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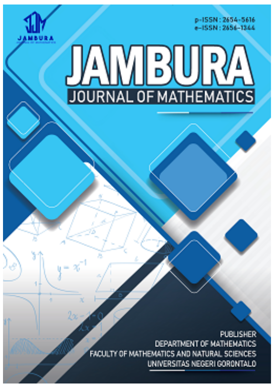

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Research Article

Group of All Taxicab Isometries: A Combinatorial Approach

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ABSTRACT. *In this work, we give a more thorough and exhaustive proof of the set of all isometries in taxicab geometry using a combinatorial approach. We show that isometries preserving taxicab distance while leaving the origin fixed are uniquely determined by how they permute the vertices of circles. Then, we use this principle to identify all isometries in taxicab geometry.*

1. Introduction

Geometry of space is determined or characterized by its group of symmetry, which in turn depends on the distance or metric defined on the space. Minkowski distance is a large family of mathematical distance functions used to measure the distance between two points in a multi-dimensional space. The Minkoswski distances are keys for many applications in many areas, such as image processing [1–3], biometric, fingerprint, and face recognition [4–7], data mining [8, 9], tomography [10–13], urban planning [14], and network analysis [15].

Let $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$ be two points in $\mathbb{R}^n$ and $\mathbf{u}_1, \ldots, \mathbf{u}_n$ where $\mathbf{u}_i = (u_{i1}, \ldots, u_{in})$ be linearly independent n unit vectors. For each positive real numbers, $\lambda_1, \ldots, \lambda_n$, we define the function $d_{p(\mathbf{u}_1,\ldots,\mathbf{u}_n)} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as

$$d_{p(\mathbf{u}_1,\ldots,\mathbf{u}_n)}(X,Y) = \sqrt[p]{\sum_{i=1}^{n} (\lambda_i \, |\mathbf{u}_i \bullet (X-Y)|)^p}.$$

The function $d_{p(\mathbf{u}_1,\ldots,\mathbf{u}_n)}$ is called the $(\mathbf{u}_1, \ldots, \mathbf{u}_n)$-Minkowski distance function in $\mathbb{R}^n$ and real numbers $d_{p(\mathbf{u}_1,\ldots,\mathbf{u}_n)}(X,Y)$ is called the $(\mathbf{u}_1, \ldots, \mathbf{u}_n)$-Minkowski distance between X and Y. If $p = 1$ and $p = 2$, then $d_{p(\mathbf{u}_1,\ldots,\mathbf{u}_n)}(X,Y)$ is called the $(\mathbf{u}_1, \ldots, \mathbf{u}_n)$-taxicab distance and $(\mathbf{u}_1, \ldots, \mathbf{u}_n)$-Euclidean distance respectively. Therefore, if we choose $\mathbf{u}_i$, $i = 1, \ldots, n$, to be standard unit vectors on $\mathbb{R}^n$, the $(\mathbf{u}_1, \ldots, \mathbf{u}_n)$-taxicab distance and $(\mathbf{u}_1, \ldots, \mathbf{u}_n)$-Euclidean distance, respectively, are called generalized taxicab-distance and generalized Euclidean-distance. The taxicab-distance and Euclidean-distance are, respectively, generalized taxicab-distance and generalized Euclidean-distance with $\lambda_i = 1$ for $i = 1, \ldots, n$.

The group of all isometries first studied by Kocayusufo ğlu and Ozdamar, see [16]. In [17], Ekmekci, et. al., identifies group of all isometries for plane generalized taxicab geometry. In [18, 19], Çolakoğlu defines a more general taxicab distance, called *m*-generalized taxicab metric. Çolakoğlu also provides us with isometries preserving the metric. Every isometry with a fixed-point can be decomposed into translation and an isometry leaving the origin fixed. Here, we present an entirely new approach for identifying isometries leaving the origin fixed. We show that such isometries permute vertices of circles, especially the unit circle. Then we show that isometries are uniquely determined by their behavior on the vertices of circle. However, none of the references [17–19], or [16] make any reference to this key result, which is essential to identifying all isometries.

