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On the Solution of Volterra Integro-differential Equations using a Modified Adomian Decomposition Method

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ABSTRACT

The Adomian decomposition method's effectiveness has been demonstrated in recent research, the process requires several iterations and can be time-consuming. By breaking down the source term function into series, the current work introduced a new decomposition approach to the Adomian decomposition method. As compared to the conventional Adomian decomposition approach, the newly devised method hastens the convergence of the solution. Numerical experiments were provided to show the superiority qualities.

Keywords:
Infinite Series; Source Term; Convergence; Decomposition Methods

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1. Introduction

The Adomian Decomposition Technique (ADM), developed by Manafianheris [1], separates an equation into linear and nonlinear components. Equations involving nonlinear functions are solved using this technique. To provide solutions that take the form of recursive series, Adomian polynomials are used. Saray [2], discovered that problems involving these equations may be reduced to a set of algebraic equations by using a method for solving Volterra integro-differential equations. Olayiwola, *et al.* [3] showed how a modified variational iteration approach may be used to solve the numerical solution of the generalized Burger's-Huxley equation (MVIM). The solution proved to be more successful than similar strategies that required less computing work. Olayiwola, *et al.* [4] described how the modified variational iteration approach converges to the precise solution after an iteration for the solution of the class of initial and boundary value problems. As a result, the approach is effective and trustworthy for solving bantu-type differential equations. Alaje, *et al.* [5] discovered that an analytical strategy of modified initial guess homotopy perturbation is used to solve the Korteweg-de vries equation. The Banach fixed point theorem was used to demonstrate the method's convergence as well as a sequence of arbitrary orders.

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The multi-wavelets Galerkin method may be used to tackle second-order problems that are both linear and nonlinear. Volterra integro-differential equations are resolved using the operational integration matrices and the wavelet transform matrix. Ibrahim, *et al.* [6] looked at the original solution to the second-order nonlinear Fredholm integro-differential equation, which involved applying the Simpson method to turn the Fredholm IDE into a collection of nonlinear algebraic equations. Siweilam [7] created the Variational Iteration Methodology (VIM), which resolves the bulk of difficulties encountered in computing Adomian polynomials using the Adomian decomposition method, to solve integro-differential equations, which are difficult to solve analytically. Using the fourth-order derivatives block approach, Ogunniran, *et al.* [8] devised the collocation and interpolation of an assumed derivative and a basic function. A method for resolving ordinary differential equations' two-point singular nonlinear boundary value issues. Alaje, *et al.* [9] discovered that by combining the modified general Lagrange multiplier technique with the modified homotopy perturbation approach, the solution of linear and nonlinear fractional order integro-differential equations may be found. According to Olayiwola, *et al.* [10], a modified initial guess variational iteration approach may solve non-homogeneous variable coefficient fourth-order parabolic partial differential equations. According to Olayiwola, *et al.* [11], a comparison of numerical and analytical solutions to telegraph equations demonstrates that the numerical scheme of solving telegraph equations is successful when utilizing a modified variational iteration approach.

According to Alqarni, *et al.* [12], spectroscopic data, heat transport issues, and physical phenomena in engineering may all be solved using integral-differential equations. The third-order derivatives of unknown functions are contained in integro-differential equations (IDEs) known as third-order IDEs. Haar functions are employed in integro-differential equations, both linear and nonlinear, to approximate the third-order derivative. Lower-order derivatives and the solution to the mystery are produced through integration. Several partial differential equations are both linear and nonlinear that Chen [13] and Rohaninasab, *et al.* [14] have been utilized to solve. It has been proven to be a successful technique for getting numerical solutions. The Legendre collocation spectral method may be used to solve high-order linear Volterra-Fredholm integro-differential equations under mixed situations. Nonlinear Volterra integral and integro-differential equations may be used to study a variety of scientific topics, including heat transport, the spread of infectious illnesses, semiconductor neutron diffusion, and others [15]. Non-orthogonal polynomials can also be decomposed using the Laplace Adomian approach. The Adomian decomposition technique is a summation of an infinite convergent series without any restrictive constraints. When solving functional equations that are no longer valid, the Laplace-Adomian decomposition method combines two efficient techniques. The modified Laplace Adomian Decomposition Technique (LADM), which uniformly distributes the source function before performing Laplace Adomian Decomposition, is used to solve the Volterra integral and integro-differential equations based on Rani and Masra [16].

