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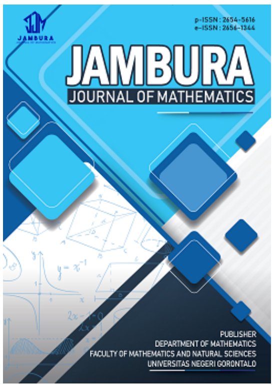

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Research Article

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Estimating Reliability for Frechet (3+1) Cascade Model

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ABSTRACT. *In this paper, the mathematical formula of the reliability function of a special (3+1) cascade model is found, where this model can work with three active components if it can cope with the stresses to which it is subjected, while the cascade 3+1 model consists of four components, where the components ($\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$) are the basic components in the model and ($\mathcal{A}_4$)is a spare component in an active state of readiness, so when any of the three core components ($\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$) fails to cope with the stress they are under working, they are replaced by the standby component ($\mathcal{A}_4$) readiness to keep the model running, it was assumed that the random variables of stress and strength follow one of the statistical life distributions, which is the Frechet distribution. The unknown parameters of the Ferchet distribution were estimated by three different estimation methods (maximum likelihood, least square, and regression), and then the reliability function of the model was estimated by these different methods. A Monte Carlo simulation was performed using MATLAB software to compare the results of different estimation methods and find out which methods are the best for estimating the reliability of the model using two statistical criteria: MSE and MAPE and using different sample sizes. After completing the comparison of the simulation results, it was found that the maximum likelihood estimator is the best for estimating the reliability function of the model among the three different estimation methods.*

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1. Introduction

At the beginning of the twentieth century, interest in industrial machines increased and developed rapidly. Their complexity increased and it became impossible to do without them, so attention was increased to to the reliability of these machines and industrial models to avoid their downtime and loss of time [1–3]. Reliability is simply defined as the working time of the component and can be found using the function $R = pr\,(X < Y)$ [4, 5], where X stands for the random variable of strength and Y stands for the random variable of stress [6, 7], where the strength of the component resists stress and if the stress is greater than the strength the component stops working [8–10]. One of the best ways to keep the model working is to use cascade models, which are considered a special type of standby redundancy system, cascade models are a hierarchical standby redundancy where the failed unit of the model can be replaced by an active standby component according to the hierarchical order of the model and thus the model continues to work and reliability increases [11, 12].

There are many papers that have dealt with this topic, such as: Khan and Jan [13] derived the reliability of a multi-component system when the factors follow the Burr distribution. Khaleel [9] found the reliability of a model of one component that has strength and is subjected to four stresses. Khaleel and Karam [12, 14] derived the reliability function of the cascade model for two primary components and a backup component. Doloi and Gogoi [15] Find the reliability function of a model consisting of n-components.

The cascade model in the form 3+1 consists of the components($\mathcal{A}_1, \mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$) where the components ($\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$) are the basic units and necessary for the model to work, while component ($\mathcal{A}_4$) is a redundancy component in an active-standby state and is activated in the event of failure of one of the basic components ($\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$) and replaces the failed component, so the model continues to work and the stress on the component t ($\mathcal{A}_4$) increases by (k) and its strength by (m), where (k and m) are attenuation factors.

This paper aims to find the mathematical formula for the reliability function of the cascade special model 3+1, assuming that random variables follow the Frechet distribution and estimate the reliability function of the model using the estimation methods ML, LS, Rg. The results will be compared using the MSE and the MAPE to find the best estimator.

2. Model Formulation

To find the mathematical formula for the Cascade Model 3+1, which contains three basic components and a standby redundant component, four cases must be found for the model to work. Suppose that the random variables of strength and stress are independent and identically and the random variable $X_\imath : \imath = 1,2,3,4$ represents the strength and the random variable $Y_j : j = 1,2,3,4$ represents the stress of the Frechet distribution with shape parameter known o and the unknown scale parameters μ and δ. The pdf and CDF functions for Frechet distribution where $X \sim Fr(o,\mu)$ and $Y \sim Fr\,(o,\ \delta)$ are:

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$$f(x) = o\mu x^{-(o+1)}e^{-\mu x^{-o}}\ ,\ x>0;o>0;\mu>0 \quad (1)$$
$$F(x) = e^{-\mu x^{-o}}\ ,\ x>0;o>0;\mu>0 \quad (2)$$
$$g(x) = o\delta y^{-(o+1)}e^{-\delta y^{-o}}\ ,\ x>0;o>0;\delta>0 \quad (3)$$
$$G(x) = e^{-\delta y^{-o}}\ ,\ x>0;o>0;\delta>0 \quad (4)$$

Cases of reliability of a model consisting of four cases can be written as follows:

1. The first case: when the three basic components ($\mathcal{A}_1, \mathcal{A}_2$ *and* $\mathcal{A}_3$) are in an active state, the mathematical formula is:

$$S_1 = p(x_1 \geq y_1, x_2 \geq y_2, x_3 \geq y_3)$$

2. The second case: when $(\mathcal{A}_1)$ fails and is replaced by component $(\mathcal{A}_4)$ while components $(\mathcal{A}_2$ *and* $\mathcal{A}_3$) remain active, so the mathematical formula is:

$$S_2 = p(x_1 < y_1, x_2 \geq y_2, x_3 \geq y_3, x_4 \geq y_4)$$

3. The third case: when $(\mathcal{A}_2)$ fails and is replaced by component $(\mathcal{A}_4)$ while components $(\mathcal{A}_1$ *and* $\mathcal{A}_3$) remain active, so the mathematical formula is:

$$S_3 = p(x_1 \geq y_1, x_2 < y_2, x_3 \geq y_3, x_4 \geq y_4)$$

4. The fourth case: when $(\mathcal{A}_3)$ fails and is replaced by component $(\mathcal{A}_4)$ while components $(\mathcal{A}_1$ *and* $\mathcal{A}_2$) remain active, so the mathematical formula is:

$$S_4 = p(x_1 \geq y_1, x_2 \geq y_2, x_3 < y_3, x_4 \geq y_4)$$

Then the general mathematical formula for the reliability of the model is as follows:

$$R = S_1 + S_2 + S_3 + S_4 \quad (5)$$

Now, the four cases of reliability will be derived successively

$$S_1 = \left[\int_0^\infty \left(\overline{F}_{x_1}(y_1)\right) g(y_1)dy_1\right]\left[\int_0^\infty \left(\overline{F}_{x_2}(y_2)\right) g(y_2)dy_2\right]\left[\int_0^\infty \left(\overline{F}_{x_3}(y_3)\right) g(y_3)dy_3\right]$$

where $\overline{F}(x) = 1 - F(x)$.

