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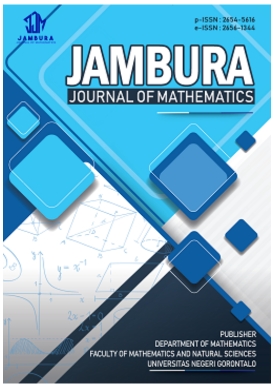
JAMBURA
JOURNAL OF MATHEMATICS

Volume 6, Issue 1, Pages 68–75, February 2024

Received 13 December 2023, Revised 24 January 2024, Accepted 30 January 2024

To Cite this Article : L. P. Sinaga, D. Kartika, and N. A. Farhana,"SEIQR Model Sensitivity and Bifurcation Analysis of SARS-CoV-2 Dynamics with International in-out Mobility Control in Indonesia", *Jambura J. Math*, vol. 6, no. 1, pp. 68–75, 2024, *https://doi.org/10.37905/jjom.v6i1.23426*

JOURNAL INFO • JAMBURA JOURNAL OF MATHEMATICS



Homepage	:	http://ejurnal.ung.ac.id/index.php/jjom/index
Journal Abbreviation	:	Jambura J. Math.
Frequency	:	Biannual (February and August)
Publication Language	:	English (preferable), Indonesia
DOI	:	https://doi.org/10.37905/jjom
Online ISSN	:	2656-1344
Editor-in-Chief	:	Hasan S. Panigoro
Publisher	:	Department of Mathematics, Universitas Negeri Gorontalo
Country	:	Indonesia
OAI Address	:	http://ejurnal.ung.ac.id/index.php/jjom/oai
Google Scholar ID	:	iWLjgaUAAAAJ
Email	:	info.jjom@ung.ac.id

Jambura Journal of Mathematics, Volume 6, Issue 1, Pages 68–75, February 2024 https://doi.org/10.37905/jjom.v6i1.23426

Research Article

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SEIQR Model Sensitivity and Bifurcation Analysis of SARS-CoV-2 Dynamics with International in-out Mobility Control in Indonesia

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ARTICLE HISTORY

Received 13 December 2023
Revised 24 January 2024
Accepted 30 January 2024

KEYWORDS

Stability
Sensitivity
Bifurcation
SEIQR Model
SARS-CoV-2
Covid-19

ABSTRACT. *This study aims to analyze the SEIQR model for the SARS-CoV-2 dynamic by considering in-out mobility. The model construction is based on the COVID-19 response strategy implemented by the Indonesian government, then analyzing the model by determining the equilibrium point and basic reproduction number, analyzing model stability, parameter sensitivity, and bifurcation. The results show that the model has stable disease-free and disease-endemic critical points when the parameter inequality conditions based on the Routh-Hurwitz criteria are satisfied. Numerical simulations show that the system takes a long time to reach equilibrium. Furthermore, the sensitivity analysis of the basic reproduction number shows that the most sensitive parameters are natural birth and death rate susceptible, contact rate of susceptible individuals with infected individuals from local and international subjects, and rate of exposed individuals who have infected. Thus, efforts to handle COVID-19 in Indonesia can be improved by focusing on controlling international in-out mobility, so that the number of exposed individuals who have been infected can be reduced. Moreover, the bifurcation analysis shows that the system undergoes forward or backward bifurcation under disease-free conditions if certain coefficient values are satisfied based on the Castillo-Chavez and Song conditions.*

1. Introduction

In March 2020, the World Health Organization (WHO) declared COVID-19 a global health emergency [1]. COVID-19 (Coronavirus Disease 2019) is a disease that causes acute respiratory system disorders due to infection with the coronavirus or SARS-CoV-2 (Severe Acute Respiratory Syndrome 2). Transmission of this virus occurs very quickly and spreads globally. Indonesia recorded 6,723,546 people confirmed positive for COVID-19 in January 2023 [2]. In dealing with the increasing rate of the spread of COVID-19 in Indonesia, the government has implemented various regulations regarding restricting community activities. Implementation of the Community Activities Restrictions Enforcement or CARE (Indonesian: Pemberlakuan Pembatasan Kegiatan Masyarakat, commonly referred to as the PPKM) is carried out to reduce direct contact between communities and control international travel [3].

Efforts to handle the spread of COVID-19 are also carried out by researchers from various fields. Health researchers analyze the nature and dynamics of SARS-CoV-2. Meanwhile, mathematics researchers construct models to describe predictions about the spread of the virus. Mathematical modeling can help to understand and characterize epidemic outbreaks, predict the spread of viruses, and offer various intervention measures. Analysis of a mathematical model of COVID-19 dynamics by Ndairou et al. [4] formulated a model to explain the spread of COVID-19 in Wuhan and some key aspects related to the pandemic.

Mathematical model analysis of disease spread is performed by observing the stability system at its equilibrium point. Next, sensitivity analysis was carried out to determine the effect of changing parameter values on the model, as in research by Peter et al. [5] analyzing factors that influence the spread and treatment of cholera. Apart from that, changes in parameter values can also lead to bifurcation, changes in the stability value of the system towards the equilibrium point. Research by Huo et al. [6] shows that a system can experience bifurcation when certain conditions are satisfied due to changes in parameter values.

Analysis of the COVID-19 mathematical model was carried out by Annas et al. [7] Resmawan et al. [8] and Zeb et al. [9] with research showing that isolating infected individuals can reduce the risk of spreading COVID-19. A study by Darti et al. [10] proposed a COVID-19 epidemic model with quarantine classes. Analysis of the model and basic reproduction number show that COVID-19 can be controlled by treating infected individuals or quarantining them.

Research by Deeb et al. [11] developed the STEIR model by considering the impact of travel adjusted to data on COVID-19 cases in Lebanon. Researchers estimate possible transmission scenarios related to different levels of implementation of social restrictions and travel inflow. The research results show that strict mitigation levels will slow the spread of the disease, whereas easing international flights will trigger an increase in infection outbreak.

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Homepage : http://ejurnal.ung.ac.id/index.php/jjom/index / E-ISSN : 2656-1344

Another study by Madubueze et al. [12], Resmawan & Yahya [13] and Hussain et al. [14] shows that parameters related to contact between susceptible and infected individuals are the most sensitive parameters for the spread of COVID-19. Other research related to bifurcation analysis in disease spread models was also carried out by Parsamanesh et al. [15] which shows that the system experiences bifurcation due to changes in parameter values and the fulfillment of certain conditions.

