JAMBURA JOURNAL OF MATHEMATICS
Jambura J. Math. Vol. 5, No. 1, pp. 139–154, February 2023
Journal Homepage: http://ejurnal.ung.ac.id/index.php/jjom
DOI: https://doi.org/10.34312/jjom.v5i1.16393

Robust Coloring Optimization Model on Electricity Circuit Problems

Viona Prisyella Balqis[1,*], Diah Chaerani[1], Herlina Napitupulu[1]

[1]*Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Padjadjaran, Sumedang, Jawa Barat 45363, Indonesia*

[*]Corresponding author. Email: viona2706@gmail.com

ABSTRACT

The Graph Coloring Problem (GCP) is assigning different colors to certain elements in a graph based on certain constraints and using a minimum number of colors. GCP can be drawn into optimization problems, namely the problem of minimizing the color used together with the uncertainty in using the color used, so it can be assumed that there is an uncertainty in the number of colored vertices. One of the mathematical optimization techniques in dealing with uncertainty is Robust Optimization (RO) combined with computational tools. This article describes a robust GCP using the Polyhedral Uncertainty Theorem and model validation for electrical circuit problems. The form of an electrical circuit color chart consists of corners (components) and edges (wires or conductors). The results obtained are up to 3 colors for the optimization model for graph coloring problems and up to 5 colors for robust optimization models for graph coloring problems. The results obtained with robust optimization show more colors because the results contain uncertainty. When RO GCP is applied to an electrical circuit, the model is used to place the electrical components in the correct path, so that the electrical components do not collide with each other.

Keywords:
Robust Optimization; Graph Coloring Problem; Electricity Problem

Citation Format:

V. P. Balqis, D. Chaerani, and H. Napitupulu, "Robust Coloring Optimization Model on Electricity Circuit Problems", *Jambura J. Math.*, vol. 5, No. 1, pp. 139–154, 2023, doi: https://doi.org/10.34312/jjom.v5i1.16393

1. Introduction

Graph coloring is a fundamental combinatorial optimization problem for coloring the vertices of a given graph with a minimum number of colors so that adjacent vertices are colored differently. There is a major challenge to constructing graph coloring, which is finding the right number for a graph that requires all the vertices to be colored differently [1]. In graph coloring, labels on graph elements are given based on some constraints or conditions and the label is a color. Colors in a graph are usually assigned to integers at the edges, vertices, or both, that is to the edges and the vertices of the graph. In graph theory, several colors are used to label edges or vertices with a constraint on using their color. If there is a color, it must be able to determine the vertex to be colored so that no two adjacent vertices have the same color. There are also some other graph coloring problems, for example, Edge Coloring and Face Coloring. In edge

e-ISSN: 2656-1344
Received: 7 September 2022 | Accepted: 17 November 2022 | Online: 1 February 2023

coloring, none of the vertices connected by two edges have the same color, and face coloring is related to Geographic map coloring [2]. There are several previous studies that discuss the optimization model for graph coloring problems, namely Diaz, et al. [3] discusses the Branch & Cut algorithm based on polyhedral studies of integer linear programming models to solve coloring problems, Nickel [4] discusses the ellipsoid method approach that is used as a tool to prove the solvency of the polynomial-time of combinatorial optimization problems, Hansen, et al. [5] discusses the consideration of two linear programming formulations of the graph coloring problem by using branch-and-cut-and-price algorithm computational experiments, Burke, et al. [6] discusses vertex coloring in combinatorial optimization problems with integer programming formulations to survey seven known vertex coloring formulations and introduces a new formulation for vertex coloring with appropriate partitions on graphs, Jabrayilov and Mutzel [7] discusses the formulation of integer linear programming with partial ordering for vertex coloring problems in graphs for a large set of benchmark graphs and randomly generated graphs of various sizes and densities, and Jovanović, et al. [8] discusses the development of optimization models using weighted node coloring combinatorial problems for strategic decision making, and so on.

Graph coloring is a traditional method and may not be able to produce an optimal solution so in graph coloring problems it often requires an approach with other optimization methods to maximize/minimize the objective function by considering the existing constraints [9]. The problem for graph coloring can be related to optimization problems, namely determining a vertex to be colored correctly, and also the number of colorings given is minimal.

In its application, not all optimization problems can be obtained with objective functions and constraint functions with certain conditions (certain), there are also optimization problems with uncertain conditions (uncertain). As in the graph coloring problem, it can be assumed that the number of vertices to be colored is indeterminate so that to obtain optimal results an optimization method is needed that considers the uncertain situation where the data obtained is unknown or incomplete, while the solution must be determined. One of the optimization methods with uncertainty is the Robust Optimization (RO) method. Robust Optimization (RO) is a methodology for solving mathematical problems by assuming that the uncertain data is in a defined set of indeterminate parameter values which is called the uncertainty set [10].

This article discusses the construction of Graph Coloring Robust Optimization models with model validation applied to electrical circuit problems. This research was conducted to obtain a new knowledge and development of an existing scientific field. The purpose of this article is to find a solution to the problem of indeterminate graph coloring so that a new way to solve the graph coloring problem is obtained. With model validation on electrical circuit problems, it is hoped that it can provide an overview of the application form of graph coloring questions to simple problems. Based on this research, it is found that by constructing a Graph Coloring Robust optimization model, can be used and applied to problems on a small to large scale, making it easier for researchers to obtain optimal results so that it can be a solution to solving a problem.

2. Method

In this section, several materials related to the research topic, namely Graph Coloring, Graph Coloring Problem Optimization Model, Graph Coloring in The Electricity Sector, Robust Optimization, Robust Counterpart, Robust Counterpart for Polyhedral Uncertainty Sets.