$$S_1 = \left[\int_0^\infty \left(1-e^{-\mu_1 y_1^{-o}}\right) o\delta_1 y_1^{-(o+1)} e^{-\delta_1 y_1^{-o}} dy_1\right]\left[\int_0^\infty \left(1-e^{-\mu_2 y_2^{-o}}\right) o\delta_2 y_2^{-(o+1)} e^{-\delta_2 y_2^{-o}} dy_2\right]\left[\int_0^\infty \left(1-e^{-\mu_3 y_3^{-o}}\right) o\delta_3 y_3^{-(o+1)} e^{-\delta_3 y_3^{-o}} dy_3\right]$$

$$S_1 = \left[\int_0^\infty o\delta_1 y_1^{-(o+1)} e^{-\delta_1 y_1^{-o}} dy_1 - \int_0^\infty o\delta_1 y_1^{-(o+1)} e^{-(\mu_1+\delta_1)y_1^{-o}} dy_1\right]\left[\int_0^\infty o\delta_2 y_2^{-(o+1)} e^{-\delta_2 y_2^{-o}} dy_2 - \int_0^\infty o\delta_2 y_2^{-(o+1)} e^{-(\mu_2+\delta_2)y_2^{-o}} dy_2\right]\left[\int_0^\infty o\delta_3 y_3^{-(o+1)} e^{-\delta_3 y_3^{-o}} dy_3 - \int_0^\infty o\delta_3 y_3^{-(o+1)} e^{-(\mu_3+\delta_3)y_3^{-o}} dy_3\right]$$

Now, S_1 can be written as the following:

$$S_1 = \left[\frac{\mu_1}{\mu_1+\delta_1}\right]\left[\frac{\mu_2}{\mu_2+\delta_2}\right]\left[\frac{\mu_3}{\mu_3+\delta_3}\right] \quad (6)$$

To find the S_2 is started as follows:

$$S_2 = \left[\int_0^\infty \left(F_{x_1}(y_1)\right)\left(\overline{F}_{x_1}\left(\frac{\mathrm{k}}{\mathrm{m}}y_1\right)\right) g(y_1)dy_1\right]\left[\int_0^\infty \left(\overline{F}_{x_2}(y_2)\right) g(y_2)dy_2\right]\left[\int_0^\infty \left(\overline{F}_{x_3}(y_3)\right) g(y_3)dy_3\right]$$
$$= \left[\int_0^\infty \left(e^{-\mu_1 y_1^{-o}}\right)\left(1-e^{-\mu_1\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o} y_1^{-o}}\right) o\delta_1 y_1^{-(o+1)} e^{-\delta_1 y_1^{-o}} dy_1\right]\left[\int_0^\infty \left(1-e^{-\mu_2 y_2^{-o}}\right) o\delta_2 y_2^{-(o+1)} e^{-\delta_2 y_2^{-o}} dy_2\right]\left[\int_0^\infty \left(1-e^{-\mu_3 y_3^{-o}}\right) o\delta_3 y_3^{-(o+1)} e^{-\delta_3 y_3^{-o}} dy_3\right]$$

$$S_2 = \left[\int_0^\infty o\delta_1 y_1^{-(o+1)} e^{-(\mu_1+\delta_1)y_1^{-o}} dy_1 - \int_0^\infty o\delta_1 y_1^{-(o+1)} e^{-(\mu_1(1+\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o})+\delta_1)y_1^{-o}} dy_1\right]\left[\int_0^\infty o\delta_2 y_2^{-(o+1)} e^{-\delta_2 y_2^{-o}} dy_2 - \int_0^\infty o\delta_2 y_2^{-(o+1)} e^{-(\mu_2+\delta_2)y_2^{-o}} dy_2\right]\left[\int_0^\infty o\delta_3 y_3^{-(o+1)} e^{-\delta_3 y_3^{-o}} dy_3 - \int_0^\infty o\delta_3 y_3^{-(o+1)} e^{-(\mu_3+\delta_3)y_3^{-o}} dy_3\right]$$

then

$$S_2 = \left[\frac{\mu_1\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o}\delta_1}{(\mu_1+\delta_1)\left(\mu_1\left(1+\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o}\right)+\delta_1\right)}\right]\left[\frac{\mu_2}{\mu_2+\delta_2}\right]\left[\frac{\mu_3}{\mu_3+\delta_3}\right] \quad (7)$$

To find S_3, track the following:

$$S_3 = \left[\int_0^\infty \left(\overline{F}_{x_1}(y_1)\right) g(y_1)dy_1\right] \left[\int_0^\infty (F_{x_2}(y_2))\left(\overline{F}_{x_2}\left(\frac{\mathrm{k}}{\mathrm{m}}y_2\right)\right) g(y_2)dy_2\right] \left[\int_0^\infty \left(\overline{F}_{x_3}(y_3)\right) g(y_3)dy_3\right]$$
$$= \left[\int_0^\infty \left(1-e^{-\mu_1 y_1^{-o}}\right) o\delta_1 y_1^{-(o+1)} e^{-\delta_1 y_1^{-o}} dy_1\right] \left[\int_0^\infty \left(e^{-\mu_2 y_2^{-o}}\right)\left(1-e^{-\mu_2\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o} y_2^{-o}}\right) o\delta_2 y_2^{-(o+1)} e^{-\delta_2 y_2^{-o}} dy_2\right] \left[\int_0^\infty \left(1-e^{-\mu_3 y_3^{-o}}\right) o\delta_3 y_3^{-(o+1)} e^{-\delta_3 y_3^{-o}} dy_3\right]$$