Next, we determine the necessary condition for a permutation to give rise to an isometry. This allows us to discriminate between the permutations that give rise to an isometry and the rest. As a result, all isometries leaving the origin fixed can be obtained by simply identifying all vertex permutations that give rise to an isometry. This gives a combinatorial flavor to the method of identifying isometries. The similar technique can also be used to find isometries of more general taxicab metrics.

2. Methods

We already know that isometries that have a fixed-point can be decomposed into a product of an isometry, leaving the origin fixed and some translations. Therefore, determining all isometries that leave the origin fixed is central to problem of identifying all isometries. We outline the approach we used to identify all such isometries in the following.

Isometries with a fixed-point leave all circles centered at the fixed-point invariant. Consequently, we need to understand the distance between any two points on a circle in order to identify isometries leaving the circle invariant. This study shows that vertices play important role in this problem: when restricted to the set of vertices, isometries turn out to be a permutation of vertices. Thus we can find all isometries by studying all possible such permutations.

Assume a permutation gives rise to an isometry. We must figure out how to identify the isometry. Is it possible to identify the isometry simply by seeing how it permutes the vertices? To be more specific, in order to identify the isometry, we need to know if isometries are uniquely determined by their behavior on the set of vertices. It can be shown that the answer is affirmative.

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To ensure that all isometries have been identified, we need to distinguish permutations that do not give rise to an isometry.

3. Results and Discussion

3.1. Distance Between Points on a Circle

A transformation on the plane is one-to-one correspondence from the set of points in the plane onto itself. A transformation that preserves distance defined on the plane is called an isometry. Thus, in this work, a bijection $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry if

$$d_T(X, Y) = d_T(\varphi(X), \varphi(Y))$$

and $d_T : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ with

$$d_T(X, Y) = |x_1 - y_1| + |x_2 - y_2|.$$

for any $X(x_1, x_2), Y(y_1, y_2) \in \mathbb{R}^2$. Throughout this work, we assume the plane is endowed with taxicab distance.

Let S_r be a circle centered at the origin with radius $r > 0$,

$$S_r = \{X : d_T(O, X) = r\}.$$

Let $A(r, 0), B(0, r), C(-r, 0)$ and $D(0, -r)$ be the vertices the circle. From now on, let $d_T(X, Y)$ denote the taxicab distance between two points, X and Y, on $\mathbb{R}^2$ plane.

Lemma 1. *Let $P(a, b)$ with $a > 0$ and $b > 0$, be a point of $\overline{AB}$. For any $X(x, y) \in S_r$,*

$$d_T(P, X) = \begin{cases} 2(a - x), & \text{if } X \in \overline{AB} \text{ and } y \geq b, \\ 2(b - y), & \text{if } X \in \overline{AB} \text{ and } 0 \leq y < b, \\ 2a, & \text{if } X \in \overline{BC} \text{ and } y \geq b, \\ -2x, & \text{if } X \in \overline{BC} \text{ and } 0 \leq y < b, \\ 2r, & \text{if } X \in \overline{CD}, \\ -2y, & \text{if } X \in \overline{AD} \text{ and } 0 \leq x < a, \\ 2b, & \text{if } X \in \overline{AD} \text{ and } x \geq a. \end{cases}$$

An illustration of lemma 1 can be seen in figure 1.

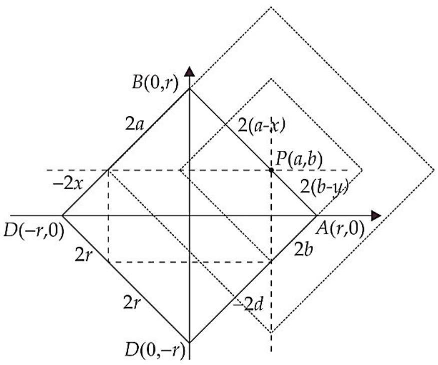

Figure 1. Taxicab distance on a circle of radius $r > 0$ between a point in the first quadrant and another point