To estimate the solutions of nonlinear partial differential equations, the Laplace transform employs the decomposition method. According to Jimoh [17], many academics have explored third-order integro-differential equations, notably the nonlinear variety in closed form. The answer is then integrated to acquire the lower-order derivatives, while the trapezoidal approach is used to derive the unknown function itself. The power of series and canonical polynomials is used to approximate the largest derivatives in the topics studied. The polynomial issues presented by Olayiwola and Kareem [18] may be solved analytically using a variety of approaches; however, some of these approaches are challenging and call for several iterations that may be challenging to solve and take a long time to arrive at an approximation. This method is applied in numerous fields, including engineering, economics, chemical kinetics, fluid mechanics, etc. Olayiwola, *et al.* [19] explained how to develop Maple code for the method and simulation of the generalized Burger-equation Fisher's solution. With less computation, the results were produced. Kareem and Olayiwola [20] discovered that the Homotopy perturbation method was used to solve Integro-differential equations with two-point boundary conditions, and the numerical results obtained proved to be a very accurate algorithm for solving problems of linear Fourth-order Integro-differential equations.

The Laplace transform employs the decomposition method to approximate the solutions of nonlinear partial differential equations. According to Amin, *et al.* [21], many scholars have explored third-order integro-differential equations, especially the nonlinear variety in closed form. The answer is then integrated to create the lower-order derivatives, while the trapezoidal approach is used to derive the unknown function itself. The power of series and canonical polynomials approximate the largest derivatives in the topics studied. This method is utilized in numerous areas, including engineering, economics, chemical kinetics, fluid mechanics, etc.

2. Model and Modification

The modification was carried out by decomposing the source term function into series of the form,

$$h(x) = \sum_{j=0}^{+\infty} h_i(x) \tag{1}$$

and the new recursive relation was obtained as the theoretical aspect of the method:

$$\begin{aligned}
u_0(x) &= h_0(x), \\
u_1(x) &= h_1(x) + h_2(x) + \lambda \int_a^x k(x,t)\left(L(u_0(x)) + P_0\right) dt, \\
u_2(x) &= h_3(x) + h_4(x) + \lambda \int_a^x k(x,t)\left(L(u_0(x) + u_1(x)) + P_1\right) dt, \\
&\vdots \\
u_{j+1}(x) &= h_{2(j+1)}(x) + h_{2(j+1)-1}(x) + \lambda \int_a^x k(x,t)\left(L\left(u_j(x) + u_{j-1}(x)\right) + P_1\right) dt.
\end{aligned} \tag{2}$$

In case of non-linear, the newly modified Adomian decomposition method (MADM) accelerates the convergence of the solution (MADM) faster than Standard Adomian Decomposition Method (SADM). Assuming that the nonlinear function is $F(u(x))$ can be evaluated by using the expression,

$$P_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n}\lambda^i u_i\right)\right]_{\lambda=0} \tag{3}$$

where, $n = 0, 1, 2, \ldots$ and $i = 2, 3, 4, \ldots$.

Therefore, below are few of Adomian polynomials:

$$\begin{aligned}
P_0 &= F(u_0), \\
P_1 &= u_1 F'(u_0), \\
P_2 &= u_2 F'(u_0) + \frac{1}{2!}u_1^2 F''(u_0), \\
P_3 &= u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!}u_1^3 F'''(u_0), \\
P_4 &= u_4 F'(u_0) + \left(\frac{1}{2!}u_2^2 + u_1 u_3\right) F''(u_0) + \frac{1}{2!}u_1^2 u_2 F'''(u_0) + \frac{1}{4}u_1^4 F^{(\mathrm{iv})}(u_0).
\end{aligned} \tag{4}$$

Two important observations can be made here. First, P_0 depends only on u_0, P_1 depends only on u_0 and u_1 , P_2 depends only on u_0 , u_1 and u_2, and so on.