$$S_3 = \left[\int_0^\infty o\delta_1 y_1^{-(o+1)} e^{-\delta_1 y_1^{-o}} dy_1 - \int_0^\infty o\delta_1 y_1^{-(o+1)} e^{-(\mu_1+\delta_1) y_1^{-o}} dy_1\right] \left[\int_0^\infty o\delta_2 y_2^{-(o+1)} e^{-(\mu_2+\delta_2) y_2^{-o}} dy_2 - \int_0^\infty o\delta_2 y_2^{-(o+1)} e^{-\left(\mu_2\left(1+\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o}\right)+\delta_2\right) y_2^{-o}} dy_2\right] \left[\int_0^\infty o\delta_3 y_3^{-(o+1)} e^{-\delta_3 y_3^{-o}} dy_3 - \int_0^\infty o\delta_3 y_3^{-(o+1)} e^{-(\mu_3+\delta_3) y_3^{-o}} dy_3\right]$$

then

$$S_3 = \left[\frac{\mu_1}{\mu_1+\delta_1}\right] \left[\frac{\mu_2\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o}\delta_2}{(\mu_2+\delta_2)\left(\mu_2\left(1+\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o}\right)+\delta_3\right)}\right] \left[\frac{\mu_3}{\mu_3+\delta_3}\right] \tag{8}$$

Also for S_4:

$$S_4 = \left[\int_0^\infty \left(\overline{F}_{x_1}(y_1)\right) g(y_1)dy_1\right] \left[\int_0^\infty \left(\overline{F}_{x_2}(y_2)\right) g(y_2)dy_2\right] \left[\int_0^\infty (F_{x_3}(y_3))\left(\overline{F}_{x_3}\left(\frac{\mathrm{k}}{\mathrm{m}}y_3\right)\right) g(y_3)dy_3\right]$$

$$S_4 = \left[\int_0^\infty \left(1-e^{-\mu_1 y_1^{-o}}\right) o\delta_1 y_1^{-(o+1)} e^{-\delta_1 y_1^{-o}} dy_1\right] \left[\int_0^\infty \left(1-e^{-\mu_2 y_2^{-o}}\right) o\delta_2 y_2^{-(o+1)} e^{-\delta_2 y_2^{-o}} dy_2\right] \left[\int_0^\infty \left(e^{-\mu_3 y_3^{-o}}\right)\left(1-e^{-\mu_3\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o} y_3^{-o}}\right) o\delta_3 y_3^{-(o+1)} e^{-\delta_3 y_3^{-o}} dy_3\right]$$

$$S_4 = \left[\int_0^\infty o\delta_1 y_1^{-(o+1)} e^{-\delta_1 y_1^{-o}} dy_1 - \int_0^\infty o\delta_1 y_1^{-(o+1)} e^{-(\mu_1+\delta_1) y_1^{-o}} dy_1\right] \left[\int_0^\infty o\delta_2 y_2^{-(o+1)} e^{-\delta_2 y_2^{-o}} dy_2 - \int_0^\infty o\delta_2 y_2^{-(o+1)} e^{-(\mu_2+\delta_2) y_2^{-o}} dy_2\right] \left[\int_0^\infty o\delta_3 y_3^{-(o+1)} e^{-(\mu_3+\delta_3) y_3^{-o}} dy_3 - \int_0^\infty o\delta_3 y_3^{-(o+1)} e^{-\left(\mu_3\left(1+\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o}\right)+\delta_3\right) y_3^{-o}} dy_3\right]$$

then

$$S_4 = \left[\frac{\mu_1}{\mu_1+\delta_1}\right] \left[\frac{\mu_2}{\mu_2+\delta_2}\right] \left[\frac{\mu_3\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o}\delta_3}{(\mu_3+\delta_3)\left(\mu_3\left(1+\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o}\right)+\delta_3\right)}\right] \tag{9}$$

Now the final mathematical formula for the reliability function of the model can be obtained by collecting Eqs. (6), (7), (8), (9) and as follows:

$$R = \left[\frac{\mu_1}{\mu_1+\delta_1}\right]\left[\frac{\mu_2}{\mu_2+\delta_2}\right]\left[\frac{\mu_3}{\mu_3+\delta_3}\right] + \left[\frac{\mu_1\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o}\delta_1}{(\mu_1+\delta_1)\left(\mu_1\left(1+\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o}\right)+\delta_1\right)}\right]\left[\frac{\mu_2}{\mu_2+\delta_2}\right]\left[\frac{\mu_3}{\mu_3+\delta_3}\right] + \left[\frac{\mu_1}{\mu_1+\delta_1}\right]\left[\frac{\mu_2\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o}\delta_2}{(\mu_2+\delta_2)\left(\mu_2\left(1+\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o}\right)+\delta_3\right)}\right]\left[\frac{\mu_3}{\mu_3+\delta_3}\right] + \left[\frac{\mu_1}{\mu_1+\delta_1}\right]\left[\frac{\mu_2}{\mu_2+\delta_2}\right]\left[\frac{\mu_3\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o}\delta_3}{(\mu_3+\delta_3)\left(\mu_3\left(1+\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o}\right)+\delta_3\right)}\right] \tag{10}$$

3. Results and Discussion

The reliability of the model will be estimated by three different estimation methods, and then a simulation will be performed and the results of the different estimation methods will be compared to indicate which methods are the best for estimating reliability.