This research uses the SEIQR model of the spread of SARS-CoV-2 based on the paper by Sinaga et al. [16]. This study assumes that deaths that occur in the infected and quarantined subpopulations are deaths due to COVID-19 infection, so natural deaths for these two subpopulations are zero. This is consistent with the Indonesian Ministry of Health's definition of death due to COVID-19, which defines a death as a confirmed case of COVID-19 who dies [17]. The researchers plan to analyze the model's stability, parameter sensitivity, and bifurcation, as the high prevalence of COVID-19 infections in Indonesia is considered to require further attention.

2. Model

This research modifies the SEIQR model of the SARS-CoV-2 dynamics. The first step is determining the SEIQR model modification scheme, critical points, and type of stability using the Routh-Hurwitz criteria. The next step is determining the basic reproduction number using the next generation matrix approach. Sensitivity analysis is performed by determining the sensitivity expression and sensitivity index of basic reproduction number (R_0), endemic critical points E, I, and Q. Next, we performed bifurcation analysis on a model that satisfies the Castillo-Chavez and Song conditions when $R_0 = 1$. The final step is to perform a numerical simulation using data on Covid-19 cases in Indonesia published by the Indonesian Ministry of Health.

The model construction consists of five subpopulations, namely, Susceptible (S), Exposed (E), Infected (I), Quarantined (Q), and Recovered (R). The model in research [16] is used by considering the control of international in-out mobility. The modification scheme for the SEIQR model of the SARS-CoV-2 dynamics is shown in Figure 1.

Based on the scheme in Figure 1, a system of differential equations is formulated in eq. (1).

$$
\begin{aligned}
\frac{dS}{dt} &= \Lambda_1 + \delta_3\Lambda_2 - \Lambda_3 - (\alpha_1 + \alpha_2) SI - d_1 S \\
\frac{dE}{dt} &= (\alpha_1 + \alpha_2) SI - (\beta_1 + \beta_2 + \beta_3 + d_2) E \\
\frac{dI}{dt} &= \delta_2\Lambda_2 + \beta_2 E - (\sigma_1 + \sigma_2 + d_3) I \\
\frac{dQ}{dt} &= \delta_1\Lambda_2 + \beta_1 E + \sigma_1 I - (\theta + d_4) Q \\
\frac{dR}{dt} &= \beta_3 E + \sigma_2 I + \theta Q - d_5 R
\end{aligned}
\tag{1}
$$

The definitions of the variables and parameters model in eq. (1) are explained in Table 1 and Table 2.

3. Results and Discussion

The parameters in the model are determined based on the distribution situation in Indonesia and estimated based on data on COVID-19 cases in Indonesia (March 2020-August 2022) published by the Ministry of Health of the Republic of Indonesia [2].

Table 1. Variable definition

Variable	Description
S	The number of healthy individuals who are susceptible to infection
E	The number of individuals who have tested positive for the virus during the incubation period but were unable to infect others
I	The number of individuals who have tested positive for the virus and may infect others
Q	The number of exposed and infected individuals who have been quarantined
R	The number of individuals who have recovered and developed immunity to the virus

3.1. Critical Points of SEIQR Model for the SARS-CoV-2 Dynamics

The critical point $E(S,\ E,\ I,\ Q,\ R)$ of system can be found if it satisfies $\frac{dS}{dt} = \frac{dE}{dt} = \frac{dI}{dt} = \frac{dQ}{dt} = \frac{dR}{dt} = 0$ [18]. By solving the eq. (1) two equilibrium points are obtained as follows:

1. The disease-free critical point (E_0):

$$
\begin{aligned}
E_0 &= (S_0^*, E_0^*, I_0^*, Q_0^*, R_0^*) \\
&= \left(\frac{\Lambda_1 + \delta_3\Lambda_2 - \Lambda_3}{d_1}, 0, 0, \frac{\delta_1\Lambda_2}{\theta + d_4}, \frac{\theta\delta_1\Lambda_2}{d_5(\theta + d_4)} \right)
\end{aligned}
\tag{2}
$$

   where $\Lambda_1 + \delta_3\Lambda_2 - \Lambda_3 \geq 0$.

2. The disease-endemic critical point (E_1):

$$
E_1 = (S_1^*, E_1^*, I_1^*, Q_1^*, R_1^*) \tag{3}
$$

   with

$$
\begin{aligned}
S_1^* &= \frac{(\Lambda_1 + \delta_3\Lambda_2 - \Lambda_3)(\sigma_1 + \sigma_2 + d_3)}{(\alpha_1 + \alpha_2)(\delta_2\Lambda_2 + \beta_2 E_1^*) + d_1(\sigma_1 + \sigma_2 + d_3)} \\
I_1^* &= \frac{\delta_2\Lambda_2 + \beta_2 E_1^*}{\sigma_1 + \sigma_2 + d_3} \\
Q_1^* &= \frac{\delta_1\Lambda_2(\sigma_1 + \sigma_2 + d_3) + \sigma_1\delta_2\Lambda_2}{(\theta + d_4)(\sigma_1 + \sigma_2 + d_3)} + \\
&\quad \frac{\beta_1(\sigma_1 + \sigma_2 + d_3) + \sigma_1\beta_2}{(\theta + d_4)(\sigma_1 + \sigma_2 + d_3)} E_1^* \\
R_1^* &= \frac{\sigma_2\delta_2\Lambda_2(\theta + d_4) + \theta\delta_1\Lambda_2(\sigma_1 + \sigma_2 + d_3) + \theta\sigma_1\delta_2\Lambda_2}{d_5(\theta + d_4)(\sigma_1 + \sigma_2 + d_3)} + \\
&\quad \frac{(\beta_3(\theta + d_4) + \theta\beta_1)(\sigma_1 + \sigma_2 + d_3) + \beta_2(\sigma_2(\theta + d_4)\theta\sigma_1)}{d_5(\theta + d_4)(\sigma_1 + \sigma_2 + d_3)} \\
E_1^* &= \frac{-B \pm \sqrt{D}}{2A}
\end{aligned}
$$

   where $D = B^2 - 4AC > 0$, $-B \pm \sqrt{D} \geq 0$, and

$$
\begin{aligned}
A &= \beta_2(\alpha_1 + \alpha_2)(\beta_1 + \beta_2 + \beta_3 + d_2) \\
B &= (\delta_2\Lambda_2(\alpha_1 + \alpha_2) + d_1(\sigma_1 + \sigma_2 + d_3))(\beta_1 + \beta_2 + \beta_3 + d_1) \\
&\quad - \beta_2(\alpha_1 + \alpha_2)(\Lambda_1 + \delta_3\Lambda_2 - \Lambda_3) \\
C &= -\delta_2\Lambda_2(\alpha_1 + \alpha_2)(\Lambda_1 + \delta_3\Lambda_2 - \Lambda_3)
\end{aligned}
$$

