathematical model for analyzing the spread of an extreme ideology in a closed society. In the case of the uncontrolled model, we have shown that the model has two equilibrium points, namely the extremist-free and endemic equilibrium points. We have also derived the basic reproduction number as the measure of the transmission potential of extreme ideology. We have then reformulated the model in the framework of optimal control. By introducing three control variables, namely preventive action, disengagement, and deradicalization, we have made the model intervention possible. To quantify the effectiveness of the control actions, we have developed several scenarios regarding the combination. We have proposed a mathematical model for analyzing the spread of an extreme ideology in a closed society of control instruments. It has been shown that disengagement contributes the most effect in suppressing extremist and recruiter populations, both by using single control and multiple controls. However, as control implementation requires cost, a cost-effectiveness analysis is recommended.

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