Secondly, substituting these $P_j's$ in (2) gives:

$$\begin{aligned}
F(u) &= P_0 + P_1 + P_2 + P_3 + \ldots \\
&= F(u_0) + (u_1 + u_2 + u_3 + \ldots)\ F'(u_0) + \frac{1}{2!}\left(u_1^2 + 2u_1u_2 + 2u_1u_3 + u_2^2\right) F''(u_0) \\
&\quad + \frac{1}{3!}\left(u_1^3 + 3u_1^2 u_3 + 6u_1u_2u_3 + \ldots\right)\ F'''(u_0) + \ldots \\
&= F(u_0) + (u - u_0) F'(u_0) + \frac{1}{2!}(u - u_0)^2 F''(u_0) + \ldots
\end{aligned}$$

In the following, we will calculate Adomian polynomials for several linear terms that may arise in nonlinear integral equations.

Case 1. The first four Adomian polynomials for $F(u) = u^2$ are given by

$$\begin{aligned}
P_0 &= u_0^2, \\
P_1 &= 2u_0 \mathrm{u}_1, \\
P_2 &= 2u_0 u_2 + u_1^2, \\
P_3 &= 2u_0 u_3 + 2u_1 u_2.
\end{aligned} \tag{5}$$

Case 2. The first four Adomian polynomials for $F(u) = u^3$ are given by

$$\begin{aligned}
P_0 &= u_0^3, \\
P_1 &= 3u_0^2 u_1, \\
P_2 &= 3u_0^2 u_2 + 3u_0 u_1^2, \\
P_3 &= 3u_0^2 u_3 + 6u_0 u_1 u_2 + u_1^3.
\end{aligned} \tag{6}$$

Case 3. The first four Adomian polynomials for $F(u) = u^4$ are given by

$$\begin{aligned}
P_0 &= u_0^4, \\
P_1 &= 4u_0^3 u_1, \\
P_2 &= 4u_0^3 u_2 + 6u_0^2 u_1^2, \\
P_3 &= 4u_0^3 u_3 + 4u_1^3 u_0 + 12u_0^2 u_1 + u_2.
\end{aligned} \tag{7}$$

Case 4. The first four Adomian polynomials for $F(u) = \sin u$ are given by

$$\begin{aligned}
P_0 &= \sin u_0, \\
P_1 &= u_1 \cos u_0, \\
P_2 &= u_2 \cos u_0 - \frac{1}{2!} u_1^2 \sin u_0, \\
P_3 &= u_3 \cos u_0 - u_1 u_2 \sin u_0 - \frac{1}{3!} u_1^3 \cos u_0.
\end{aligned} \tag{8}$$

Case 5. The first four Adomian polynomials for $F(u) = \cos u$ are given by

$$\begin{aligned}
P_0 &= \cos u_0, \\
P_1 &= -u_1 \sin u_0, \\
P_2 &= -u_2 \sin u_0 - \frac{1}{2!} u_1^2 \cos u_0, \\
P_3 &= -u_3 \sin u_0 - u_1 u_2 \cos u_0 + \frac{1}{3!} u_1^3 \sin u_0.
\end{aligned} \tag{9}$$

Case 6. The first four Adomian polynomials for $F(u) = \exp(u)$ are given by

$$\begin{aligned}
P_0 &= \exp(u_0), \\
P_1 &= u_1 \exp(u_0), \\
P_2 &= \left(u_2 + \frac{1}{2!} u_1^2\right) \exp(u_0), \\
P_3 &= \left(u_3 + u_1 u_2 + \frac{1}{3!} u_1^3\right) \exp(u_0).
\end{aligned} \tag{10}$$

3. Results and Discussions

3.1. Results

Some research results are given in 3 different case examples.

Example 1. Consider the first order linear Volterra Integro-differetial Equation:

$$u'(x) = 1 - \int_0^x u(t)\ \mathrm{dt} \quad 0 \leq x, \quad t \leq 1 \tag{11}$$

with initial condition $u(0) = 0$.