3.1. Estimation by Maximum Likelihood Method

In the maximum likelihood method, the general form function L can be used as following eq. (11)

$$\mathrm{L}(x_1, x_2, \ldots, x_n, o, \mu) = f(\mathrm{x}_1; o, \mu)\,\mathrm{f}(x_2; o, \mu)\ldots\mathrm{f}(x_n; o, \mu) = o^n \mu^n \prod_{\imath=1}^{n} x_\imath \mathrm{e}^{-\sum_{\imath=1}^{n} \mu x_\imath^{-o}} \tag{11}$$

Taking the logarithm of Eq. (11):

$$\ln \mathrm{L} = n \,\mathrm{Ln}\, o + n \,\mathrm{Ln}\, \mu + (\sigma+1)\sum_{\imath=1}^{n} \ln(\mathrm{x_i}) - \mu \sum_{\imath=1}^{n} \mathrm{x}_\imath^{-\sigma} \tag{12}$$

Derive Eq. (12) yields:

$$\frac{\partial \ln \mathrm{L}}{\partial \mu} = \frac{n}{\mu} - \sum_{\imath=1}^{n} x_\imath^{-\sigma} \tag{13}$$

By equating Eq. (13) to zero:

$$\frac{n}{\widehat{\mu}} - \sum_{\imath=1}^{n} x_\imath^{-\sigma} = 0 \tag{14}$$

The ML estimator for parameter μ :

$$\widehat{\mu}_{ML} = \frac{n}{\sum_{\imath=1}^{n} x_\imath^{-\sigma}} \tag{15}$$

In a similar way to the steps above, the ML estimator for parameter δ is obtained:

$$\widehat{\delta}_{ML} = \frac{m}{\sum_{j=1}^{m} y_j^{-\sigma}} \tag{16}$$

Now, the maximum likelihood estimator for parameters can be obtained as follows:

$$\widehat{\mu}_{\xi ML} = \frac{n_\xi}{\sum_{\imath_\xi=1}^{n_\xi} x_{\xi_{\imath_\xi}}^{-\sigma}} \ , \ \xi = 1,2,3,4 \tag{17}$$

and

$$\widehat{\delta}_{\xi ML} = \frac{m_\xi}{\sum_{j_\xi=1}^{m_\xi} y_{\xi_{j_\xi}}^{-\sigma}} \ , \ \xi = 1,2,3,4 \tag{18}$$

Substituting (17) and (18) by (10) produces:

$$\begin{aligned}\widehat{\mathrm{R}}_{ML} = {} & \left[\frac{\widehat{\mu}_{1ML}}{\widehat{\mu}_{1ML}+\widehat{\delta}_{1ML}}\right]\left[\frac{\widehat{\mu}_{2ML}}{\widehat{\mu}_{2ML}+\widehat{\delta}_{2ML}}\right]\left[\frac{\widehat{\mu}_{3ML}}{\widehat{\mu}_{3ML}+\widehat{\delta}_{3ML}}\right] + \\ & \left[\frac{\widehat{\mu}_{1ML}\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o}\widehat{\delta}_{1ML}}{\left(\widehat{\mu}_{1ML}+\widehat{\delta}_{1ML}\right)\left(\widehat{\mu}_{1ML}\left(1+\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o}\right)+\widehat{\delta}_{1ML}\right)}\right] \\ & \left[\frac{\widehat{\mu}_{2ML}}{\widehat{\mu}_{2ML}+\widehat{\delta}_{2ML}}\right]\left[\frac{\widehat{\mu}_{3ML}}{\widehat{\mu}_{3ML}+\widehat{\delta}_{3ML}}\right] + \left[\frac{\widehat{\mu}_{1ML}}{\widehat{\mu}_{1ML}+\widehat{\delta}_{1ML}}\right] \\ & \left[\frac{\widehat{\mu}_{2ML}\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o}\widehat{\delta}_{2ML}}{\left(\widehat{\mu}_{2ML}+\widehat{\delta}_{2ML}\right)\left(\widehat{\mu}_{2ML}\left(1+\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o}\right)+\widehat{\delta}_{3ML}\right)}\right] \\ & \left[\frac{\widehat{\mu}_{3ML}}{\widehat{\mu}_{3ML}+\widehat{\delta}_{3ML}}\right] + \left[\frac{\widehat{\mu}_{1ML}}{\widehat{\mu}_{1ML}+\widehat{\delta}_{1ML}}\right]\left[\frac{\widehat{\mu}_{2ML}}{\widehat{\mu}_{2ML}+\widehat{\delta}_{2ML}}\right] \\ & \left[\frac{\widehat{\mu}_{3ML}\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o}\delta_{3ML}}{\left(\widehat{\mu}_{3ML}+\widehat{\delta}_{3ML}\right)\left(\widehat{\mu}_{3ML}\left(1+\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o}\right)+\widehat{\delta}_{3ML}\right)}\right]\end{aligned} \tag{19}$$

3.2. Estimation by Least Squares Method

To find an estimator of parameter μ by the method of least square, start using the minimize equation:

$$\begin{aligned} S(o,\mu) &= \sum_{\imath=1}^{r}\left(\hat{F}\left(x_{(\imath)}\right) - F\left(x_{(\imath)}\right)\right)^2 \\ &= \sum_{\imath=1}^{\mathrm{n}}\left(\widehat{\mathrm{F}}\left(x_{(\imath)}\right) - \left(\mathrm{e}^{-\mu x_{(\imath)}^{-o}}\right)\right)^2 \end{aligned} \tag{20}$$

Linear equation of the CDF:

$$\begin{aligned} \mathrm{F}\left(x_{(\imath)}\right) &= \mathrm{e}^{-\mu x_{(\imath)}^{-o}} \\ -\ln\left(\mathrm{F}\left(x_{(\imath)}\right)\right) &= \mu x_{(\imath)}^{-o} \end{aligned} \tag{21}$$

$\widehat{\mathrm{F}}\left(\mathrm{x}_{(\imath)}\right)$ is replaced by:

$$\hat{F}\left(x_{(\imath)}\right) = \frac{\imath}{n+1} \ ; \imath = 1,2,3,\ldots,n$$

Eq. (20) becomes:

$$S(o,\mu) = \sum_{\imath=1}^{n}\left(\mathrm{q}_\imath - \mu x_{(\imath)}^{-o}\right)^2 \tag{22}$$

where $\mathrm{q}_{(\imath)} = -\ln\left(\widehat{\mathrm{F}}\left(x_{(\imath)}\right)\right) = -\ln\left(\mathrm{P}_\imath\right)$. Eq.(22) is partially derived for parameter μ:

$$\begin{aligned} \frac{\partial S(2,\lambda)}{\mu} &= \sum_{\imath=1}^{n} 2\left(\mathrm{q}_\imath - \mu x_{(\imath)}^{-o}\right)\left(-x_{(\imath)}^{-o}\right) = 0 \\ & -\sum_{\imath=1}^{n}\mathrm{q}_\imath x_{(\imath)}^{-o} + \mu\sum_{\imath=1}^{n} x_{(\imath)}^{-2o} = 0. \end{aligned}$$

then the least square estimator of the parameter μ is:

$$\widehat{\mu}_{LS} = \frac{\sum_{\imath=1}^{n}\mathrm{q}_\imath\, x_{(\imath)}^{-o}}{\sum_{\mathrm{i}=1}^{n} x_{(\imath)}^{-2o}} \tag{23}$$