Proof. Firstly, let us consider the case $X(x, y) \in \overline{AB}$, i.e. $x + y = r$. Thus

$$x + y = a + b.$$

We subdivide the case into two subcases: $y \geq b$ and $0 < y < b$. If $y \geq b$, since $b = a - x$, then $0 \leq x \leq a$ and, hence,

$$\begin{aligned} d_T(P, X) &= |x - a| + |y - b| \\ &= a - x + y - b \\ &= a - b + y - x \\ &= (r - b - b) + (y - (r - y)) \\ &= r - 2b + 2y - r \\ &= 2(y - b) \\ &= 2(a - x). \end{aligned}$$

If $0 \leq y < b$, then $a < x$ and, hence

$$\begin{aligned} d_T(P, X) &= |x - a| + |y - b| = x - a + b - y \\ &= b - a + b - y \\ &= (b - (r - b)) + ((r - y) - y) \\ &= 2b - r + r - 2y \\ &= 2(b - y). \end{aligned}$$

Let $X(x, y) \in \overline{BC}$. Then $y = x + r$ and $-a \leq x \leq 0$. Again, we subdivide case into two subcases: $y \geq b$ and $0 \leq y < b$. If $y \geq b$, then

$$\begin{aligned} d_T(P, X) &= |x - a| + |y - b| \\ &= a - x + y - b \\ &= a - b + y - x \\ &= a - (r - a) + r \\ &= 2a. \end{aligned}$$

On the other hand, if $0 \leq y < b$, then

$$\begin{aligned} d_T(P, X) &= |x - a| + |y - b| \\ &= a - x + b - y \\ &= a + b - x - (x + r) \\ &= r - 2x - r \\ &= -2x. \end{aligned}$$

Now, we consider the case, $X \in \overline{CD}$. Since $y = -x - r$, we have that

$$\begin{aligned} d_T(P, X) &= |x - a| + |y - b| \\ &= a - x + b - y \\ &= a + b - x - (-x - r) \\ &= r - x + x + r \\ &= 2r. \end{aligned}$$

Suppose that $X \in \overline{AD}$, which means $y = x - r$. If $0 \leq x < a$ then

$$\begin{aligned} d_T(P,X) =& \ |x-a|+|y-b| \\ &= a-x+b-y \\ &= a+b-x-(x-r) \\ &= r-x-x+r \\ &= 2(r-x) \\ &= \ -2y. \end{aligned}$$

If $a < x \le r$, then

$$\begin{aligned} d_T(P,X) =& \ |x-a|+|y-b| \\ &= x-a+b-y \\ &= b-(r-b)+x-(x-r) \\ &= 2b-r+r=2b. \end{aligned}$$

□

The distance between any two circle points can be determined using lemma 1.

Lemma 2. *If $P = A(r,0) \in S_r$, then for any $X \in S_r$,*

$$d_T(P,X) = r-x+|y| = \begin{cases} 2y, & \text{if } x\geq 0, y\geq 0, \\ 2r, & \text{if } x<0, \\ -2y, & \text{if } x\geq 0, y<0. \end{cases}$$

An illustration of lemma 2 can be seen in figure 2.

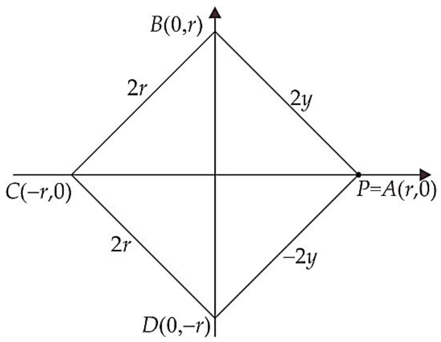

Figure 2. Taxicab distance on a circle with radius $r > 0$ between the vertex $A(r,0)$ and another point

Proof. Let $X(x,y)$ be any point the circle S_r. If $x < 0$, then

$$\begin{aligned} d_T(P,X) =& \ |r-x|+|y| \\ &= r-x+r-|x| \\ &= r-x+r-(-x) \\ &= 2r. \end{aligned}$$

Assuming $x \geq 0$, we have that

$$\begin{aligned} d_T(P,X) =& \ |r-x|+|y| \\ &= r-x+r-|x| \\ &= r-x+r-x \\ &= 2r-2x \\ &= 2(r-x). \end{aligned}$$

If $y \geq 0$, then $|x|+|y| = x+y = r$ and $r-x=y$. As a result,

$$d_T(P,X) = 2y.$$

If $y \geq 0$, then $|x|+|y| = x-y = r$ and $r-x=-y$. Thus,

$$d_T(P,X) = -2y.$$

□

The distance from any circle's vertex to any point on the circle can be found using lemma 2.