2.1. Graph Coloring

Graph coloring is one type of graph labeling and is a branch of graph labeling in special cases. In graph coloring, labeling is given based on existing constraints or conditions. The label used is a color. In the graph, labeling can usually be done by assigning a value (weight) to the edges, vertices, or both the edges and vertices of the graph [2].

Graph coloring is a fundamental combinatorial optimization problem to color the vertices of a particular graph with the minimum number of colors so that adjacent vertices can be colored differently [1]. Based on this, an optimization problem arises to determine the minimum number of colors for nodes, this problem is usually referred to as the Graph Coloring Problem (GCP).

There are three types of graph coloring, namely vertex coloring, edge coloring, and region coloring [11]. Node coloring is the assignment of color to the vertices with every two adjacent vertices having a different color. Side coloring is the assignment of colors to the sides with every two adjacent sides having a different color. Regional coloring is giving color to an area with every two neighboring areas having a different color [12].

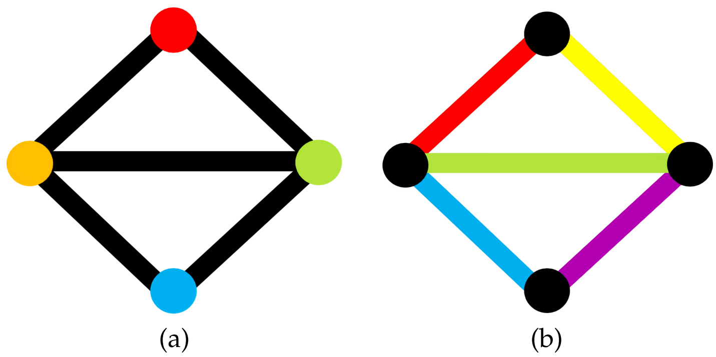

Figure 1. Example of: (a) Node Coloring, (b) Edge Coloring

Figure 2. Example of Region Coloring

2.2. Graph Coloring Problem Optimization Model

Jovanović, et al., in 2020 developed an optimization model for strategic decision making using well-known combinatorial problems, such as the weighted node coloring problem and the Traveling Salesman Problem. In this study, an optimization model for the vertex coloring problem in the graph will be developed as follows [8]:

$$\beta = \min \sum_{j} y_j \quad (1)$$

where β is the minimum number of colored points (y_j).

There are four constraints from the optimization model of the node coloring problem above, the first constraint defines if all nodes must be colored. These constraints are formulated as follows:

$$\sum_{j} x_{ij} = 1,\ i\ \in V \quad (2)$$

where x_{ij} is the node variable i which is colored j and $V = \{v_1, v_2, \ldots, v_n\}$.

The second constraint defines that there is at most one of the pair of vertices and edges that receive a certain color. The second constraint is formulated as follows:

$$x_{ij} + x_{kj} \leq y_j,\ \forall\,(i,k) \in E,\ j = 1,..,n \quad (3)$$

where x_{ij} and x_{kj} are node variables i and k which are colored j and $E = \{e_1, e_2, \ldots, e_n\}$.

The third and fourth constraints are conditions if the variables x_{ij} and y_j are binary variables. The third and fourth constraints are formulated as follows:

$$x_{ij} \in 0,1\ ,\ i\ \in V,\ j = 1,..,n \quad (4)$$

$$y_j \in 0,1\ , j = 1,..,n \quad (5)$$

The description of the set, parameters, and objective variables used in the model is as follows

1. Set
   V : Set of vertices that are not empty
   E : The set of edges that connect a pair of nodes
2. Decision Variables
   $x_{ij} = 1$ declares variable to have value 1 if and only if color j is assigned to node i;
   x_{ij} : Node variable i which is colored j
   $y_j = 1$ declares variable to have value 1 if and only if color j is set to at least one of the vertices;
   y_j : The variable that determines whether the sign (color) of j used.

2.3. Graph Coloring in The Electricity Sircuit

Maiti and Tripathy [13] in 2012 conducted a study on the application of graph coloring for matching electrical circuits in electrical repositories. In this article, we discuss a colored graph isomorphism-based model for matching two electrical circuits. The procedure for matching two electrical circuits consists of two steps, first, saving the circuit in the database in the form of a colored graph and then matching the input circuit using an isomorphism graph. This study aims to determine the compatibility of two circuits using Luellau's Algorithm and Ohlrich's Algorithm [13].

An electrical circuit consists of a set of different components and is connected by wires or conducting materials. Electrical circuits can be converted into graphs where the vertices are the components of the electrical circuit and the edges are the wires or conductors. In this model, the series is represented as a color-weighted undirected graph:

1. Nodes based on each component or terminal section such as a diode consists of 2 terminals so that it can be represented by two D+ and D- nodes connected with W as the connection node.
2. The color of the node is determined based on the electrical component it represents. For example, Resistors are always red in Figure 4.
3. The weight of the node is the value of the component itself. For example, a resistor is weighed by its resistance value. However, this is optional if the component is generic like Ground.
4. The nodes are connected in the graph if they are connected in the circuit.

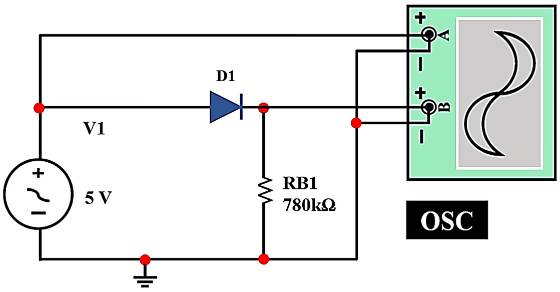

Figure 3. Half-wave rectifier circuit diagram. Source: [13]

The half-wave rectifier circuit diagram is converted into a graph consisting of 5 components, Ground (G), Oscilloscope (O), Signal Generator (V), Diode (D), and Resistor (R).