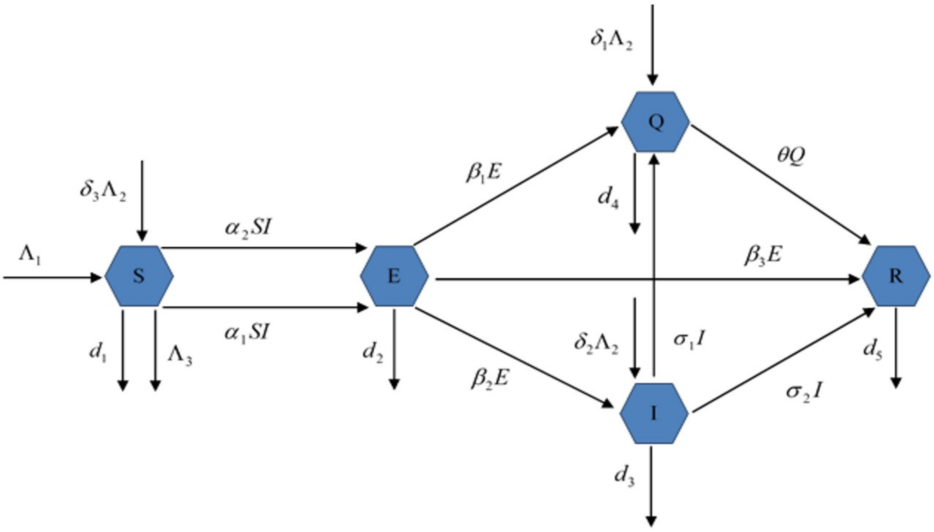

Figure 1. SEIQR model modification scheme

Table 2. Parameter definition

Parameter	Description	Value	
d_1, d_2, d_5	Natural death rate of susceptible, exposed, and recovered	0.00025	Estimated
d_3, d_4	Death rate due to SARS-CoV-2 infection for subpopulation infected and quarantined	0.00025	Estimated
Λ_1	Susceptible addition parameter	0.85836	Estimated
Λ_2	Susceptible addition parameter through in international mobility	0.01575	Estimated
Λ_3	Susceptible reduction parameter through out international mobility	0.01575	Estimated
α_1	Rate of contacts susceptible to infection from local subjects	0.01677	Estimated
α_2	Rate of contacts susceptible to infection from international subjects		Estimated
β_1	Rate of exposed individuals who have quarantined	0.01210	Estimated
β_2	Rate of exposed individuals who have infected	0.05117	Estimated
β_3	Rate of exposed individuals who have recovered	0.01650	Estimated
σ_1	Rate of infected individuals who have quarantined	0.01210	Estimated
σ_2	Rate of infected individuals who have recovered	0.01650	Estimated
θ	Rate of quarantined individuals who have recovered	0.01650	Estimated
δ_1	Proportion of the number of infected individuals from in international mobility and quarantined	0.25	Assumed
δ_2	Proportion of the number of infected individuals from in international mobility who are not quarantined	0.25	Assumed
δ_3	Proportion of increase in population suspect by mobility in international	0.5	Assumed

3.2. Local Stability Analysis of Critical Points

Based on model (1), the Jacobian matrix at the disease-free conditions $J(E_0)$ is represented by eq. (4):

$$J(E_0) = \begin{pmatrix} J_{11} & 0 & J_{13} & 0 & 0 \\ 0 & J_{21} & J_{22} & 0 & 0 \\ 0 & J_{32} & J_{33} & 0 & 0 \\ 0 & J_{42} & J_{43} & J_{44} & 0 \\ 0 & J_{52} & J_{53} & J_{54} & J_{55} \end{pmatrix} \tag{4}$$

where

$$\begin{aligned} J_{11} &= -d_1 \\ J_{13} &= -\frac{(\alpha_1+\alpha_2)(\Lambda_1+\delta_2\Lambda_2-\Lambda_3)}{d_1} \\ J_{21} &= -(\beta_1+\beta_2+\beta_3+d_2) \\ J_{22} &= \frac{(\alpha_1+\alpha_2)(\Lambda_1+\delta_2\Lambda_2-\Lambda_3)}{d_1} \\ J_{32} &= \beta_2 \quad J_{33} = -(\sigma_1+\sigma_2+d_3) \\ J_{42} &= \beta_1 \quad J_{43} = \sigma_1 \quad J_{44} = -(\theta+d_4) \\ J_{52} &= \beta_3 \quad J_{53} = \sigma_2 \quad J_{54} = \theta \quad J_{55} = -d_5 \end{aligned}$$

By solving $|\lambda I - J(E_0)| = 0$, the characteristic equation for disease-free conditions is obtained:

$$(\lambda+d_1)(\lambda+d_5)(\lambda+\theta+d_4)\,r(\lambda) = 0 \tag{5}$$

with $r(\lambda) = a_2\lambda^2 + a_1\lambda + a_0$, where

$$\begin{aligned} a_2 &= 1, \\ a_1 &= \beta_1+\beta_2+\beta_3+\sigma_1+\sigma_2+d_2+d_3, \\ a_0 &= (\beta_1+\beta_2+\beta_3+d_2)(\sigma_1+\sigma_2+d_3) - \\ &\quad \frac{\beta_2}{d_1}(\alpha_1+\alpha_2)(\Lambda_1+\delta_3\Lambda_2-\Lambda_3) \end{aligned}$$

Using the Routh-Hurwitz criteria [18], the disease-free critical point is locally asymptotically stable if it was satisfied $d_1(\beta_1+\beta_2+\beta_3+d_2)(\sigma_1+\sigma_2+d_3) > \beta_2(\alpha_1+\alpha_2)(\Lambda_1+\delta_3\Lambda_2-\Lambda_3)$.