3.2. Isometries Leaving The Origin Fixed

The main result is that isometries which leave the origin fixed can be determined by how they permute the vertices of circles centered at the origin. Thus, we have a combinatorial flavor of isometries. Let Fix_O (2) be the set of all taxicab isometries leaving the origin fixed.

The following lemma 3, that isometries in taxicab geometry preserve order along lines.

Lemma 3. *Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ by any isometry and A, B are two distinct points. For any $X \in \overleftrightarrow{AB}$,*

$$\frac{d_T(A,X)}{d_T(A,B)} = \frac{d_T(\varphi(A),\varphi(X))}{d_T(\varphi(A),\varphi(B))}.$$

Proof. Let (a_1,a_2), $B(b_1,b_2)$, and $X(t) = A+t(B-A) \in \overleftrightarrow{AB}$ with $t \in \mathbb{R}$.

$$\begin{aligned} X(t) =& \ (a_1,a_2)+t(b_1-a_1,b_2-a_2) \\ =& \ (1-t)a_1+tb_1,(1-t)a_1+tb_2. \end{aligned}$$

In general,

$$\begin{aligned} d_T(A,X) =& \ |(1-t)a_1+tb_1-a_1|+|(1-t)a_2+tb_2-a_2| \\ &= t(|b_1-a_1|+|b_2-a_2|) \\ &= td_T(A,B) \end{aligned} \tag{1}$$

$$\begin{aligned} d_T(B,X) =& \ |(1-t)a_1+tb_1-a_1|+|(1-t)a_2+tb_2-a_2| \\ &= (1-t)(|b_1-a_1|+|b_2-a_2|) \\ &= (1-t)d_T(A,B) \end{aligned} \tag{2}$$

Since φ is an isometry, we have that

$$\begin{aligned} d_T(\varphi(A),\varphi(X)) &= d_T(A,X) \\ &= td_T(A,B) \\ &= td_T(\varphi(A),\varphi(B)) \end{aligned} \tag{3}$$

$$\begin{aligned} d_T(\varphi(B),\varphi(X)) &= d_T(B,X) \\ &= \ (1-\mathrm{t})\,d_T(A,B) \\ &= \ (1-\mathrm{t})\,d_T(\varphi(A),\varphi(B)) \end{aligned} \tag{4}$$

which shows that, in general, isometries preserve order along lines and also the relative position of points along lines. □

Theorem 1 shows that isometries leaving the origin fixed permute the vertices of circles.

Theorem 1. *Let $r>0$ and $A(r,0)\,,\ B(0,r)\,,\ C(-r,0)$, and $D(0,\ -r)$ are vertices of the circles S_r. If $\varphi\in \mathrm{Fix}_O(2)$, then φ permutes $\{A,B,C,D\}$, the set all vertices of S_r.*

Proof. Given $\varphi(O)=O$, it follows $\varphi(S_r)\subset S_r$. Let

$$A'=\varphi(A)\,,B'=\varphi(B)\,,C'=\varphi(C)\,,\ D'=\varphi(D)\,.$$

It suffices to show that $\varphi(A)=A'\in\{A,B,C,D\}$. Suppose that $A'\notin\{A,B,C,D\}$. Lemma 1 yields

$$d_T(A,B)=d_T(A,C)=d_T(A,D)=2r.$$

and thus,

$$\begin{aligned} d_T(A',B') &= T(A',C') \\ &= d_T(A',D') \\ &= 2r. \end{aligned} \tag{5}$$

Since A' is a not vertex, using Lemma 1, we obtain that $B',\ C'$, and D' must be on a segment of S_r across the segment that contain A'. The fact that φ is isometry, and thus injective, gives us that $B',\ C'$, and D' are three distinct points. Since $B',\ C'$, and D' are points on the same segment of the circle, consequently, for some X and Y in $\{B',C',D'\}$,

$$d_T(X,Y)<2r,$$

which contradicts with eq. (5). Thus, φ maps vertex to a vertex. Furthermore, since φ is injective, φ permutes $\{A,B,C,D\}$. □