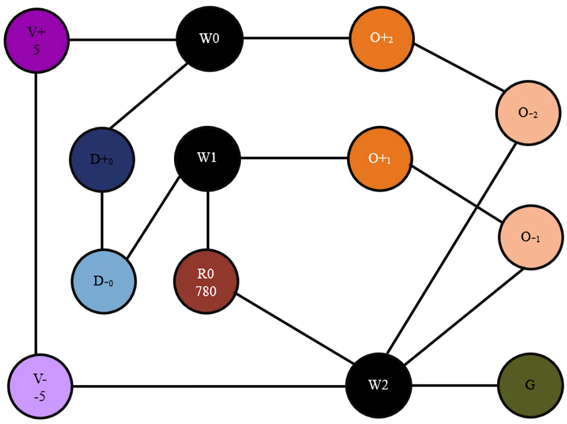

Figure 4. Electrical circuit graph. Source: [13]

Each component in the circuit is represented as a vertex in the graph. For example, for diodes D+ and D-, an oscilloscope with two channels is represented by O+1, O+2, and O-1 and O-2. Resistor and Ground have orientations that are not important so they are simply represented by one node. The anode of the oscilloscope channel has the same components and functions so that it has the same color. The sides in the graph represent the relationship between each component in the series.

2.4. Robust Optimization

In real-world problems, optimization problems in the presence of uncertainty in their parameters can be solved in a way to find out the result of the computed solution being highly inappropriate, suboptimal, or both (potentially worthless). The optimization related to the uncertainty parameter is Robust Control. Robust optimization focuses on the theoretical concepts of traditional optimization, especially algorithms, geometry, and tractability, in addition to using modeling will obtain results that can generally be used to calculate Robustness [14].

Robust Optimization (RO) is a model, combined with computational tools, to solve optimization problems with uncertain data and multiple uncertainty sets [15]. RO considers a deterministic uncertainty model consisting of sets with the ability to change the parameters of a function to obtain the best decision [16]. Several things can cause optimization problems to have indeterminate parameters, including measurement/rounding errors, prediction/estimation errors, implementation errors [17].

A linear optimization model, in general, can be expressed in the following equation

$$\min_{x} \{\boldsymbol{c}^T\boldsymbol{x} \ : A\boldsymbol{x} \leq \boldsymbol{b}\} \tag{6}$$

where $\boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{x} \in \mathbb{R}^n$, $A \in M_{m\times n}(\mathbb{R})$, $\boldsymbol{b} \in \mathbb{R}^m$.

Referring to [15] three parameters can be assumed to be indeterminate, namely parameters $\boldsymbol{c}, A, \boldsymbol{b}$, so that model (6) can be expressed in the general form of an indefinite linear optimization model as follows

$$\min_{x}\{\boldsymbol{c}^T\boldsymbol{x} \ : A\boldsymbol{x} \leq \boldsymbol{b} \ | \ (\boldsymbol{c}, A, \boldsymbol{b}) \in \mathcal{U}\} \tag{7}$$

There are three basic assumptions in Robust Optimization (RO) [18], namely:

1. All decision variables are "here and now" namely the acquisition of numerical values as a result of solving problems before the actual value of the original data is known.
2. The decision-maker is responsible for the decision to be made with the actual data in the uncertainty set $\mathcal{U}$.
3. All constraints of the problem are not necessarily "hard", that is, the decision-maker cannot violate the existing constraints even though all data are in the uncertainty set $\mathcal{U}$.

In addition to the three basic assumptions above, there are several additional assumptions, namely:

1. If the uncertainty is in the objective function, then the problem can be changed by adding the variable t into the objective function so that the uncertainty can appear in the constraint function. From the general model (7), the objective function $\boldsymbol{c}^T\boldsymbol{x}$ is replaced by an additional variable with $t \geq \boldsymbol{c}^T\boldsymbol{x}$ and $t \in \mathbb{R}$ so that it becomes

$$\min_{x}\{t : \boldsymbol{c}^T\boldsymbol{x} \leq t, \ A\boldsymbol{x} \leq \boldsymbol{b} \ | \ (\boldsymbol{c}, A, \boldsymbol{b}) \in \mathcal{U}\} \tag{8}$$

or

$$\min_x \{t : c^T x - t \leq 0,\ Ax \leq b \mid (c, A, b) \in \mathcal{U}\} \tag{9}$$

Therefore, the uncertainty has disappeared from the objective function.

2. If there is any uncertainty on the right side, it can be translated by adding variable $x_{n+1} = -1$.
3. The set of uncertainty $\mathcal{U}$ can be replaced with convex hull $\mathcal{U}$, which is the smallest convex set of $\mathcal{U}$. To obtain a feasible solution to $\mathcal{U}$, you can take the supremum of the left-hand side constraints $\mathcal{U}$, so that the objective function will produce an optimal value if the convex hull $\mathcal{U}$ has a maximum value.
4. Robustness to $\mathcal{U}$ can be illustrated constraint-wise.

2.5. Robust Counterpart

The robust Counterpart (RC) is part of the uncertainty problem. A feasible/optimal solution for RC is called a Robust feasible/optimal solution for an indeterminate problem. RC is used as a decision based on "real life" problems to obtain the Robust optimal solution [19].