The estimator for parameter δ can be obtained in the same previous steps:

$$\widehat{\delta}_{LS} = \frac{\sum_{\mathrm{j}=1}^{m}\mathrm{q}_\mathrm{i}\, y_{(\mathrm{j})}^{-o}}{\sum_{j=1}^{m} y_{(j)}^{-2o}} \tag{24}$$

The estimator for parameters are:

$$\widehat{\mu}_{\xi LS} = \frac{\sum_{\imath_\xi=1}^{n_\zeta}\mathrm{q}_{\imath_\xi}\, x_{\zeta_{(\imath_\xi)}}^{-o}}{\sum_{\imath_\xi=1}^{n_\xi} x_{\xi_{(\imath_\xi)}}^{-2o}} \ , \ \xi = 1,2,3,4 \tag{25}$$

and

$$\widehat{\delta}_{\xi LS} = \frac{\sum_{j_\xi=1}^{m_\xi}\mathrm{q}_{j_\xi}\, y_{\xi_{(j_\xi)}}^{-o}}{\sum_{j_\xi=1}^{m_\xi} y_{\xi_{(j_\xi)}}^{-2o}} \ , \ \xi = 1,2,3,4. \tag{26}$$

Substituting (25) and (26) by (10) produces:

$$\begin{aligned}\widehat{\mathrm{R}}_{LS} = & \left[\frac{\widehat{\mu}_{1LS}}{\widehat{\mu}_{1LS}+\widehat{\delta}_{1LS}}\right]\left[\frac{\widehat{\mu}_{2LS}}{\widehat{\mu}_{2LS}+\widehat{\delta}_{2LS}}\right]\left[\frac{\widehat{\mu}_{3LS}}{\widehat{\mu}_{3LS}+\widehat{\delta}_{3LS}}\right]+\\ & \left[\frac{\widehat{\mu}_{1LS}\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o}\widehat{\delta}_{1LS}}{\left(\widehat{\mu}_{1LS}+\widehat{\delta}_{1LS}\right)\left(\widehat{\mu}_{1LS}\left(1+\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o}\right)+\widehat{\delta}_{1LS}\right)}\right]\\ & \left[\frac{\widehat{\mu}_{2LS}}{\widehat{\mu}_{2LS}+\widehat{\delta}_{2LS}}\right]\left[\frac{\widehat{\mu}_{3LS}}{\widehat{\mu}_{3LS}+\widehat{\delta}_{3LS}}\right]+\left[\frac{\widehat{\mu}_{1LS}}{\widehat{\mu}_{1LS}+\widehat{\delta}_{1LS}}\right]\\ & \left[\frac{\widehat{\mu}_{2LS}\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o}\widehat{\delta}_{2LS}}{\left(\widehat{\mu}_{2LS}+\widehat{\delta}_{2LS}\right)\left(\widehat{\mu}_{2LS}\left(1+\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o}\right)+\widehat{\delta}_{3LS}\right)}\right]\\ & \left[\frac{\widehat{\mu}_{3LS}}{\widehat{\mu}_{3LS}+\widehat{\delta}_{3LS}}\right]+\left[\frac{\widehat{\mu}_{1LS}}{\widehat{\mu}_{1LS}+\widehat{\delta}_{1LS}}\right]\left[\frac{\widehat{\mu}_{2LS}}{\widehat{\mu}_{2LS}+\widehat{\delta}_{2LS}}\right]\\ & \left[\frac{\widehat{\mu}_{3LS}\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o}\delta_{3LS}}{\left(\widehat{\mu}_{3LS}+\widehat{\delta}_{3LS}\right)\left(\widehat{\mu}_{3LS}\left(1+\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o}\right)+\widehat{\delta}_{3LS}\right)}\right]\end{aligned} \tag{27}$$

3.3. *Estimation by Regression Method*

To start by finding the estimator of parameter μ by the regression method, the standard equation of regression is used [16]:

$$z_i = a + bu_\imath + e_\imath \tag{28}$$

where $(\mathrm{z}_\imath)$ is the "dependent variable" and $(\mathrm{u}_\imath)$ is the "independent variable" while $(\mathrm{e}_\imath)$ is "error random variable independent".

Let $x_1, x_2, \ldots, x_n$ of $\mathrm{Fr}(o, \mu)$. By taking the logarithm of the cumulative distribution function for Frechet distribution:

$$\begin{aligned}\mathrm{F}\left(x_{(\imath)}\right) &= e^{-\mu x_{(\imath)}^{-o}}\\ \mathrm{Ln}\left[\mathrm{F}\left(x_{(\imath)}\right)\right] &= -\mu x_{(\imath)}^{-o}.\end{aligned} \tag{29}$$

Replace $\mathrm{F}\left(x_{(\imath)}\right)$ by plotting position $\mathrm{P}_\imath$:

$$\mathrm{Ln}\ \mathrm{P}_\imath = -\mu x_{(\imath)}^{-o}. \tag{30}$$

Comparing eq. (28) and eq. (30) shows:

$$\mathrm{z}_\imath = \mathrm{LnP}_\imath,\ \ \mathrm{a}=0,\ \ \mathrm{b}=\mu,\ \mathrm{u}_\imath = -x_{(\imath)}^{-o}; \imath = 1, 2, \ldots, n \tag{31}$$

Now it is possible to estimate b by reducing the sum of the quadratic error as follows:

$$\widehat{\mathrm{b}} = \frac{n\sum_{\mathrm{i}=1}^{n}\mathrm{z}_\imath\mathrm{u}_\imath - \sum_{\imath=1}^{n}\mathrm{z}_\imath\sum_{\imath=1}^{n}\mathrm{u}_\imath}{n\sum_{\imath=1}^{n}(\mathrm{u}_\imath)^2 - \left(\sum_{\imath=1}^{n}\mathrm{u}_\imath\right)^2} \tag{32}$$