Solution. The exact solution is $u(x) = \sin x$. By using New Modification Adomian Decomposition Method (NMADM) give:

$$\int_0^x u'\ \mathrm{dx} = u(x),$$
$$\int_0^x 1\ \mathrm{dx} = x,$$

then

$$u(x) = x - \int_0^x \int_0^x u(t)\ \mathrm{dtdx}.$$

Let $r = x$. Expand taylor (r, x,10):

$$\begin{aligned}
&x\\
a_0 &= x\\
u_0 &= t,\\
a_1 &= -\int_0^x \int_0^x u_0\ \mathrm{dtdx} = -\frac{1}{6}x^3\\
u_1 &= -\frac{1}{6}t^3,\\
a_2 &= -\int_0^x \int_0^x u_1\ \mathrm{dtdx} = \frac{1}{120}x^5\\
u_2 &= \frac{1}{120}t^5,\\
a_3 &= -\int_0^x \int_0^x u_2\ \mathrm{dtdx}\\
u_3 &= -\frac{1}{5040}t^7,\\
a_4 &= -\int_0^x \int_0^x u_3\ \mathrm{dtdx} = \frac{1}{362880}x^9\\
u_4 &= \frac{1}{362880}t^9.
\end{aligned}$$

then

$$u_n = u_0 + u_1 + u_2 + u_3 + u_4.$$

Hence

$$u_n\left(t\right) = t - \frac{1}{6}t^3 + \frac{1}{120}t^5 - \frac{1}{5040}t^7 + \frac{1}{3628800}t^9, \text{ and}$$
$$u_n\left(x\right) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{3628800}x^9.$$

Comparison of results and curve for Example 1 are represented in Table 1 and Figure 1.

Table 1. Comparison of results for Example 1

X	Exact	NADM
0	0	0
0.1	0.099833416646828	0.099833416646828
0.2	0.198669330795061	0.198669330795061
0.3	0.295520206661340	0.295520206661340
0.4	0.389418342308651	0.389418342308651
0.5	0.479425538604203	0.479425538604203
0.6	0.564642473395035	0.564642473395035
0.7	0.644217687237691	0.644217687237691
0.8	0.717356090899523	0.717356090899523
0.9	0.783326909627483	0.783326909627483
1.0	0.841470984807897	0.841470984807897

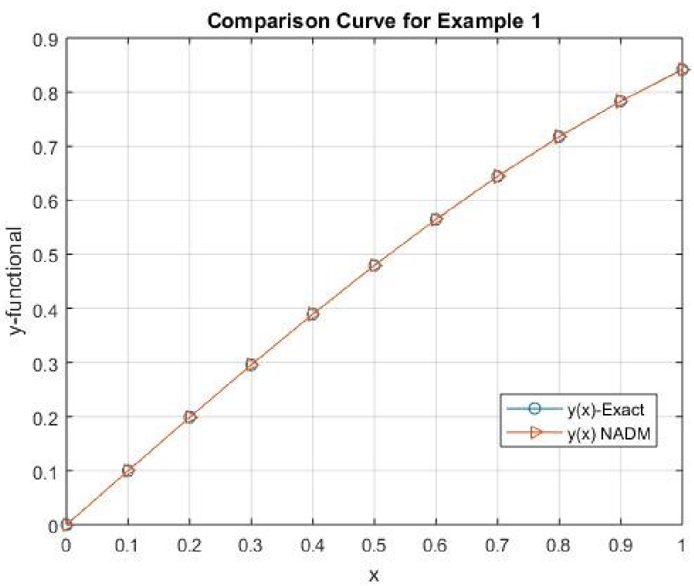

Figure 1. Comparison curve for Example 1

Example 2. Consider the second order linear Volterra Integro-differential Equation:

$$u''\left(x\right) = 1 + \int_0^x \left(x - t\right) u\left(t\right) \, \mathrm{d}t \tag{12}$$

with initial condition $u(0) = 1,\ u'(0) = 0.$

Solution. The exact solution is $u(x) = \cosh x$.
By using Modified Adomian Decomposition Method (MADM) give:

$$\int_0^x \int_0^x u''\ \mathrm{dxdx} = u(x) - 1,$$
$$\int_0^x \int_0^x 1\ \mathrm{dxdx} = \frac{1}{2}x^2,$$

then

$$u(x) - 1 = \frac{1}{2}x^2 + \int_0^x \int_0^x \int_0^1 (x-t)\, u(t)\ \mathrm{dtdxdx},$$
$$u(x) = 1 + \frac{1}{2}x^2 + \int_0^x \int_0^x \int_0^1 (x-t)\, u(t)\ \mathrm{dtdxdx}.$$