Assuming that $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ are definite, so that model (7) can be reformulated and called the RC problem is as follows [18]

$$\min_x \{c^T x : A(\zeta)x \leq b \mid \forall \zeta \in \mathcal{Z}\} \tag{10}$$

$$\min_x \{c^T x : \bar{a}_i^T(\zeta)x \leq b_i \mid i = 1, \ldots, m,\ \forall \iota \in \mathcal{Z}\} \tag{11}$$

where $\mathcal{Z} \in \mathbb{R}^L$ which denotes a user-defined set of primitive uncertainties. Solution $x \in \mathbb{R}^n$ is a feasible solution which is called Robust feasible if it satisfies the indefinite constraint $A(\zeta)x \leq b$ for all realizations $\in \mathcal{Z}$.

Based on the previous assumptions, namely $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ is certain, then the indefinite parameter is $A \in M_{m\times n}(\mathbb{R})$. Because Robust is assumed to be constraint-wise, the single constraint in the model (11) becomes

$$(\bar{a} + P\zeta)^T x \leq b,\ \forall \zeta \in \mathcal{Z} \tag{12}$$

where $(\bar{a} + P\zeta)$ is an affine function of the primitive uncertainty parameter $\zeta \in \mathcal{Z}, \bar{a} \in \mathbb{R}^n$ and $P \in M_{n\times L}(\mathbb{R})$.

Furthermore, there are two ways to determine ζ, namely first, by applying the Robust reformulation technique to exclude for all quantifications. The second is to apply an adversarial approach. In this study, only the first method is used, which consists of three steps and produces a computationally tractable RC with a limited number of constraints.

2.6. Robust Counterpart for Polyhedral Uncertainty Sets

The set of uncertainty in the determination of ζ consists of the set of box, ellipsoidal, and polyhedral uncertainties. In this study, we only assume that the uncertainty parameter is in the polyhedral uncertainty set. According to [18] the set of polyhedral indeterminacy can be defined as follows

$$\mathcal{Z}=\{\zeta : D\zeta + \boldsymbol{q} \geq \boldsymbol{0}\} \tag{13}$$

where $D\in\mathbb{R}^{m\times L}$, $1\in\mathbb{R}^{L}$ and $\boldsymbol{q}\in\mathbb{R}^{m}$. So for the set of uncertainty $\mathcal{U}$ can be formulated as follows:

$$\mathcal{U}=\{\boldsymbol{a}|\boldsymbol{a} = (\overline{\boldsymbol{a}} + P\zeta)\,,\; D\zeta + \boldsymbol{q} \geq \boldsymbol{0}\} \tag{14}$$

To obtain an RC formulation with a polyhedral uncertainty set, equation (13) can be applied to equation (12) so that it becomes

$$(\overline{\boldsymbol{a}} + P\zeta)^{T}\boldsymbol{x} \leq \boldsymbol{b},\; \forall\, \zeta : D\zeta + \boldsymbol{q} \geq \boldsymbol{0} \tag{15}$$

There are three steps to the approach using the first method, namely:

1. Worst Case Reformulation

$$\begin{aligned} \max_{\zeta:D\zeta+\boldsymbol{q}\geq\boldsymbol{0}} (\overline{\boldsymbol{a}} + P\zeta)^{T}\boldsymbol{x} &\leq \boldsymbol{b} \\ \overline{\boldsymbol{a}}^{T}\boldsymbol{x} + \max_{\zeta:D\zeta+\boldsymbol{q}\geq\boldsymbol{0}} \left(P^{T}\boldsymbol{x}\right)^{T}\zeta &\leq \boldsymbol{b} \end{aligned} \tag{16}$$

2. Duality Convert the primal shape of the model (16) to the dual shape. To make it easier to convert the primal to dual form, the rules of the reflection transformation or the reflection of the primal problem to the dual problem can be used as shown in Table 1 which refers to [20].

Table 1. Correspondence rules for primal-dual relations

	Primal	**Dual**
Objective Function	$\min \boldsymbol{c}^{T}\boldsymbol{x}$	$\max \boldsymbol{b}^{T}\boldsymbol{y}$
Variable	$x_i \geq 0$	i th constraint $A_i^T\boldsymbol{y} \leq c_i$
Variable	x_i unrestricted	i th constraint $A_i^T\boldsymbol{y} = c_i$
Constraints	j th constraint $A_j\boldsymbol{x} = b_j$	j th variable y_j unrestricted
Constraints	j th constraint $A_j\boldsymbol{x} \geq b_j$	j th variable $y_j \geq 0$
Coefficient Matrix	$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix}$	$A^T = \begin{bmatrix} A_1 & A_2 & \cdots & A_m \end{bmatrix}$
Right-hand-side	Vector b	Vector c
Cost Coefficient	Vector c	Vector b

The primal form of the model (16) is

$$\max_{\zeta:D\zeta+\boldsymbol{q}\geq\boldsymbol{0}} \left(P^{T}\boldsymbol{x}\right)^{T}\zeta\,. \tag{17}$$

So the dual problem form for model (17) is

$$\min_{w}\{\boldsymbol{q}^{T}\boldsymbol{w} : -D^{T}\boldsymbol{w} = P^{T}\boldsymbol{x},\; \boldsymbol{w} \geq \boldsymbol{0}\} \text{ or } \min_{w}\{\boldsymbol{q}^{T}\boldsymbol{w} : D^{T}\boldsymbol{w} = -P^{T}\boldsymbol{x},\; \boldsymbol{w} \geq \boldsymbol{0}\}\,. \tag{18}$$

By substituting model (18) into the model (16) it becomes

$$\overline{a}^T x + \min_{w}\{q^T w : D^T w = -P^T x,\ w \geq \mathbf{0}\} \leq b. \tag{19}$$

3. Robust Counterpart (RC) Formulation
   After obtaining the minimization form in the model (19), the constraint applies to at least one w. The final formulation of RC was obtained

$$\exists w:\ \overline{a}^T x + q^T w \leq b, D^T w = -P^T x,\ w \geq \mathbf{0}. \tag{20}$$

   The constraint in equation (20) is in the form of Linear Programming (LP). If the set can be described well with linear constraints then it can be traced computationally tractable [21]. For the Tractable Formulation with Polyhedral Uncertainty Set, it can be seen in Table 2.