By substitution eq. (31) in eq. (32), the regression estimator for μ is :

$$\widehat{\mu}_{Rg} = \frac{n\sum_{\imath=1}^{n}\left[-\mathrm{x}_{(\imath)}^{-o}\right]\ \ln\left(\mathrm{P}_\imath\right) - \sum_{\imath=1}^{n}\left[-\mathrm{x}_{(\imath)}^{-o}\right]\ \sum_{\imath=1}^{n}\ln\left(\mathrm{P}_\imath\right)}{n\sum_{\imath=1}^{n}\left[-\mathrm{x}_{(\imath)}^{-o}\right]^2 - \left[\sum_{\imath=1}^{n}\left(-\mathrm{x}_{(\imath)}^{-o}\right)\right]^2} \tag{33}$$

The Rg of δ is:

$$\widehat{\delta}_{Rg} = \frac{m\sum_{\mathrm{j}=1}^{m}\left[-\mathrm{y}_{(j)}^{-o}\right]\ \ln\left(\mathrm{P}_j\right) - \sum_{j=1}^{m}\left[-\mathrm{y}_{(j)}^{-o}\right]\ \sum_{j=1}^{m}\ln\left(\mathrm{P}_j\right)}{m\sum_{j=1}^{m}\left[-\mathrm{y}_{(j)}^{-o}\right]^2 - \left[\sum_{j=1}^{m}\left(-\mathrm{y}_{(j)}^{-o}\right)\right]^2} \tag{34}$$

The regression estimator for parameters are:

$$\widehat{\mu}_{\xi Rg} = \frac{R_1}{n_\xi\sum_{\imath_\xi=1}^{n_\xi}\left[-\mathrm{x}_{\xi_{(\imath_\xi)}}^{-\sigma}\right]^2 - \left[\sum_{\imath_\xi=1}^{n_\xi}\left(-\mathrm{x}_{\xi_{(\imath_\xi)}}^{-\sigma}\right)\right]^2} \tag{35}$$

where

$$R_1 = n_\xi\sum_{\imath_\xi=1}^{n_\xi}\left[-\mathrm{x}_{\xi_{(\imath_\xi)}}^{-\sigma}\right]\ \ln\left(\mathrm{P}_{\imath_\xi}\right) - \sum_{\imath_\xi=1}^{n_\xi}\left[-\mathrm{x}_{\xi_{(\imath_\xi)}}^{-\sigma}\right]\ \sum_{\imath_\xi=1}^{n_\xi}\ln\left(\mathrm{P}_{\imath_\xi}\right),$$
$$\xi = 1, 2, 3, 4,$$

and

$$\widehat{\delta}_{\xi Rg} = \frac{R_1}{m_\xi\sum_{j_\xi=1}^{m_\xi}\left[-\mathrm{y}_{\xi_{(j_\xi)}}^{-\sigma}\right]^2 - \left[\sum_{j_\xi=1}^{m_\xi}\left(-\mathrm{y}_{\xi_{(j_\xi)}}^{-\sigma}\right)\right]^2} \tag{36}$$

where

$$m_\xi\sum_{j_\xi=1}^{m_\xi}\left[-\mathrm{y}_{\xi_{(j_\xi)}}^{-\sigma}\right]\ \ln\left(\mathrm{P}_{j_\xi}\right) - \sum_{j_\xi=1}^{m_\xi}\left[-\mathrm{y}_{\xi_{(j_\xi)}}^{-\sigma}\right]\ \sum_{j_\xi=1}^{m_\xi}\ln\left(\mathrm{P}_{j_\xi}\right),$$
$$\xi = 1, 2, 3, 4.$$

Substituting eq. (35) and eq. (36) by eq. (10) produces:

$$\begin{aligned}\widehat{\mathrm{R}}_{Rg} = & \left[\frac{\widehat{\mu}_{1Rg}}{\widehat{\mu}_{1Rg}+\widehat{\delta}_{1Rg}}\right]\left[\frac{\widehat{\mu}_{2Rg}}{\widehat{\mu}_{2Rg}+\widehat{\delta}_{2Rg}}\right]\left[\frac{\widehat{\mu}_{3Rg}}{\widehat{\mu}_{3Rg}+\widehat{\delta}_{3Rg}}\right]+\\ & \left[\frac{\widehat{\mu}_{1Rg}\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o}\widehat{\delta}_{1Rg}}{\left(\widehat{\mu}_{1Rg}+\widehat{\delta}_{1Rg}\right)\left(\widehat{\mu}_{1Rg}\left(1+\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o}\right)+\widehat{\delta}_{1Rg}\right)}\right]\\ & \left[\frac{\widehat{\mu}_{2Rg}}{\widehat{\mu}_{2Rg}+\widehat{\delta}_{2Rg}}\right]\left[\frac{\widehat{\mu}_{3Rg}}{\widehat{\mu}_{3Rg}+\widehat{\delta}_{3Rg}}\right]+\left[\frac{\widehat{\mu}_{1Rg}}{\widehat{\mu}_{1Rg}+\widehat{\delta}_{1Rg}}\right]\\ & \left[\frac{\widehat{\mu}_{2Rg}\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o}\widehat{\delta}_{2Rg}}{\left(\widehat{\mu}_{2Rg}+\widehat{\delta}_{2Rg}\right)\left(\widehat{\mu}_{2Rg}\left(1+\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o}\right)+\widehat{\delta}_{3Rg}\right)}\right]\\ & \left[\frac{\widehat{\mu}_{3Rg}}{\widehat{\mu}_{3Rg}+\widehat{\delta}_{3Rg}}\right]+\left[\frac{\widehat{\mu}_{1Rg}}{\widehat{\mu}_{1Rg}+\widehat{\delta}_{1Rg}}\right]\left[\frac{\widehat{\mu}_{2Rg}}{\widehat{\mu}_{2Rg}+\widehat{\delta}_{2Rg}}\right]\\ & \left[\frac{\widehat{\mu}_{3Rg}\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o}\delta_{3Rg}}{\left(\widehat{\mu}_{3Rg}+\widehat{\delta}_{3Rg}\right)\left(\widehat{\mu}_{3Rg}\left(1+\left(\frac{\mathrm{k}}{\mathrm{m}}\right)^{-o}\right)+\widehat{\delta}_{3Rg}\right)}\right]\end{aligned} \tag{37}$$

3.4. *Simulation*

Monte Carlo simulation of estimation methods was conducted with the results compared using two criteria, MSE and MAPE. It is repeated 10,000 times with samples of different sizes (small sample, medium sample, large sample) and independently [11].