Theorem 2. *Suppose $\varphi\in\mathrm{Fix}_O(2)$. If φ leaves all vertices of S_r fixed, then each point on the x-axis and y-axis is a fixed-point of φ.*

Proof. Let φ be any isometry such that $\varphi(O)=O$. Let A,B,C,D be vertices of S_r. Suppose that all vertices of S_r are also fixed-points of φ. It suffices to show that all points on the *x*-axis are fixed-point of φ.

Let $X(x,0)$ be any point on the *x*-axis. Without loss of generality, we may assume that $X\neq A$ and $X\neq C$. Suppose that $\varphi(X)=X'(u,v)$. Since φ isometry, we have that

$$\begin{aligned} d_T(B,X) &= |x|+|r| \\ &= d_T(B,X') \\ &= |u|+|v-r| \\ d_T(D,X) &= |x|+|r| \\ &= d_T(D,X') \\ &= |u|+|v+r|\,. \end{aligned}$$

Since $r>0$, it follows that $v=0$. Thus, X' must be a point on the *x*-axis.

$$\begin{aligned} d_T(A,X) &= |r-x| \\ &= d_T(A,X') \\ &= |r-u| \end{aligned} \tag{6}$$

$$\begin{aligned} d_T(C,X) &= |r+x| \\ &= d_T(C,X') \\ &= |-r-u| \\ &= |r+u|\,. \end{aligned} \tag{7}$$

First, consider the case where $|x|<r$. Eq. (6) and eq. (7), respectively, yields

$$\begin{aligned} d_T(A,X) &= r-x = d_T(A,X') = |r-u| \\ d_T(C,X) &= r+x = d_T(D,X') = |r+u|\,. \end{aligned}$$

Since $|x|<r$, it follows that $u\neq r$ and $u\neq -r$. If $u>r$, then $d_T(A,X')=u-r$ and $d_T(A,X')=r+u$. The above equations give us

$$\begin{aligned} r-x &= u-r \\ x+r &= r+u \end{aligned}$$

and therefore $r=u=x$. This contradicts the assumption that $|x|<r$. Thus $u<r$. Similarly, we can show that $u>-r$ and consequently, $|u|<r$. Using eq. (6) and eq. (7) we obtain that

$$\begin{aligned} r-x &= r-u \\ x+r &= r+u. \end{aligned}$$

We just show that $u=x$ and hence $X'=X$.

Now consider the case where $x>r$. In this case, the eq. (6) and eq. (7), respectively, yield

$$\begin{aligned} d_T(A,X) &= x-r = d_T(A,X') = |r-u| \\ d_T(D,X) &= r+x = d_T(D,X') = |r+u|\,. \end{aligned}$$

If $|u|<r$, then

$$\begin{aligned} x-r &= r-u \\ x+r &= r+u. \end{aligned}$$

So, $r=u=x$. This contradicts the fact that $x>r$. On the other hand, if we assume that $u<-r$, the eq. (6) and eq. (7) simplifies to

$$\begin{aligned} x-r &= r-u \\ x+r &= -r-u \end{aligned}$$

and hence $r=0$ and $u=-x$. This also contradicts the fact that $r>0$. Therefore, we have that $u\geq r$ and thus using eq. (6) and eq. (7), we can infer that

$$
\begin{aligned}
x - r &= r - u \\
x + r &= r + u
\end{aligned}
$$

which shows that $x = u$. Therefore $X = X'$.

Similar to the case $x > r$, we can show that if $x < -r$, that $X' = X$. □

Theorem 3 sheds light on the importance of isometry, which leaves all the vertices of a circle fixed.