Let

$$r = 1 + \frac{1}{2}x^2.$$

Expand taylor (r, x,10):

$$1 + \frac{1}{2}x^2 a_0 = 1$$

$$\begin{aligned}
u_0 &= 1\\
h_0 &= \frac{1}{2}x^2\\
a_1 &= h_0 + \int_0^x \int_0^x \int_0^x (x-t)\, u_0\ \mathrm{dtdxdx} = \frac{1}{2}x^2 + \frac{1}{24}x^4\\
u_1 &= \frac{1}{2}t^2 + \frac{1}{24}t^4\\
a_2 &= \int_0^x \int_0^x \int_0^x (x-t)\, u_1\ \mathrm{dtdxdx} = \frac{1}{40320}x^8 + \frac{1}{720}x^6\\
u_2 &= \frac{1}{40320}t^8 + \frac{1}{720}t^6\\
a_3 &= \int_0^x \int_0^x \int_0^x (x-t)\, u_2\ \mathrm{dtdxdx} = \frac{1}{479001600}x^{12} + \frac{1}{3628800}x^{10}\\
u_3 &= \frac{1}{479001600}t^{12} + \frac{1}{3628800}t^{10}.
\end{aligned}$$

then

$$u_n = u_0 + u_1 + u_2 + u_3.$$

Hence

$$u_n(t) = 1 + \frac{1}{2}t^2 + \frac{1}{24}t^4 + \frac{1}{720}t^6 + \frac{1}{40320}t^8 + \frac{1}{3628800}t^{10} + \frac{1}{479001600}t^{12},$$
$$u_n(x) = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6 + \frac{1}{40320}x^8 + \frac{1}{3628800}x^{10} + \frac{1}{479001600}x^{12}.$$

Comparison of results and curve for Example 1 are represented in Table 2 and Figure 2.

Table 2. Comparison of results for Example 2

X	Exact	NADM
0	1.000000000000000	1.000000000000000
0.1	1.005004168055804	1.005004168055804
0.2	1.020066755619076	1.020066755619076
0.3	1.045338514128861	1.045338514128861
0.4	1.081072371838455	1.081072371838455
0.5	1.127625965206381	1.127625965206381
0.6	1.185465218242268	1.185465218242268
0.7	1.255169005630943	1.255169005630943
0.8	1.337434946304845	1.337434946304845
0.9	1.433086385448775	1.433086385448775
1.0	1.543080634815244	1.543080634815244

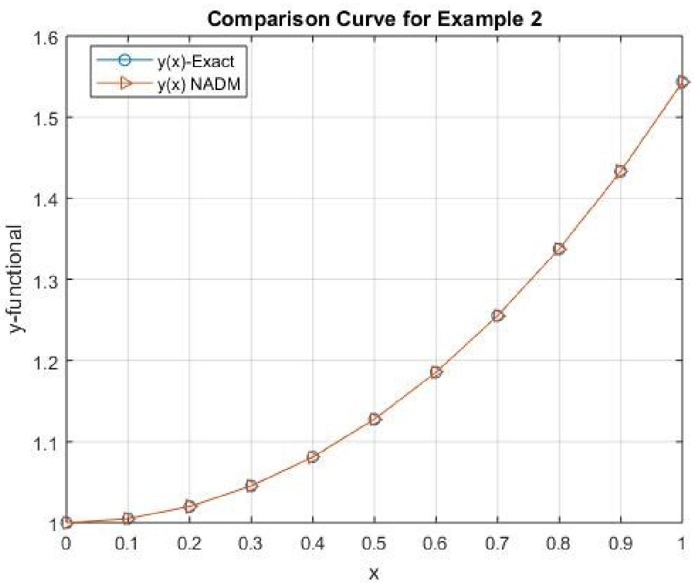

Figure 2. Comparison curve for Example 2

Example 3. Consider the third order linear Volterra Integro-differential Equation:

$$u'''(x) = 1 - \frac{1}{2}x^2 + \int_0^x u(t)\ \mathrm{d}t \tag{13}$$

with initial condition $u(0) = 1,\ u(1) = e + 1, \quad u'(0) = 2,\ u'(1) = e + 1.$