The Jacobian matrix at the disease-endemic conditions $J(E_1)$ is represented by eq. (6).

$$J(E_1) = \begin{pmatrix} J_{11} & 0 & J_{13} & 0 & 0 \\ J_{21} & J_{22} & J_{23} & 0 & 0 \\ 0 & J_{32} & J_{33} & 0 & 0 \\ 0 & J_{42} & J_{43} & J_{44} & 0 \\ 0 & J_{52} & J_{53} & J_{54} & J_{55} \end{pmatrix} \tag{6}$$

where

$$\begin{aligned}
J_{11} &= -\frac{(\alpha_1+\alpha_2)(\delta_2\Lambda_2+\beta_2E_1^*)}{\sigma_1+\sigma_2+d_3} - d_1\\
J_{13} &= -\frac{(\alpha_1+\alpha_2)(\Lambda_1+\delta_3\Lambda_2-\Lambda_3)(\sigma_1+\sigma_2+d_3)}{(\alpha_1+\alpha_2)(\delta_2\Lambda_2+\beta_2E_1^*)+d_1(\sigma_1+\sigma_2+d_3)}\\
J_{21} &= \frac{(\alpha_1+\alpha_2)(\delta_2\Lambda_2+\beta_2E_1^*)}{\sigma_1+\sigma_2+d_3}\\
J_{22} &= -\beta_1-\beta_2-\beta_3-d_2\\
J_{23} &= \frac{(\alpha_1+\alpha_2)(\Lambda_1+\delta_3\Lambda_2-\Lambda_3)(\sigma_1+\sigma_2+d_3)}{(\alpha_1+\alpha_2)(\delta_2\Lambda_2+\beta_2E_1^*)+d_1(\sigma_1+\sigma_2+d_3)}\\
J_{32} &= \beta_2 \quad J_{33} = -(\sigma_1+\sigma_2+d_3)\\
J_{42} &= \beta_1 \quad J_{43} = \sigma_1 \quad J_{44} = -(\theta+d_4)\\
J_{52} &= \beta_3 \quad J_{53} = \sigma_2 \quad J_{54} = \theta \quad J_{55} = -d_5.
\end{aligned}$$

By solving $|\lambda I - J(E_1)| = 0$, the characteristic equation for disease-endemic conditions is obtained:

$$(\lambda+d_5)(\lambda+\theta+d_4)\,s(\lambda) = 0 \tag{7}$$

with $s(\lambda) = b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0$, where

$$\begin{aligned}
b_3 &= 1,\\
b_2 &= \beta_1+\beta_2+\beta_3+\sigma_1+\sigma_2+d_1+d_2+d_3\\
&\quad + \frac{(\alpha_1+\alpha_2)(\delta_2\Lambda_2+\beta_2E_1^*)}{\sigma_1+\sigma_2+d_3},\\
b_1 &= (\beta_1+\beta_2+\beta_3+d_2)(\sigma_1+\sigma_2+d_3)\\
&\quad + \frac{(\alpha_1+\alpha_2)(\delta_2\Lambda_2+\beta_2E_1^*)}{\sigma_1+\sigma_2+d_3} + d_1\\
&\quad - \frac{\beta_2(\alpha_1+\alpha_2)(\Lambda_1+\delta_3\Lambda_2-\Lambda_3)(\sigma_1+\sigma_2+d_3)}{(\alpha_1+\alpha_2)(\delta_2\Lambda_2+\beta_2E_1^*)+d_1(\sigma_1+\sigma_2+d_3)},\\
b_0 &= (\alpha_1+\alpha_2)(\delta_2\Lambda_2+\beta_2E^*)(\beta_1+\beta_2+\beta_3+d_2)\\
&\quad + d_1(\beta_1+\beta_2+\beta_3+d_2)(\sigma_1+\sigma_2+d_3)\\
&\quad - \frac{d_1\beta_2(\alpha_1+\alpha_2)(\Lambda_1+\delta_3\Lambda_2-\Lambda_3)(\sigma_1+\sigma_2+d_3)}{(\alpha_1+\alpha_2)(\delta_2\Lambda_2+\beta_2E_1^*)+d_1(\sigma_1+\sigma_2+d_3)}.
\end{aligned}$$

Using the Routh-Hurwitz criteria [19], the disease-endemic critical point is locally stable if it was satisfied $b_1 > 0$, $b_2 > 0$, and $b_1b_2 > b_3b_0$.

The initial value of stability simulation of model is taken from the first data, $S(0) = 6777$, $E(0) = 0$, $I(0) = 1528$, $Q(0) = 1311$, $R(0) = 81$, and parameter values in Table 2. The number of individuals uses a scale of 1:1000, as shown by Figure 2.

The graph in Figure 2 shows that the number of individuals in subpopulation *S* will increase until the middle of the 11^{th} month, then will continue to decrease due to the infection process. The number of individuals in subpopulation *E* also increased until the end of the 35^{th} month and then decreased due to movements in subpopulation *I*. The number of individuals in subpopulation *I* will continue to increase and then decrease over a long period. The transmission rate continues to increase because of the insignificant intervention efforts, so the transmission process in March 2020-August 2022 is very high. The decrease in infection cases could occur due to an increase in *Q* and *R* subpopulations due to the quarantine and treatment efforts carried out.

3.3. Basic Reproduction Number

The basic reproduction number (R_0) is the estimated number of infections per unit of time determined using the next-generation matrix approach. The determination of R_0 is based on infective subpopulations *E*, *I*, and *Q* [20].