> **Theorem 3.** *Let φ be any isometry in $\text{Fix}_O(2)$. If all vertices of S_r are fixed-points of φ, then every point on S_r is a fixed-points of φ.*

Proof. It suffices to show that every interior of $\overline{AB}$ is fixed-point of φ. Let

$$X(t) = A + t(B - A) = (r, 0) + t(-r, r) = ((1-t)r, tr),$$

with $0 < t < 1$, be any interior point of $\overline{AB}$. Since φ is an isometry,

$$
\begin{aligned}
d_T(A, X) = \; d_T(A, X') &= |(1-t)r - r| + |0 - tr| \\
&= 2tr < 2r \\
d_T(B, X) = \; d_T(B, X') &= |(1-t)r - r| + |tr - r| \\
&= 2(1-t)r < 2r
\end{aligned}
$$

Since $d_T(A, X') < 2r$, using Lemma 2, we have that $X' \in \overline{AB} \cup \overline{DA}$. Similarly, we also have that $X' \in \overline{AB} \cup \overline{BC}$. Therefore, we have that $X' \in \overline{AB}$. Since $d_T(A, X) = d_T(A, X')$ dan $d_T(B, X) = d_T(B, X')$, it is easy to see that $X = X'$. □

> **Corollary 1.** *Let φ be any isometry in $\text{Fix}_O(2)$. If all vertices of S_r are fixed-points of φ, for some $r > 0$, then φ is an identity.*

Proof. Let $r > 0$ and $A(r, 0)$, $B(0, r)$, $C(-r, 0)$ and $D(0, -r)$ are vertices of S_r. Suppose that $O(0, 0)$, $A(r, 0)$, $B(0, r)$, $C(-r, 0)$ and $D(0, -r)$ are fixed-points of φ. Theorem 3 shows that all points on the *x*-axis and *y*-axis are also fixed-points of φ. Then, for all $t > 0$, all vertices of S_t are fixed-points of φ and consequently, all points S_t are also fixed-points of φ. We just show that φ is identity. □

Corollary 1 enables us to identify an isometry based on how it behaves on the vertices of a circle.

> **Theorem 4.** *For any φ and ψ isometries in $\text{Fix}_O(2)$, if there exist $r > 0$ such that*
>
> $$\varphi(r, 0) = \psi(r, 0), \varphi(0, r) = \psi(0, r), \varphi(-r, 0) = \psi(-r, 0),$$
>
> *and* $\varphi(0, -r) = \psi(0, -r)$*, then $\varphi = \psi$.*

Proof. If $\alpha = \psi^{-1} \circ \varphi$, then $O(0, 0)$, $(r, 0)$, $(0, r)$, $(-r, 0)$ and $(0, -r)$ are fixed-points of α and thus α identity. Consequently, $\varphi = \psi$. □

Theorem 4 says that isometries are uniquely determined by their behavior on the vertices of a circle. This result is essential to the problem of identifying isometries. We can locate an isometry, leaving the origin fixed based on how it permutes the vertices of any circle.

Let A, B, C, and D be the vertices of the circle S_r and μ be any permutation of $\{A, B, C, D\}$. We need to find a condition that there exists an isometry φ such that the

$$\varphi|_{A,B,C,D} = \mu.$$

In this case, we say that μ gives rise to the isometry φ. This would enable us to completely determine all isometries in $\text{Fix}_O(2)$. Two points X and Y on the circle S_r are called **neighbors** if the line segment $\overline{XY} \subset S_r$. Two points X and Y are **opposite**, if they are not neighbors.

> **Theorem 5.** *Let $\varphi \in \text{Fix}_O(2)$ and $\mathbb{C}$ be the set of vertices of S_r. If $\varphi|_{\mathbb{C}}$ permutes $\mathbb{C}$, then φ preserves neigborliness.*

Proof. Suppose vertices A and B are neighbors. Assume that $\varphi(A)$ and $\varphi(B)$ opposites. Then

$$\{\varphi(A), \varphi(B)\} = \{A, C\} \text{ or } \{\varphi(A), \varphi(B)\} = \{B, D\}.$$

Without loss of generality, we may assume that $\varphi(A), \varphi(B) = A, C$. Furthermore, assume that $\varphi(A) = C$ and $\varphi(B) = A$. Let

$$M = \frac{1}{2}A + \frac{1}{2}B = \left(\frac{r}{2}, \frac{r}{2}\right) \in \overline{AB}$$

be midpoint of A and B.