Solution. The analytical solution is $u(x) = e^x + x$. By using Modified Adomian Decomposition Method (MADM):

$$\int_0^x \int_0^x \int_0^x u'''(x)\ \mathrm{dxdxdx} = u(x) - \frac{1}{2}x^2 - 2x - 1,$$
$$\int_0^x \int_0^x \int_0^x 1\ \mathrm{dxdxdx} = \frac{1}{6}x^3,$$
$$\int_0^x \int_0^x \int_0^x \frac{1}{2}x^2\ \mathrm{dxdxdx} = \frac{1}{120}x^5.$$

then

$$u(x) = 1 + 2x + \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{120}x^5 + \int_0^x \int_0^x \int_0^x \int_0^x u(t)\ \mathrm{dtdxdxdx}.$$

Let

$$r = 1 + 2x + \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{120}x^5.$$

Expand taylor (r, x,10):

$$1 + 2x + \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{120}x^5$$

$$a_0 = 1$$
$$u_0 = 1$$
$$h_0 = 2x + \frac{1}{2}x^2$$
$$a_1 = h_0 + \int_0^x \int_0^x \int_0^x \int_0^x u_0\ \mathrm{dtdxdxdx} = 2x + \frac{1}{2}x^2 + \frac{1}{24}x^4$$
$$u_1 = 2t + \frac{1}{2}t^2 + \frac{1}{24}t^4$$
$$h_1 = \frac{1}{6}x^3 - \frac{1}{120}x^5$$
$$a_2 = h_1 + \int_0^x \int_0^x \int_0^x \int_0^x u_1\ \mathrm{dtdxdxdx} = \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{40320}x^8$$
$$u_2 = \frac{1}{6}t^3 + \frac{1}{120}t^5 + \frac{1}{720}t^6 + \frac{1}{40320}t^8$$
$$a_3 = h_2 + \int_0^x \int_0^x \int_0^x \int_0^x u_2\ \mathrm{dtdxdxdx}$$
$$a_3 = \frac{1}{5040}x^7 + \frac{1}{362880}x^9 + \frac{1}{3628800}x^{10} + \frac{1}{479001600}x^{12}$$
$$u_3 = \frac{1}{5040}t^7 + \frac{1}{362880}t^9 + \frac{1}{3628800}t^{10} + \frac{1}{479001600}t^{12}$$

then

$$u_n = u_0 + u_1 + u_2 + u_3.$$

Hence

$$
\begin{aligned}
u_n(t) = {} & 1+2t+\frac{1}{2}t^2+\frac{1}{6}t^3+\frac{1}{24}t^4+\frac{1}{120}t^5+\frac{1}{720}t^6+\frac{1}{5040}t^7+\frac{1}{40320}t^8+ \\
& \frac{1}{362880}t^9+\frac{1}{3628800}t^{10}, \\
u_n(x) = {} & 1+2x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5+\frac{1}{720}x^6+\frac{1}{5040}x^7+\frac{1}{40320}x^8+ \\
& \frac{1}{362880}x^9+\frac{1}{3628800}x^{10}.
\end{aligned}
$$

Comparison of results and curve for Example 1 are represented in Table 3 and Figure 3.