Simulation algorithms programmed using MATLAB software to estimate model reliability, according to the steps outlined below:

1. Let be the random variables $X_{1\imath}; \imath = 1, 2, \ldots, n_1$, $X_{2\imath}; \imath = 1, 2, \ldots, n_2$, $X_{3\imath}; \imath = 1, 2, \ldots, n_3$, $X_{4\imath}; \imath = 1, 2, \ldots, n_4$, $Y_{1j}; j = 1, 2, \ldots, m_1$, $Y_{2j}; j = 1, 2, \ldots, m_2$, $Y_{3j}; j = 1, 2, \ldots, m_3$ and $Y_{4j}; j = 1, 2, \ldots, m_4$ for sample sizes $n_1, n_2, n_3, n_4, m_1, m_2, m_3, m_4 = (25, 25, 25, 25, 25, 25, 25, 25)$ small samples, (50,50,50,50,50,50,50,50) middle samples and (80,80,80,80,80,80,80,80) large are generated from Ferchet distribution.
2. Parameter values were selected for ten trials in Table 1:

Table 1. The parameters and reliability

Trial	k	m	o	μ_1	μ_2	μ_3	δ_1	δ_2	δ_3	R
1	1.9	0.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	0.1372
2	1.9	0.2	0.8	1.2	1.2	1.2	1.2	1.2	1.2	0.1536
3	1.9	0.2	1.6	1.2	1.2	1.2	1.2	1.2	1.2	0.1300
4	1.9	0.2	1.2	1.0	1.0	1.0	1.2	1.2	1.2	0.1039
5	1.9	0.2	1.2	2.6	2.6	1.2	1.2	1.2	1.2	0.3398
6	1.9	0.2	1.2	1.2	1.2	1.2	0.4	0.4	0.4	0.4421
7	1.9	0.2	1.2	1.2	1.2	1.2	1.4	1.4	1.4	0.1087
8	1.8	0.4	1.2	1.2	1.2	1.2	1.2	1.2	1.2	0.1535
9	1.5	0.6	1.2	1.2	1.2	1.2	1.2	1.2	1.2	0.1785
10	1.1	0.9	1.2	1.2	1.2	1.2	1.2	1.2	1.2	0.2308

3. Eq. (17), (18), (25), (26), (35), (36) included the estimation of parameters $(\mu_1, \mu_2, \mu_3, \mu_4, \delta_1, \delta_2, \delta_3, \delta_4)$ according to the estimation methods (ML, LS, Rg) respectively.
4. Eq. (19), (27), (37) included estimating the reliability of the model according to estimation methods (ML, LS and Rg).
5. The mathematical formula for the mean used is:

$$\text{Mean} = \frac{\sum_{\imath=1}^{L} \hat{R}_\imath}{L}$$

6. The mathematical formula for the "mean square error" used is :

$$\text{MSE}\left(\hat{R}\right) = \frac{1}{\text{L}} \sum_{\imath=1}^{\text{L}} \left(\hat{R}_\imath - R\right)^2.$$

7. The mathematical formula for the " mean absolute percentage error "used is:

$$\text{MAPE}(\hat{R}) = \frac{1}{L} \sum_{\imath=1}^{L} \left| \frac{\hat{R}_\imath - R}{R} \right|.$$

After completing the previous steps, the results of the simulation are produced, which are as shown in Tables 2-11, where each table contained the results of one of the ten trials.

Table 2. Results of simulation trial 1

Sizes of simples	Criterion	ML	LS	Rg	Best
25,25,25,25,	Mean	0.1370	0.1371	0.1371	
25,25,25,25	MSE	0.0010	0.0013	0.0019	
	MAPE	0.1877	0.2073	0.2530	
50,50,50,50,	Mean	0.1372	0.1373	0.1373	
50,50,50,50	MSE	0.0005	0.0007	0.0010	ML
	MAPE	0.1337	0.1504	0.1869	
80,80,80,80,	Mean	0.1372	0.1372	0.1371	
80,80,80,80	MSE	0.0003	0.0004	0.0006	
	MAPE	0.1058	0.1198	0.1505	

Table 3. Results of simulation trial 2

Sizes of simples	Criterion	ML	LS	Rg	Best
25,25,25,25,	Mean	0.1532	0.1530	0.1527	
25,25,25,25	MSE	0.0012	0.0014	0.0021	
	MAPE	0.1779	0.1963	0.2401	
50,50,50,50,	Mean	0.1534	0.1530	0.1526	
50,50,50,50	MSE	0.0006	0.0008	0.0012	ML
	MAPE	0.1298	0.1453	0.1803	
80,80,80,80,	Mean	0.1537	0.1534	0.1531	
80,80,80,80	MSE	0.0003	0.0004	0.0007	
	MAPE	0.1019	0.1148	0.1440	

Table 4. Results of simulation trial 3

Sizes of simples	Criterion	ML	LS	Rg	Best
25,25,25,25,	Mean	0.1300	0.1301	0.1301	
25,25,25,25	MSE	0.0010	0.0012	0.0018	
	MAPE	0.1937	0.2133	0.2586	
50,50,50,50,	Mean	0.1300	0.1299	0.1298	
50,50,50,50	MSE	0.0005	0.0006	0.0009	ML
	MAPE	0.1371	0.1523	0.1882	
80,80,80,80,	Mean	0.1303	0.1303	0.1303	
80,80,80,80	MSE	0.0002	0.0004	0.0006	
	MAPE	0.1078	0.1203	0.1504	