$$d_T(A, M) = d_T(M, B) = \frac{r}{2}.$$

Suppose $\varphi(M) = N(u, v)$. Since φ is an isometry,

$$
\begin{aligned}
\frac{r}{2} = \; d_T(A, M) &= d_T(\varphi(A), \varphi(M)) \\
&= d_T(C, \varphi(M)) \\
&= |u + r| + |v| \\
\frac{r}{2} = \; d_T(M, B) &= d_T(\varphi(M), \varphi(B)) \\
&= d_T(\varphi(M), A) \\
&= |u - r| + |v|.
\end{aligned}
$$

We conclude that $u = 0$, since $r > 0$. Given that $\varphi(O) = O$, we may deduce that

$$\varphi(O, M) = r = \varphi(O, \varphi(M)) = |v|.$$

Hence $v = \pm r$ implies that either $\varphi(M) = B$ or $\varphi(M) = D$. This contradicts the fact that φ permutes $\{A, B, C, D\}$.

Assuming that $\phi(A) = A$ and $\phi(B) = C$, we can also arrive at contradiction using similar strategy. The same result, is also obtained if we suppose that $\{\varphi(A), \varphi(B)\} = \{B, D\}$. As a result, $\varphi(A)$ and $\varphi(B)$ are neighbors.

Similar reasoning can be used to show isometries neighborliness of vertices. For a pair X and Y, that neighbors on S_r, we can use Lemma 3 to show that $\varphi(X)$ and $\varphi(Y)$ are neighbors on S_r. □

We use Theorem 4 and Theorem 5 to determine the set of all isometries, leaving the origin fixed. First, use Theorem 5 to determine whether one permutation of vertices gives rise to isometry or not and, secondly, use Theorem 4 to determine the isometry that arises from one particular permutation.

The number of permutations of n distinct objects is $n! = 24$. By using Theorem 5, we can show there are 16 permutations that do not give rise to isometries. The following is the list of all such permutations,

$$\begin{gathered}(AB), (BC), (CD), (AD),\\ (ABC), (ACB), (ABD), (ADB),\\ (ACD), (ADC), (BCD), (BDC),\\ (ABDC), (ACBD), (ACDB), (ADBC).\end{gathered}$$

which is further represented in Figure 3.

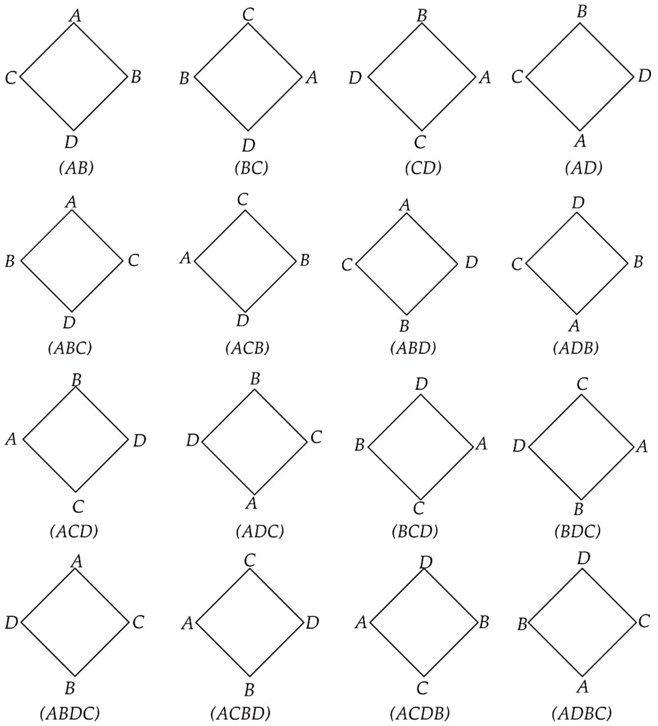

Figure 3. Permutations that do not give rise to isometries

The following are eight permutations that give rise to isometries,

$$\begin{array}{llllll}(AC) & \Leftrightarrow & \Omega_{y-axis} & (ABCD) & \Leftrightarrow & \rho_{\frac{\pi}{2}}\\ (BD) & \Leftrightarrow & \Omega_{x-axis} & (AC)(BD) & \Leftrightarrow & \rho_{\pi}\\ (AB)(CD) & \Leftrightarrow & \Omega_{y=x} & (ADCB) & \Leftrightarrow & \rho_{\frac{3\pi}{2}}\\ (AD)(BC) & \Leftrightarrow & \Omega_{y=-x} & (A) & \Leftrightarrow & \rho_{2\pi}\end{array}$$

which is further represented in Figure 4.