Table 2. Formulation of tractable polyhedral uncertainty set

Uncertainty Set	$\mathcal{Z}$	Robust Counterpart	Tracktibility
Polyhedral	$D\zeta + q \geq \mathbf{0}$	$\overline{a}^T x + q^T w \leq b$ $D^T w = -P^T x$ $w \geq \mathbf{0}$	Linear Programming (LP)

3. Results and Discussions

This section discusses the Graph Coloring Problem Optimization Model Formulation, Optimization Uncertainty Model Formulation Graph Coloring Problem, Robust Counterpart Formulation of Graph Coloring with Polyhedral Uncertainty Sets, Numerical Experiments, and Analysis of Numerical Experimental Results.The difference between this study and previous research which is similar to the research conducted by Jabrayilov and Mutzel [7] is that this study discusses integer linear programming on vertex coloring probles using partial ordering, while in this study using integer linear programming models specifically for vertex coloring problems and as a development it will viewed against the problem of uncertainty with the help of Robust Optimization model.

3.1. Graph Coloring Problem Optimization Model Formulation

The optimization model for the graph coloring problem consists of an objective function to minimize the number of colors at the vertices which is formulated as follows:

$$\beta = \min \sum_{j} y_j\ . \tag{21}$$

This study uses an objective function because the main problem of coloring the graph vertices is to determine the minimum number of colors in the vertices. In addition, the graph coloring optimization model consists of four constraints as in equation (2)-(5). So the optimization model of the graph coloring problem is as follows:

$$
\begin{aligned}
\beta = & \min \sum_j y_j \,, \\
s.t \quad & \sum_j x_{ij} = 1, \; i \in V, \\
x_{ij} + x_{kj} \leq y_j, \; & \forall (i,k) \in E, \; j = 1,..,n, \\
x_{ij} \in 0,1 \,, \; i \in V, \; & j = 1,..,n, \\
y_j \in 0,1 \,, j = 1,..,n&.
\end{aligned}
\tag{22}
$$

3.2. Optimization Uncertainty Model Formulation Graph Coloring Problem

The challenge in the graph coloring problem is to determine the number of color vertices to be colored precisely so that all neighboring vertices have different colors. Therefore, the uncertainty parameter in this study is β. The uncertainty that lies in the objective function causes a variable to be given to limit the objective function, for example the variable is t, so that the optimization model for the graph coloring problem in (22) can be rewritten into the following equation:

$$
\begin{aligned}
& \min t \\
s.t \quad & \boldsymbol{c}^T \boldsymbol{y} \leq t, \\
& \sum_j x_{ij} = 1, \; i \in V, \\
& x_{ij} + x_{kj} \leq y_j, \; \forall (i,k) \in E, \; j = 1,..,n, \\
& x_{ij} \in 0,1 \,, \; i \in V, \; j = 1,..,n, \\
& y_j \in 0,1 \,, j = 1,..,n, \\
& \boldsymbol{c} \in \mathcal{U}.
\end{aligned}
\tag{23}
$$

3.3. Robust Counterpart Formulation of Graph Coloring with Polyhedral Uncertainty Sets

Assume that $\boldsymbol{c}$ is uncertain and the uncertainty set $\mathcal{U}$ has the form $\mathcal{U} = \boldsymbol{c} \mid \boldsymbol{c} = \bar{\boldsymbol{c}} + P\zeta, \; \zeta \in \mathcal{Z}\}$, so it can be formulated as follows

$$
\begin{aligned}
& \min t \,, \\
s.t \quad & (\bar{\boldsymbol{c}} + P\zeta)^T \boldsymbol{y} \leq t, \\
& \sum_j x_{ij} = 1, \; i \in V, \\
& x_{ij} + x_{kj} \leq y_j, \; \forall (i,k) \in E, \; j = 1,..,n, \\
& x_{ij} \in 0,1 \,, \; i \in V, \; j = 1,..,n, \\
& y_j \in 0,1 \,, j = 1,..,n, \\
& \zeta \in \mathcal{Z}.
\end{aligned}
\tag{24}
$$

Next, assume that $\mathcal{Z}$ is a polyhedral uncertainty set of the form $\mathcal{Z} = \{\zeta : D\zeta + \boldsymbol{q} \geq \boldsymbol{0}\}$, where $\mathcal{Z}\in\mathbb{R}^L$ denotes the primitive uncertainty set. Solution $\boldsymbol{x}\in\mathbb{R}^n$ is a feasible solution called Robust feasible if it satisfies the indefinite constraint $(\bar{c}+P\zeta)^T\boldsymbol{y} \leq t$ for all realizations $\zeta\in\mathcal{Z}$.

Robust Counterpart formulation with Polyhedral uncertainty set can be obtained through three steps as follows.