Let φ_i and ψ_i represent the rate of increase in infection and disease transfer, death, and recovery from the *i*-th compartment respectively. Based on the infective subpopulations *E*, *I*, and *Q*, the vectors φ and ψ are obtained as follows:

$$\begin{aligned}
\varphi &= \begin{pmatrix} (\alpha_1+\alpha_2)SI \\ 0 \\ 0 \end{pmatrix},\\
\psi &= \begin{pmatrix} (\beta_1+\beta_2+\beta_3+d_2)E \\ (\sigma_1+\sigma_2+d_3)I - (\delta_2\Lambda_2+\beta_2E) \\ (\theta+d_4)Q - (\delta_1\Lambda_2+\beta_2E+\sigma_1 I) \end{pmatrix}.
\end{aligned} \tag{8}$$

Next, the linearization of vectors φ and ψ produces vectors F and V:

$$\begin{aligned}
F &= \begin{pmatrix} 0 & (\alpha_1+\alpha_2)S & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\\
V &= \begin{pmatrix} \beta_1+\beta_2+\beta_3+d_2 & 0 & 0 \\ -\beta_2 & \sigma_1+\sigma_2+d_3 & 0 \\ -\beta_2 & -\sigma_1 & \theta+d_4 \end{pmatrix}.
\end{aligned} \tag{9}$$

By solving $K = FV^{-1}$ and substituting the disease-free critical point (E_0), the next-generation matrix is obtained as follows:

$$K = \begin{pmatrix} k_{11} & k_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{10}$$

where

$$\begin{aligned}
k_{11} &= -\frac{\beta_2(\alpha_1+\alpha_2)(\Lambda_1+\delta_3\Lambda_2-\Lambda_3)}{d_1}\\
k_{12} &= \frac{(\alpha_1+\alpha_2)(\sigma_1+\sigma_2+d_3)(\Lambda_1+\delta_3\Lambda_2-\Lambda_3)}{d_1}
\end{aligned}$$

The R_0 value is the largest eigenvalue of the matrix K [19]. The eigenvalue of K is acquired by satisfying the equation $|\lambda I - K| = 0$. Thus, R_0 is obtained as follows:

$$R_0 = \frac{\beta_2(\alpha_1+\alpha_2)(\Lambda_1+\delta_3\Lambda_2-\Lambda_3)}{d_1}. \tag{11}$$

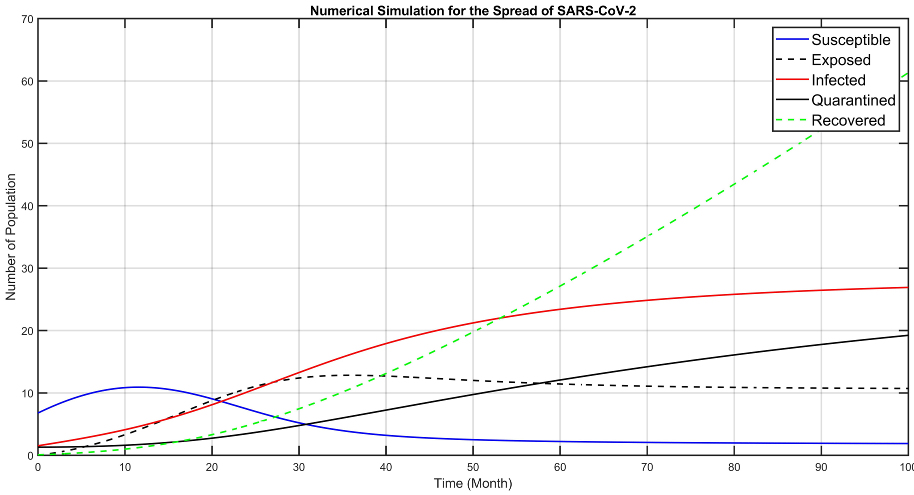

Figure 2. Stability of SEIQR model for the spread of SARS-CoV-2

3.4. *Sensitivity Analysis*

The sensitivity expression R_0 obtained by satisfying the following equation [21]:

$$C_p^{R_0} = \frac{\partial R_0}{\partial p} \times \frac{p}{R_0} \tag{12}$$

where p is the parameter tested for its sensitivity to R_0. Next, the sensitivity index R_0 calculated by substituting the parameter values into the sensitivity expression $C_p^{R_0}$. The sensitivity index shown in Table 3.

Table 3. Sensitivity indices of the parameters of the basic reproduction number

Parameter	Baseline Value	Sensitivity Indices
d_1	0.00025	-1
d_2	0.00025	0
d_3	0.00025	0
d_4	0.00025	0
d_5	0.00025	0
Λ_1	0.85836	1.009259423
Λ_2	0.01575	0.009259423
Λ_3	0.01575	-0.0018518845
$\alpha_1 + \alpha_2$	0.01677	1
β_1	0.01210	0
β_2	0.05117	1
β_3	0.01650	0
σ_1	0.01210	0
σ_2	0.01650	0
θ	0.01650	0
δ_1	0.25	0
δ_2	0.25	0
δ_3	0.5	0.009259423

Based on Table 3, it is obtained that d_1, Λ_1, $\alpha_1 + \alpha_2$, and β_2 are the most sensitive parameters to changes in the R_0 value.

We intended to simulate different values of the selected parameters that reduce the basic reproduction number by 0%, 5%, 15% and 20%. The next section shows the effect of various values of the most sensitive parameter on the number of exposed cases at the peak.

Graph (a) in Figure 3 shows the relationship between parameter d_1 and the exposed cases. Parameter d_1 has a negative relationship with the number of exposed cases but did not significantly change the number of exposed cases. The value $d_1 = 0.0003125$ (which can reduce R_0 by 20%) has the most influence in reducing the number of exposed cases.

Graph (b) in Figure 3 shows the relationship between parameter Λ_1 and the exposed cases. Parameter Λ_1 has a positive relationship with the exposed cases as shown in the graph. The value $\Lambda_1 = 0.6882330$ (which can reduce R_0 by 20%) has the most influence in reducing the number of exposed cases.

Graph (c) in Figure 3 shows the relationship between parameters $\alpha_1 + \alpha_2$ and the exposed cases. From the start of the simulation until the 34^{th} month, the value $\alpha_1 + \alpha_2 = 0.0134160$ (which can reduce R_0 by 20%) has the most influence in reducing the number of exposed cases. However, at the end of the simulation, the value $\alpha_1 + \alpha_2 = 0.01677$ (which does not reduce R_0) shows the best effect in reducing the number of exposed cases.