Table 5. Results of simulation trial 4

Sizes of simples	Criterion	ML	LS	Rg	Best
25,25,25,25,	Mean	0.1046	0.1047	0.1050	
25,25,25,25	MSE	0.0007	0.0009	0.0014	
	MAPE	0.2076	0.2307	0.2822	
50,50,50,50,	Mean	0.1042	0.1043	0.1045	
50,50,50,50	MSE	0.0003	0.0004	0.0007	ML
	MAPE	0.1455	0.1626	0.2024	
80,80,80,80,	Mean	0.1039	0.1038	0.1039	
80,80,80,80	MSE	0.0002	0.0003	0.0004	
	MAPE	0.1156	0.1308	0.1644	

4. Conclusion

From Table 1, the following was concluded that Comparing trial 2 with trial 1 as well as trial 3 with trial 2, it is observed that the increases the value of the coefficient (o), the decreases reliability value and vice versa, and this can be shown when looking at the reliability values in the mentioned trials. When comparing trial 4 with trial 1, as well as trial 5 with trial 4, it is observed that the increases the value of the parameters $(\mu_1, \mu_1 \ and \ \mu_1)$, the more the reliability value increases exponentially, and this can be seen when looking at the reliability values in the trials mentioned. When comparing trial 6 with trial 1, as well as trial 7 with trial 6, it is observed that with the increase in the value of parameters $(\delta_1, \delta_1$ and $\delta_1)$, the reliability value decreases, and this can be seen when looking at the reliability values in the mentioned trials. Comparing trials 1, 8, 9, 10, it is noted that the decreases value of $\left(\frac{\text{k}}{\text{m}}\right)$, the increase in the reliability function, and this can

Table 6. Results of simulation trial 5

Sizes of simples	Criterion	ML	LS	Rg	Best
25,25,25,25,	Mean	0.3346	0.3339	0.3313	
25,25,25,25	MSE	0.0026	0.0031	0.0046	
	MAPE	0.1197	0.1311	0.1598	
50,50,50,50,	Mean	0.3371	0.3364	0.3348	
50,50,50,50	MSE	0.0013	0.0016	0.0025	ML
	MAPE	0.0835	0.0935	0.1168	
80,80,80,80,	Mean	0.3382	0.3376	0.3364	
80,80,80,80	MSE	0.0008	0.0010	0.0016	
	MAPE	0.0657	0.0741	0.0931	

Table 7. Results of simulation trial 6

Sizes of simples	Criterion	ML	LS	Rg	Best
25,25,25,25,	Mean	0.4353	0.4336	0.4295	
25,25,25,25	MSE	0.0028	0.0034	0.0051	
	MAPE	0.0958	0.1059	0.1297	
50,50,50,50,	Mean	0.4384	0.4377	0.4356	
50,50,50,50	MSE	0.0014	0.0017	0.0027	ML
	MAPE	0.0666	0.0746	0.0929	
80,80,80,80,	Mean	0.4398	0.4391	0.4375	
80,80,80,80	MSE	0.0009	0.0011	0.0017	
	MAPE	0.0529	0.0593	0.0744	

Table 8. Results of simulation trial 7

Sizes of simples	Criterion	ML	LS	Rg	Best
25,25,25,25,	Mean	0.1088	0.1088	0.1090	
25,25,25,25	MSE	0.0008	0.0010	0.0014	
	MAPE	0.2055	0.2261	0.2746	
50,50,50,50,	Mean	0.1091	0.1092	0.1093	
50,50,50,50	MSE	0.0003	0.0004	0.0007	ML
	MAPE	0.1430	0.1605	0.2005	
80,80,80,80,	Mean	0.1089	0.1089	0.1089	
80,80,80,80	MSE	0.0002	0.0003	0.0004	
	MAPE	0.1142	0.1279	0.1599	

Table 9. Results of simulation trial 8

Sizes of simples	Criterion	ML	LS	Rg	Best
25,25,25,25,	Mean	0.1529	0.1527	0.1523	
25,25,25,25	MSE	0.0012	0.0015	0.0021	
	MAPE	0.1825	0.1992	0.2410	
50,50,50,50,	Mean	0.1530	0.1530	0.1529	
50,50,50,50	MSE	0.0006	0.0008	0.0012	ML
	MAPE	0.1289	0.1432	0.1784	
80,80,80,80,	Mean	0.1534	0.1534	0.1532	
80,80,80,80	MSE	0.0003	0.0004	0.0007	
	MAPE	0.1004	0.1131	0.1414	

be seen when looking at the values of reliability in the trials mentioned.

From Tables 2-11 it was concluded that the best estimator for model reliability is the maximum likelihood estimator for all experiments and all sample sizes.

Table 10. Results of simulation trial 9

Sizes of simples	Criterion	ML	LS	Rg	Best
25,25,25,25,	Mean	0.1775	0.1775	0.1771	
25,25,25,25	MSE	0.0015	0.0018	0.0028	
	MAPE	0.1719	0.1919	0.2349	
50,50,50,50,	Mean	0.1777	0.1776	0.1771	
50,50,50,50,	MSE	0.0007	0.0009	0.0014	ML
	MAPE	0.1214	0.1369	0.1706	
80,80,80,80,	Mean	0.1781	0.1781	0.1780	
80,80,80,80	MSE	0.0004	0.0005	0.0009	
	MAPE	0.0976	0.1091	0.1359	

Table 11. Results of simulation trial 10

Sizes of simples	Criterion	ML	LS	Rg	Best
25,25,25,25,	Mean	0.2284	0.2279	0.2267	
25,25,25,25	MSE	0.0020	0.0024	0.0036	
	MAPE	0.1559	0.1717	0.2091	
50,50,50,50,	Mean	0.2294	0.2292	0.2286	
50,50,50,50	MSE	0.0010	0.0013	0.0020	ML
	MAPE	0.1115	0.1244	0.1545	
80,80,80,80,	Mean	0.2302	0.2302	0.2300	
80,80,80,80	MSE	0.0006	0.0008	0.0012	
	MAPE	0.0873	0.0977	0.1225	

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