- **Step 1.** Worst case reformulation
  In equation (24) it is equivalent to the worst case reformulation as follows

$$\bar{\boldsymbol{c}}^T\boldsymbol{y} + \max_{\zeta:D\zeta+\boldsymbol{q}\geq\boldsymbol{0}} ((P\boldsymbol{y})^T\zeta) \leq t\,. \tag{25}$$

- **Step 2**. Duality
  Take the dual of the maximization problem in equation (25) so that it is equivalent to

$$\bar{\boldsymbol{c}}^T\boldsymbol{y} + \min_{w} \boldsymbol{q}^T\boldsymbol{w} : D^T\boldsymbol{w} = -P^T\boldsymbol{y},\ \boldsymbol{w}\geq\boldsymbol{0} \ \leq t. \tag{26}$$

- **Step 3.** Robust Couterpart
  After the minimization form of equation (26) is obtained, the constraint holds for at least one $\boldsymbol{w}$. So that the final RC formulation is obtained:

$$\exists\boldsymbol{w} : \bar{\boldsymbol{c}}^T\boldsymbol{y} + \boldsymbol{q}^T\boldsymbol{w} \leq t, D^T\boldsymbol{w} = -P^T\boldsymbol{y},\ \boldsymbol{w}\geq\boldsymbol{0}. \tag{27}$$

  The form of equation (27) can be re-expressed in the form of sigma and index, so that the Robust Counterpart model with polyhedral uncertainty set graph coloring problem is as follows:

$$\begin{aligned}
&\min t\,, \\
s.t \quad &\sum_{j=1}^{n}\bar{c}_j y_j + \sum_{m=1}^{M} q_m w_m \ \leq t, \\
&\sum_{j} x_{ij} = 1,\ i\ \in V, \\
&x_{ij} + x_{kj} \leq y_j,\ \forall\,(i,k)\in E,\ j = 1,..,n, \\
&\sum_{m=1}^{M} D_{ml} w_m = -\ \sum_{j=1}^{n} P_j y_j,\ l = 1,..,L \\
&x_{ij} \in \ 0,1\ ,\ i\ \in V,\ j = 1,..,n, \\
&y_j \in \ 0,1\ , j = 1,..,n, \\
&w_m \geq 0.
\end{aligned} \tag{28}$$

3.4. Numerical Experiments on Electrical Circuit Problems

Referring to [13], there is an electrical circuit that is converted into a color graph with 13 vertices and 15 edges as shown in Figure 5. The complete vertices and edges in Figure 5 are as follows:

Table 3. Table of vertex variables and its components

Vertex	Component	Vertex	Component	Vertex	Component
1	V^+	6	D_0^-	11	W_2
2	W_0	7	W_1	12	G
3	O_2^+	8	O_1^+	13	V^-
4	O_2^-	9	O_1^-		
5	D_0^+	10	R_0		

Table 4. Edges table and description

Edge	Description	Edge	Descriptio	Edge	Description
e_1	(1,2)	e_6	(13,1)	e_{11}	(8,9)
e_2	(2,3)	e_7	(2,5)	e_{12}	(9,11)
e_3	(3,4)	e_8	(5,6)	e_{13}	(7,10)
e_4	(4,11)	e_9	(6,7)	e_{14}	(10,11)
e_5	(11,13)	e_{10}	(7,8)	e_{15}	(11,12)

Based on the data in Table 3 and Table 4, the results of the calculation using the Graph Coloring Optimization Model in equation (23) are presented in Table 5.

Table 5. Results of numerical experiments on optimization of graph coloring problems in electrical circuits

Vertex (i)	Color (j)	Variable (x_{ij})	Vertex (i)	Color (j)	Variable (x_{ij})
1	Green	x_{14}	8	Red	x_{86}
2	Yellow	x_{25}	9	Yellow	x_{95}
3	Green	x_{34}	10	Yellow	x_{105}
4	Yellow	x_{45}	11	Green	x_{114}
5	Green	x_{54}	12	Yellow	x_{125}
6	Yellow	x_{65}	13	Yellow	x_{135}
7	Green	x_{74}			

Based on Table 5, it can be seen that the number of colors used is 3 colors, so the minimum value for β is 3. Suppose the color code used is 4 = Green, 5 = Yellow, 6 = Red. In Table 5, the value of the variable x_{ij} means that node i is given the color j, for example x_{14} means that node 1 is given color 4, namely Green. The number of color codes in the calculation is as many as the existing vertices, but the number of colors chosen to color the graph in this case is 3. A graph whose vertices are colored based on the results from Table 5 is presented in Figure 5.

Based on the data in Table 3 and Table 4, the results of the calculations using the Robust Coloring Optimization Model with the Polyhedral Uncertainty Set in equation (28) are presented in Table 6.

Based on Table 6, it can be seen that the number of colors used is 5 colors, so the minimum value for t is 5. Suppose the color code used is 1 = Purple, 2 = Yellow, 7 = Green, 10 = Orange, 11 = Brown. In Table 6, the value of the variable x_{ij} means that node i is given the color j, for example x_{17} means that node 1 is given the color 7 which is Green. The number of color codes in the calculation is as many as there are vertices, but the number

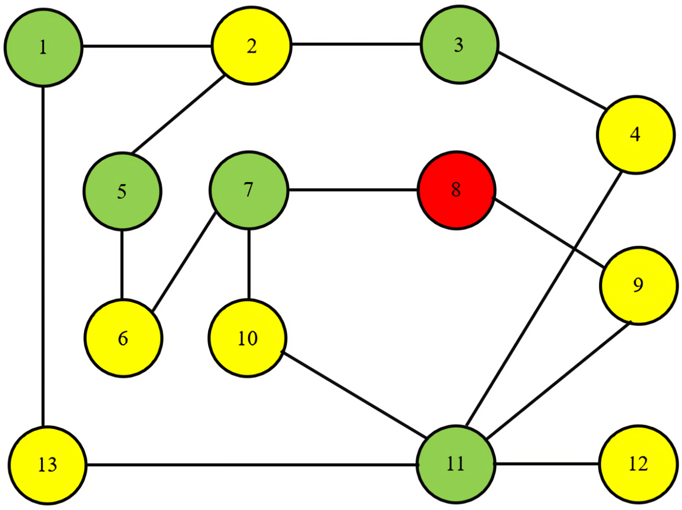

Figure 5. Optimization of graph coloring problems in electric circuits

Table 6. Results of numerical experiments on robust optimization models for graph coloring problems in electric circuits