Graph (d) in Figure 3 shows the relationship between parameter β_2 and the exposed cases. The value of $\beta_2 = 0.05177$ (which does not reduce R_0) increase the number of exposed cases significantly. Meanwhile, the value of β_2 which can reduce R_0 by 5%, 15%, and 20% decreases the number of exposed cases significantly.

3.5. *Bifurcation Analysis*

The bifurcation analysis is used to see changes in the stability orbit towards a critical point which is influenced by changes in certain parameter values. The bifurcation of the SEIQR model is using the well-known Castillo-Song bifurcation theorem at $R_0 = 1$ [22]. Let eq. (1) be formulated as follows, where $x_1,\ x_2,\ x_3,\ x_4,\ x_5$ respectively being $S,\ E,\ I,\ Q,\ R$:

$$\begin{aligned} g_1 &= \frac{dS}{dt} = \Lambda_1 + \delta_3\Lambda_2 - \Lambda_3 - (\alpha_1 + \alpha_2)\, SI - d_1 x_1 \\ g_2 &= \frac{dE}{dt} = (\alpha_1 + \alpha_2)\, x_1 x_3 - (\beta_1 + \beta_2 + \beta_3 + d_2)\, x_2 \\ g_3 &= \frac{dI}{dt} = \delta_2\Lambda_2 + \beta_2 x_2 - (\sigma_1 + \sigma_2 + d_3)\, x_3 \end{aligned} \tag{13}$$

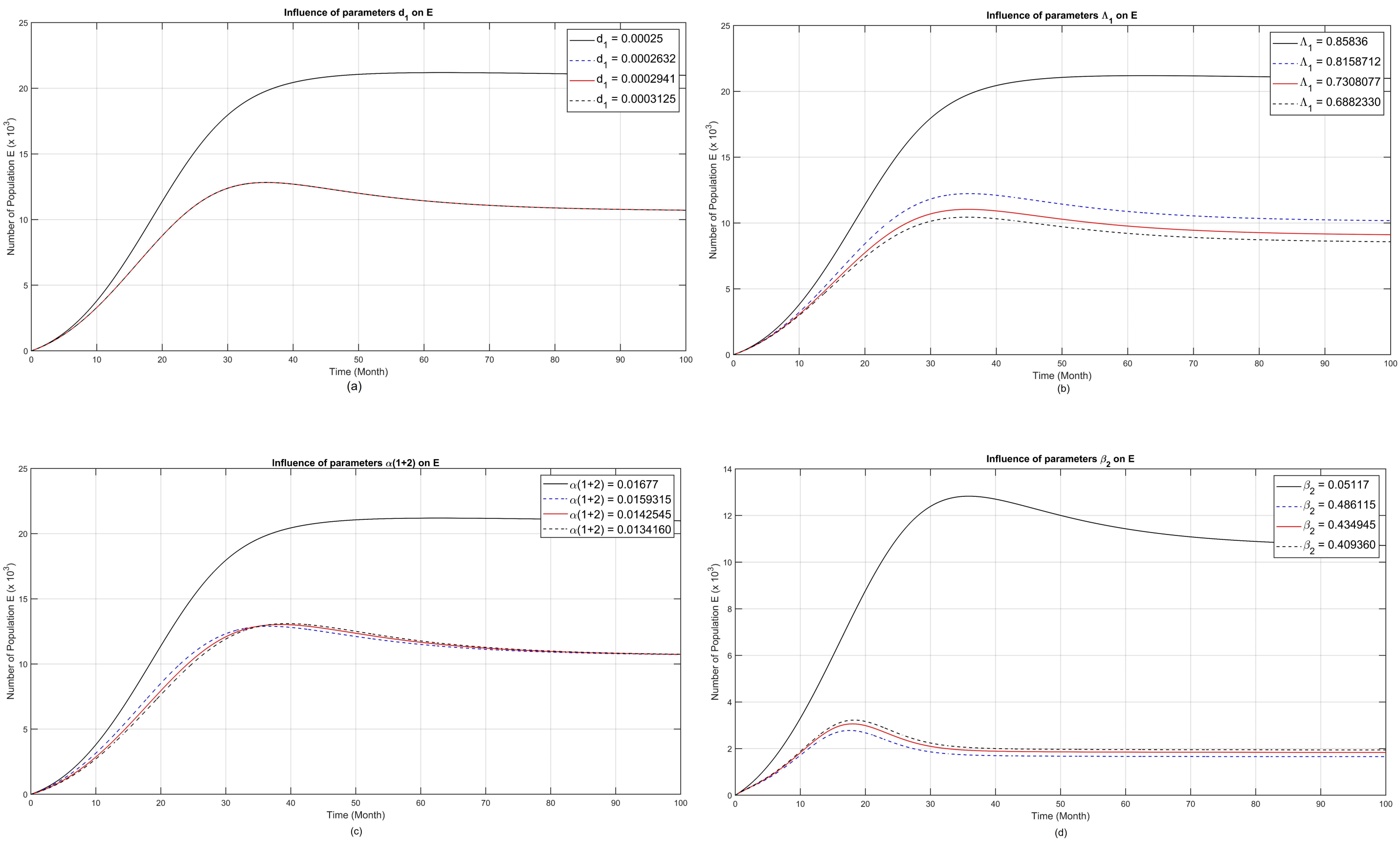

Figure 3. The number of exposed cases for various values of the most sensitive parameter

$$g_4 = \frac{dQ}{dt} = \delta_1 \Lambda_2 + \beta_1 x_2 + \sigma_1 x_3 - (\theta + d_4) x_4$$
$$g_5 = \frac{dR}{dt} = \beta_3 x_2 + \sigma_2 x_3 + \theta x_4 - d_5 x_5$$

The parameter $(\alpha_1 + \alpha_2)$ is chosen as the bifurcation parameter when $R_0 = 1$, so it is obtained:

$$(\alpha_1 + \alpha_2) = (\alpha_1 + \alpha_2)^* = \frac{d_1}{\beta_2 (\Lambda_1 + \delta_3 \Lambda_2 - \Lambda_3)}. \quad (14)$$