Table 3. Comparison of results for Example 3

X	Exact	NADM
0	1.000000000000000	1.000000000000000
0.1	1.205170918075648	1.205170918075648
0.2	1.421402758160170	1.421402758160170
0.3	1.649858807576003	1.649858807576003
0.4	1.891824697641270	1.891824697641270
0.5	2.148721270700128	2.148721270700128
0.6	2.422118800390509	2.422118800390509
0.7	2.713752707470476	2.713752707470476
0.8	3.025540928492468	3.025540928492468
0.9	3.359603111156950	3.359603111156950
1.0	3.718281828459046	3.718281828459046

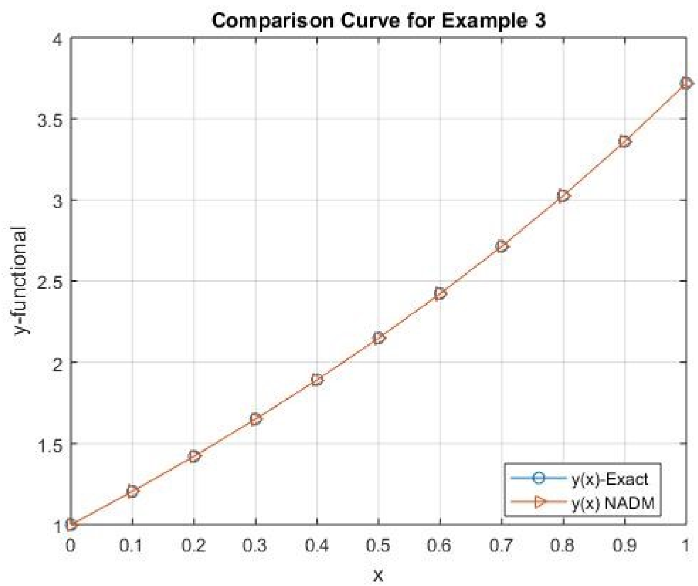

Figure 3. Comparison curve for Example 3

3.2. Discussion

This research work has introduced a new approach to the modification of Adomian Decomposition Method. This new method applies effectively to the solution of Volterra Integro-differential equations. The proposed method converges faster. Also, selection of the Taylor series expansion of the source term needs to be of high order to make the selection of the Taylor series expansion of the source term. The accuracy is also improve through an increase in the selection of terms of the Taylor series expansion. The result obtained compared well with the exact and in most cases they converge directly to the exact in low number of iterations. It is therefore worthy to state that the method is elegant and sufficiently applicable to the solution of Volterra integro-differential equations.

4. Conclusion

This study developed a new method for modifying the Adomian Decomposition Technique. The Volterra Integro-Differential Equation is easily solved using this novel approach. The Taylor series extension of the source word must be chosen with care to broaden the selection as much as feasible.To boost the convergence tendency, we broaden the Taylor series of the source term with additional options. The proposed method converges more quickly to exact than existing methods.

References

[1] J. Manafianheris, "Solving the integro-differential equations usingthe modified laplace adomian decomposition method," *Journal of Mathematical Extension*, vol. 6, no. 1, pp. 1–15, 2012.

[2] B. N. Saray, "An efficient algorithm for solving volterra integro-differential equations based on alpert's multi-wavelets galerkin method," *Journal of Computational and Applied Mathematics*, vol. 348, pp. 453–465, 2019, doi: 10.1016/j.cam.2018.09.016.

[3] M. Olayiwola, A. Gbolagade, and F. Akinpelu, "Numerical solutions of generalized burger's-huxley equation by modified variational iteration method," *Journal of the Nigerian Association of Mathematical Physics*, vol. 17, pp. 433–438, 2010.

[4] M. Olayiwola, F. Akinpelu, and A. Gbolagade, "Modified variational iteration method for the solution of a class of differential equations," *American Journal of Computational and Applied Mathematics*, vol. 2, no. 5, pp. 228–231, 2012, doi: 10.5923/j.ajcam.20120205.05.

[5] A. I. Alaje, M. O. Olayiwola, K. A. Adedokun, J. A. Adedeji, and A. O. Oladapo, "Modified homotopy perturbation method and its application to analytical solitons of fractional-order korteweg–de vries equation," *Beni-Suef University Journal of Basic and Applied Sciences*, vol. 11, no. 1, p. 139, 2022, doi: 10.1186/s43088-022-00317-w.

[6] A. A.-E. Ibrahim, A. A. S. Zaghrout, K. R. Raslan, and K. K. Ali, "On the analytical and numerical study for nonlinear fredholm integro-differential equations," *Applied Mathematics & Information Sciences*, vol. 14, no. 5, pp. 921–929, 2020, doi: 10.18576/amis/140520.