Vertex (i)	Color (j)	Variable (x_{ij})	Vertex (i)	Color (j)	Variable (x_{ij})
1	Green	x_{17}	8	Brown	x_{811}
2	Purple	x_{21}	9	Purple	x_{91}
3	Green	x_{37}	10	Yellow	x_{102}
4	Purple	x_{41}	11	Orange	x_{1110}
5	Green	x_{57}	12	Purple	x_{121}
6	Brown	x_{611}	13	Yellow	x_{132}
7	Purple	x_{71}			

of colors chosen to color the graph in this case is 5. A graph whose vertices are colored based on the results from Table 6 is presented in Figure 6.

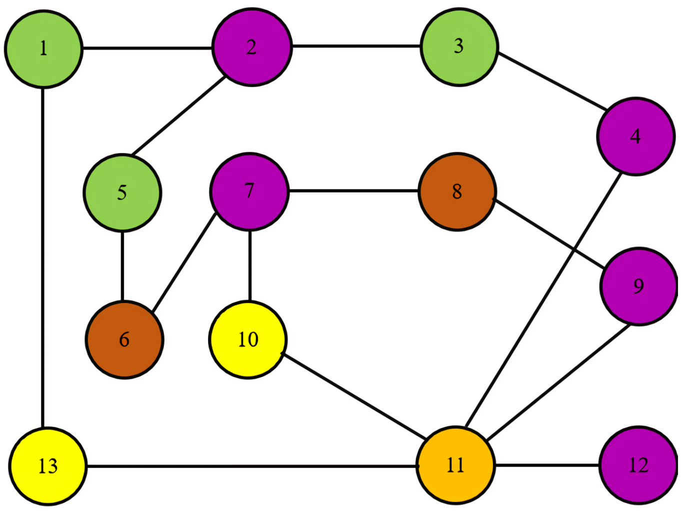

Figure 6. Robust optimization of graph coloring problems in electric circuits

3.5. Analysis of Numerical Experiments Results

Graph Coloring Problem using Robust Optimization Model with Polyhedral Uncertainty Set produces Linear Programming form. Based on this, the Robust Optimization Model on the Graph Coloring Problem produces a computationally tractable solution. The optimal solution of the objective function of Graph Coloring Problem Optimization Model and Robust Optimization Model of Graph Coloring Problems with Polyhedral Uncertainty Sets is in Table 7.

Table 7. Numerical experiment optimal solution

	Graph Coloring Problem Optimization Model	Robust Optimization Model with Polyhedral Uncertainty Set on Graph Coloring Problems
Number of colors used	3 colors	5 colors

Based on the table above, the acquisition of the number of colors using the Robust Optimization Model produces more numbers than the Graph Coloring Problem Optimization Model. The difference in the number of colors used in the RC Optimization Model is due to the uncertainty in the model.

In the problem of Electrical Circuits, these colors are used as a way to arrange electrical components on the appropriate path. If the two electrical components that are connected to each other use different colors, then the electrical components are on the right track. Based on this, the Robust Optimization Model on Graph Coloring Problems can be used as a way to form electrical circuits with optimal paths.

4. Conclusion

The Graph Coloring problem optimization model can be formulated into a Robust Optimization Model by using the uncertainty parameter in the number of colors used. Robust Optimization Model is solved by using Polyhedral Uncertainty Set to produce a computationally tractable solution.

The Electric Circuit Problem can be solved using Robust Optimization Model with Polyhedral Uncertainty Set. The results of numerical experiments using the Graph Coloring Problem Optimization Model obtained the minimum number of colors used as many as 3 colors, while the results of numerical experiments using the Graph Coloring Problem Robust Optimization Model obtained the minimum number of colors used as many as 5 colors. Obtained if the objective function of the Graph Coloring Robust Optimization Model has been met. This is because, the minimum number of colors used in the node has been obtained and the optimal solution has been obtained.

Acknowledgement

This research is supported by the Indonesian Ministry of Education, Culture, Research, and Technology under project with Basic Research Scheme 2022 entitled ”Adjustable Robust Counterpart Optimization Model and Social Media Analysis for Internet Shopping Online Problem” under contract number 2064/UN6.3.1/PT.00/2022.

Reference

[1] W.-J. van Hoeve, "Graph Coloring Lower Bounds from Decision Diagrams," in *Lect. Notes Comput. Sci. (including Subser. Lect. Notes Artif. Intell. Lect. Notes Bioinformatics)*, 2020, pp. 405–418, doi: 10.1007/978-3-030-45771-6_31. [Online]. Available: http://link.springer.com/10.1007/978-3-030-45771-6_31

[2] A. Elumalai, "Graph coloring and its implementation," *Malaya Journal of Matematik*, vol. S, no. 2, pp. 1672–1674, 1672, doi: https://doi.org/10.26637/MJM0S20/0445.

[3] I. M. Dıaz, G. Nasini, and D. Severın, "A Linear Integer Programming Approach for The Equitable Coloring Problem," *Information Sciences*, pp. 2–5, 2004.