Next, using the Center-Manifold theorem, the existence of zero eigenvalues when $R_0 = 1$ and E_0 is analyzed [23]. The linearization of eq. (13) is obtained as follows:

$$J(E_0, (\alpha_1 + \alpha_2)) = \begin{pmatrix} J_{11} & 0 & J_{13} & 0 & 0 \\ 0 & J_{22} & J_{23} & 0 & 0 \\ 0 & J_{32} & J_{33} & 0 & 0 \\ 0 & J_{42} & J_{43} & J_{44} & 0 \\ 0 & J_{52} & J_{53} & J_{54} & J_{55} \end{pmatrix} \quad (15)$$

where

$$J_{11} = -d_1$$
$$J_{13} = -\frac{(\alpha_1 + \alpha_2)(\Lambda_1 + \delta_3 \Lambda_2 - \Lambda_3)}{d_1}$$
$$J_{22} = -(\beta_1 + \beta_2 + \beta_3 + d_2)$$
$$J_{23} = \frac{(\alpha_1 + \alpha_2)(\Lambda_1 + \delta_3 \Lambda_2 - \Lambda_3)}{d_1}$$
$$J_{32} = \beta_2 \quad J_{33} = -(\sigma_1 + \sigma_2 + d_3)$$
$$J_{42} = \beta_1 \quad J_{43} = \sigma_1 \quad J_{44} = -(\theta + d_4)$$
$$J_{52} = \beta_3 \quad J_{53} = \sigma_2 \quad J_{54} = \theta \quad J_{55} = -d_5.$$

By solving equation $|\lambda I - J(E_0, (\alpha_1 + \alpha_2))| = 0$, five eigen values are obtained:

$$\lambda_1 = 0,$$
$$\lambda_2 = -d_1,$$
$$\lambda_3 = -d_5,$$
$$\lambda_4 = -(\theta + d_4),$$
$$\lambda_5 = -(\beta_1 + \beta_2 + \beta_3 + \sigma_1 + \sigma_2 + d_2 + d_3).$$

Because there is a zero eigenvalue and the other eigenvalues are negative, the right and left eigenvectors respected to the zero eigenvalue will be calculated.

The right eigenvectors $w = (w_1, w_2, w_3, w_4, w_5)^T$ of the zero eigenvalue is obtained by satisfying condition $J(E_0, (\alpha_1 + \alpha_2)) \bullet w = 0$, so that:

$$w_1 = \frac{\beta_2 (\alpha_1 + \alpha_2) x_1}{d_1 (\beta_2 + \sigma_1 + \sigma_2 + d_3)} w_5,$$
$$w_2 = \frac{\beta_2 (\alpha_1 + \alpha_2) x_1}{(\beta_1 + \beta_2 + \beta_3 + d_2)(\beta_2 + \sigma_1 + \sigma_2 + d_3)} w_5,$$
$$w_3 = -\frac{\beta_2}{(\beta_2 + \sigma_1 + \sigma_2 + d_3)} w_5,$$
$$w_4 = 0,$$
$$w_5 = w_5.$$

The left eigenvectors $v = (v_1, v_2, v_3, v_4, v_5)$ of the zero eigenvalue is obtained by satisfying condition $v \bullet J\left(E_0, (\alpha_1 + \alpha_2)\right) = 0$, such that:

$$\begin{aligned}
v_1 &= 0,\\
v_2 &= \frac{\beta_3 - \beta_2\left((\alpha_1+\alpha_2)x_1 - \sigma_2\right)}{(\beta_1+\beta_2+\beta_3+d_2)\left((\alpha_1+\alpha_2)x_1 + (\sigma_1+\sigma_2+d_3)\right)}v_5,\\
v_3 &= -\frac{(\alpha_1+\alpha_2)x_1 - \sigma_2}{(\alpha_1+\alpha_2)x_1 + (\sigma_1+\sigma_2+d_3)}v_5,\\
v_4 &= 0,\\
v_5 &= v_5.
\end{aligned}$$

The values of v_5 and w_5 are obtained by satisfying the equation $v \bullet w = 1$, such that:

$$\begin{aligned}
v_5 &= (\beta_1+\beta_2+\beta_3+d_2)^2(\beta_2+\sigma_1+\sigma_2+d_3)\\
&\quad\left((\alpha_1+\alpha_2)x_1 + (\sigma_1+\sigma_2+d_3)\right)\\
w_5 &= \frac{1}{p_1+p_2+p_3}
\end{aligned}$$

where

$$\begin{aligned}
p_1 &= \beta_2(\alpha_1+\alpha_2)x_1\left(\beta_3 - \beta_2\left((\alpha_1+\alpha_2)x_1 - \sigma_2\right)\right)\\
p_2 &= \beta_2(\beta_1+\beta_2+\beta_3+d_2)^2\left((\alpha_1+\alpha_2)x_1 - \sigma_2\right)\\
p_3 &= (\beta_1+\beta_2+\beta_3+d_2)^2(\beta_2+\sigma_1+\sigma_2+d_3)\\
&\quad\left((\alpha_1+\alpha_2)x_1 + (\sigma_1+\sigma_2+d_3)\right).
\end{aligned}$$

Because of the parameter is positive and $x_1 = S_0 = \frac{\Lambda_1+\delta_3\Lambda_2-\Lambda_3}{d_1} > 0$, then $v_5 > 0$ and

1. $w_5 < 0$ if $(\alpha_1+\alpha_2)x_1 - \sigma_2 < 0$ and

$$\begin{aligned}
&\beta_2(\beta_1+\beta_2+\beta_3+d_2)^2\\
&\quad\left((\alpha_1+\alpha_2)x_1 - \sigma_2\right) > \beta_2(\alpha_1+\alpha_2)\\
&\qquad\left(\beta_3 - \beta_2\left((\alpha_1+\alpha_2)x_1 - \sigma_2\right)\right)x_1\\
&\qquad + (\beta_1+\beta_2+\beta_3+d_2)^2\\
&\qquad(\beta_2+\sigma_1+\sigma_2+d_3)\\
&\qquad\left((\alpha_1+\alpha_2)x_1 + (\sigma_1+\sigma_2+d_3)\right)
\end{aligned}$$