[7] N. Sweilam, "Fourth order integro-differential equations using variational iteration method," *Computers & Mathematics with Applications*, vol. 54, no. 7, pp. 1086–1091, 2007, variational Iteration Method for Nonlinear Problems.

[8] M. Ogunniran, A. Alabi, Q. Amobi, and S. Emmanuel, "Fourth derivative block method for solving two-point singular boundary value problems and related stiff problems," *Turkish Journal of Science*, vol. 6, no. 2, pp. 50 – 60, 2021.

[9] A. Alaje, M. Olayiwola, M. Ogunniran, J. Adedeji, and K. Adedokun, "Approximate analytical methods for the solution of fractional order integro-differential equations," *Nigerian Journal of Mathematics and Applications*, vol. 31, p. 175, 2021.

[10] M. Olayiwola, A. Gbolagade, and A. Adesanya, "Solving variable coefficient fourth-order

parabolic equation by modified initial guess variational iteration method," *Journal of the Nigerian Association of Mathematical Physics*, vol. 16, pp. 205–210, 2010.

[11] ——, "An efficient algorithm for solving the telegraph equation," *Journal of the Nigerian Association of Mathematical Physics*, vol. 16, pp. 199–2014, 2010.

[12] M. Alqarni, R. Amin, K. Shah, S. Nazir, M. Awais, N. A. Alshehri, and E. E. Mahmoud, "Solution of third order linear and nonlinear boundary value problems of integro-differential equations using haar wavelet method," *Results in Physics*, vol. 25, p. 104176, 2021, doi: 10.1016/j.rinp.2021.104176.

[13] J. Chen, M. He, and Y. Huang, "A fast multiscale galerkin method for solving second order linear fredholm integro-differential equation with dirichlet boundary conditions," *Journal of Computational and Applied Mathematics*, vol. 364, p. 112352, 2020, doi: 10.1016/j.cam.2019.112352.

[14] N. Rohaninasab, K. Maleknejad, and R. Ezzati, "Numerical solution of high-order volterra–fredholm integro-differential equations by using legendre collocation method," *Applied Mathematics and Computation*, vol. 328, pp. 171–188, 2018, doi: 10.1016/j.amc.2018.01.032.

[15] S. M. Yassein, "Application of iterative method for solving higher order integro-differential equations," *Ibn AL-Haitham Journal For Pure and Applied Sciences*, vol. 32, no. 2, pp. 51–61, 2019, doi: 10.30526/32.2.2139.

[16] D. Rani and V. Mishra, "Solutions of volterra integral and integro-differential equations using modified laplace adomian decomposition method," *Journal of Applied Mathematics, Statistics and Informatics*, vol. 15, no. 1, pp. 5–18, 2019, doi: 10.2478/jamsi-2019-0001.

[17] A. K. Jimoh, "Integrated trapezoidal collocation method for solving third order integro-differential equations," *International Journal of Scientific & Engineering Research*, vol. 10, no. 7, pp. 1–14, 2019.

[18] M. O. Olayiwola and K. Kareem, "A new decomposition method for integro-differential equations," *Cumhuriyet Science Journal*, vol. 43, no. 2, pp. 283–288, 2022, doi: 10.17776/csj.986019.

[19] M. Olayiwola, A. Gbolagade, and F. Akinpelu, "An efficient algorithm for solving the nonlinear pde," *International Journal of Scientific and Engineering Research*, vol. 2, no. 10, pp. 1–10, 2011.

[20] K. Kareem and M. Olayiwola, "Some results on fourth-order integro-differential equations. annals computer science series," *Anale. Seria Informatică. Annals. Computer Science Series*, vol. 18, no. 1, pp. 183–185, 2020.

[21] R. Amin, I. Mahariq, K. Shah, M. Awais, and F. Elsayed, "Numerical solution of the second order linear and nonlinear integro-differential equations using haar wavelet method," *Arab Journal of Basic and Applied Sciences*, vol. 28, no. 1, pp. 12–20, 2021.