[4] R. Nickel, "Graph Coloring: Application of the Ellipsoid Method in Combinatorial Optimization," *FernUni Hagen*, pp. 408–438, 2005, doi: 10.1201/b16132-29.

[5] P. Hansen, M. Labbé, and D. Schindl, "Set covering and packing formulations of graph coloring: Algorithms and first polyhedral results," *Discrete Optimization*, vol. 6, no. 2, pp. 135–147, may 2009, doi: 10.1016/j.disopt.2008.10.004. [Online]. Available: https://linkinghub.elsevier.com/retrieve/pii/S1572528608000716

[6] E. K. Burke, J. Mareček, A. J. Parkes, and H. Rudová, "A supernodal formulation of vertex colouring with applications in course timetabling," *Annals of Operations Research*, vol. 179, no. 1, pp. 105–130, sep 2010, doi: 10.1007/s10479-010-0716-z. [Online]. Available: http://link.springer.com/10.1007/s10479-010-0716-z

[7] A. Jabrayilov and P. Mutzel, "New Integer Linear Programming Models for the Vertex Coloring Problem," in *Lect. Notes Comput. Sci. (including Subser. Lect. Notes Artif. Intell. Lect. Notes Bioinformatics),*, 2018, pp. 640–652, doi: 10.1007/978-3-319-77404-6_47. [Online]. Available: http://link.springer.com/10.1007/978-3-319-77404-6_47

[8] P. Jovanović, N. Pavlović, I. Belošević, and S. Milinković, "Graph coloring-based approach for railway station design analysis and capacity determination," *European Journal of Operational Research*, vol. 287, no. 1, pp. 348–360, nov 2020, doi: 10.1016/j.ejor.2020.04.057. [Online]. Available: https://linkinghub.elsevier.com/retrieve/pii/S0377221720304100

[9] D. Marx, "Graph colouring problems and their applications in scheduling," *Periodica Polytechnica Electrical Engineering*, vol. 48, no. 1-2, pp. 11–16, 2004.

[10] h. Yanıkoğlu, B. L. Gorissen, and D. den Hertog, "A survey of adjustable robust optimization," *European Journal of Operational Research*, vol. 277, no. 3, pp. 799–813, sep 2019, doi: 10.1016/j.ejor.2018.08.031. [Online]. Available: https://linkinghub.elsevier.com/retrieve/pii/S0377221718307264

[11] N. K. Kabang, Y. Yundari, and F. Fran, "Bilangan kromatik lokasi pada graf bayangan dan graf middle dari graf bintang," *Bimaster : Buletin Ilmiah Matematika, Statistika dan Terapannya*, vol. 9, no. 2, pp. 329–336, mar 2020, doi: 10.26418/bbimst.v9i2.39977. [Online]. Available: https://jurnal.untan.ac.id/index.php/jbmstr/article/view/39977

[12] M. A. K. Arsyad, L. Yahya, D. Wungguli, and N. I. Yahya, "Pewarnaan Graf pada Penjadwalan Pengangkutan Sampah di Kota Gorontalo," *OSF Preprints*, pp. 1–18, 2020, doi: 10.31219/osf.io/rq6cg. [Online]. Available: 10.31219/osf.io/rq6cg

[13] A. Maiti and B. Tripathy, "Applying Colored-Graph Isomorphism for Electrical Circuit Matching in Circuit Repository," *International Journal of Computer Science*, vol. 9, no. 3, pp. 391–395, 2012. [Online]. Available: http://www.doaj.org/doaj?func=fulltext&aId=1158414

[14] D. Bertsimas, D. B. Brown, and C. Caramanis, "Theory and Applications of Robust Optimization," *SIAM Review*, vol. 53, no. 3, pp. 464–501, jan 2011, doi: 10.1137/080734510. [Online]. Available: http://epubs.siam.org/doi/10.1137/080734510

[15] A. Ben-Tal and A. Nemirovski, "Robust optimization methodology and applications," *Mathematical Programming*, vol. 92, no. 3, pp. 453–480, may 2002, doi: 10.1007/s101070100286. [Online]. Available: http://link.springer.com/10.1007/s101070100286

[16] H. Xu, C. Caramanis, and S. Mannor, "A Distributional Interpretation of Robust Optimization," *Mathematics of Operations Research*, vol. 37, no. 1, pp. 95–110, feb 2012, doi: 10.1287/moor.1110.0531. [Online]. Available: http://pubsonline.informs.org/doi/10.1287/moor.1110.0531

[17] D. D. Hertog, "Practical Robust Optimization - an introduction," in *NGB/LNMB Seminar*. Tilburg: Tilburg University, 2013, pp. 1–53.

[18] B. L. Gorissen, h. Yanıkoğlu, and D. den Hertog, “A practical guide to robust optimization,” *Omega*, vol. 53, pp. 124–137, jun 2015, doi: 10.1016/j.omega.2014.12.006. [Online]. Available: https://linkinghub.elsevier.com/retrieve/pii/S0305048314001698

[19] A. Ben-Tal, L. E. Ghaoui, and A. Nemirovski, “Robust Optimization,” in *Princeton Series in Applied Mathematics*. New Jersey: Princeton University Press, 2009.

[20] S. S. Rao, *Engineering Optimization Theory and Practice*, 4th ed. New Jersey: John Wiley & Sons, Inc., 2009. ISBN 9780470183526

[21] D. Chaerani and C. Roos, “Handling Optimization under Uncertainty Problem Using Robust Counterpart Methodology,” *Jurnal Teknik Industri*, vol. 15, no. 2, pp. 111–118, dec 2013, doi: 10.9744/jti.15.2.111-118. [Online]. Available: http://puslit2.petra.ac.id/ejournal/index.php/ind/article/view/18848