2. $w_5 > 0$ if $(\alpha_1+\alpha_2)x_1 - \sigma_2 > 0$ and

$$\begin{aligned}
&\beta_2(\beta_1+\beta_2+\beta_3+d_2)^2\\
&\quad\left((\alpha_1+\alpha_2)x_1 - \sigma_2\right) < \beta_2(\alpha_1+\alpha_2)\\
&\qquad\left(\beta_3 - \beta_2\left((\alpha_1+\alpha_2)x_1 - \sigma_2\right)\right)x_1\\
&\qquad + (\beta_1+\beta_2+\beta_3+d_2)^2\\
&\qquad(\beta_2+\sigma_1+\sigma_2+d_3)\\
&\qquad\left((\alpha_1+\alpha_2)x_1 + (\sigma_1+\sigma_2+d_3)\right).
\end{aligned}$$

Next, calculate the second-order partial derivatives for g_1, g_2, g_3, g_4, g_5 with respect to $x_1, x_2,\ x_3,\ x_4,\ x_5,\ (\alpha_1+\alpha_2)$, it is found that all second-order partial derivatives have the value 0, except:

$$\begin{aligned}
\frac{\partial^2 g_1}{\partial x_1 \partial x_3} = \frac{\partial^2 g_1}{\partial x_3 \partial x_1} &= -(\alpha_1+\alpha_2);\\
\frac{\partial^2 g_2}{\partial x_1 \partial x_3} = \frac{\partial^2 g_2}{\partial x_3 \partial x_1} &= \alpha_1+\alpha_2;\\
\frac{\partial^2 g_1}{\partial x_1 \partial(\alpha_1+\alpha_2)} &= -x_3;\\
\frac{\partial^2 g_1}{\partial x_3 \partial(\alpha_1+\alpha_2)} &= -x_1;\\
\frac{\partial^2 g_2}{\partial x_1 \partial(\alpha_1+\alpha_2)} &= x_3;\\
\frac{\partial^2 g_2}{\partial x_3 \partial(\alpha_1+\alpha_2)} &= x_1.
\end{aligned}$$

The bifurcation coefficients A and B are obtained by satisfying the Castillo-Song bifurcation theorem, so that:

$$\begin{aligned}
A &= \sum_{k,i,j=1}^{5} v_k w_i w_j \frac{\partial^2 g_k}{\partial x_i \partial x_j}\left(E_0, (\alpha_1+\alpha_2)^*\right)\\
&= -v_5 w_5^2 \frac{2\beta_2^2(\alpha_1+\alpha_2)^2\left(\beta_3 - \beta_2\left((\alpha_1+\alpha_2)x_1 - \sigma_2\right)\right)x_1}{d_1 {q_1}^2 q_2\left((\alpha_1+\alpha_2)x_1 + q_3\right)}
\end{aligned}$$

and

$$\begin{aligned}
B &= \sum_{k,i=1}^{5} v_k w_i \frac{\partial^2 g_k}{\partial x_i \partial(\alpha_1+\alpha_2)}\left(E_0, (\alpha_1+\alpha_2)^*\right)\\
&= -v_5 w_5 \frac{\beta_2\left(\beta_3 - \beta_2\left((\alpha_1+\alpha_2)x_1 - \sigma_2\right)\right)x_1}{q_1 q_2\left((\alpha_1+\alpha_2)x_1 + q_3\right)}
\end{aligned}$$

where

$$\begin{aligned}
q_1 &= \beta_2+\sigma_1+\sigma_2+d_3\\
q_2 &= \beta_1+\beta_2+\beta_3+d_2\\
q_3 &= \sigma_1+\sigma_2+d_3.
\end{aligned}$$

The parameter value is positive and $x_1 = S_0 = \frac{\Lambda_1+\delta_3\Lambda_2-\Lambda_3}{d_1} > 0$, then:

1. $A < 0$ if $\beta_3 - \beta_2\left((\alpha_1+\alpha_2)x_1 - \sigma_2\right) > 0$ and $w_5 < 0$
   $A > 0$ if $\beta_3 - \beta_2\left((\alpha_1+\alpha_2)x_1 - \sigma_2\right) < 0$ and $w_5 < 0$
2. B will always be positive
   If $\beta_3 - \beta_2\left((\alpha_1+\alpha_2)x_1 - \sigma_2\right) > 0$ and $w_5 < 0$, then $B > 0$
   If $\beta_3 - \beta_2\left((\alpha_1+\alpha_2)x_1 - \sigma_2\right) < 0$ and $w_5 < 0$, then $B > 0$

Based on the above discussion, we can see that under the conditions $R_0 = 1$ and $S_0 > 0$, the system experiences a forward bifurcation when $A < 0$ and $B > 0$. Under the same conditions, the system experiences backward bifurcation when $A > 0$ and $B > 0$.

4. Conclusion

The SEIQR model for the SARS-CoV-2 dynamics has disease-free and disease-endemic critical points that are stable when the inequality condition of parameter values based on the Routh-Hurwitz criteria was satisfied. Numerical simulations of stability analysis show that the system will be stable for a long time.

Furthermore, sensitivity analysis of the basic reproduction number shows that natural birth and death rate S (susceptible), the contact rate of susceptible individuals with infected individuals from local and international subjects, and the rate of exposed individuals who infected are the most sensitive parameters. Thus, based on the results of the sensitivity analysis, efforts to handle COVID-19 in Indonesia can be improved by focusing on controlling international in-out mobility, so that the number of exposed individuals who have been infected can be reduced. In addition, bifurcation analysis shows that the system has forward or backward bifurcation by satisfying certain coefficient values based on the conditions stated by Castillo-Chavez and Song when $R_0 = 1$. For further research development, numerical simulations of bifurcation analysis in the model can be demonstrated so that further analysis can be carried out.

Author Contributions. **Lasker P. Sinaga:** Conceptualization, methodology, formal analysis, software, investigation, resources, writing—original draft preparation, and visualization. **Dinda Kartika:** Conceptualization, validation, writing—review and editing, and supervision. **Nurul A. Farhana:** Conceptualization, validation, writing—review and editing, supervision, and project administration. All authors have read and agreed to the published version of the manuscript.

Acknowledgement. The authors express their gratitude to the editor and reviewers for their meticulous reading, insightful critiques, and practical recommendations, all of which have greatly enhanced the quality of this work.

Funding. This research has been supported by "Lembaga Penelitian dan Pengabdian kepada Masyarakat (LPPM)," Universitas Negeri Medan fiscal year 2023.

Conflict of interest. The authors declare no conflict of interest related to this article.

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