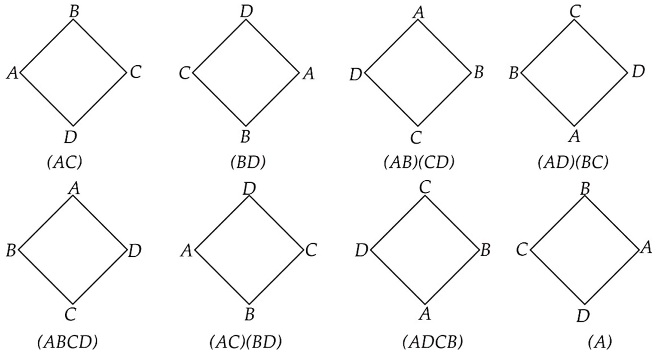

Figure 4. Permutations that give rise to isometries

Theorem 6.

$$\text{Fix}_O(2) = \{\Omega_{x-axis}, \Omega_{y-axis}, \Omega_{y=x}, \Omega_{y=-x}, \rho_{\frac{\pi}{2}}, \rho_{\pi}, \rho_{\frac{3\pi}{2}}, Id\}$$

The set $\text{Fix}_O(2)$ under the function composition is a group isomorphic to the dihedral group, $\text{Fix}_O(2) \cong D4$.

3.3. Isometries of Taxicab Geometry

Let $\tau_{\mathbf{v}} : \mathbb{R}^2 \to \mathbb{R}^2$ be a translation by vector $\mathbf{v} = (a, b)$. For any $X(x_1, x_2)$ and $Y(y_1, y_2)$

$$\begin{aligned}d_T(\tau_{\mathbf{v}}(X), \tau_{\mathbf{v}}(Y)) &= |(y_1 + a) - (x_1 + a)| + |(y_2 + b) - (x_2 + b)|\\ &= |x_1 - y_1| + |x_2 - y_2| = d_T(X, Y).\end{aligned}$$

Thus, we show the Theorem 7.

Theorem 7. *Every translation is an isometry*

Proof. We identify all isometries by showing isometries are motions and vice versa. □

Definition 1. *An isometry that is a product of a finite number of reflections is called a motion.*

We have shown that reflections in axes are isometries. We can show that reflections on vertical lines and reflections on horizontal lines are isometries. For any vertical line $x = a$,

$$\begin{aligned}d_T(\Omega_{x=a}(x_1, x_2), \Omega_{x=a}(y_1, y_2)) &= d_T((2a - x_1, x_2),\\ &\quad (2a - y_1, y_2))\\ &= |2a - y_1 - (2a - x_1)| +\\ &\quad |y_2 - x_2|\\ &= |x_1 - y_1| + |y_2 - x_2|\\ &= d_T((x_1, x_2), (y_1, y_2)).\end{aligned}$$

Let $\varphi = \Omega_{y=x}$ be a reflection in the line $y = x$. For any $X(x_1, x_2)$ and $Y(y_1, y_2)$

$$\begin{aligned} d_T\left(\Omega_{y=x}(X), \Omega_{y=x}(Y)\right) &= d_T\left((x_1, x_2), \Omega_{x=p}(y_1, y_2)\right) \\ &= |y_1 - y_2| + |x_1 - x_2| \\ &= d_T(X, Y). \end{aligned}$$

Similarly, reflection in the line $y = -x$ is also an isometry.

Theorem 8. *Every non trivial translation is a product of either two or four reflections.*

Proof. Let $\varphi = \tau_{\mathbf{v}}$ be a translation by a vector $\mathbf{v} = (a, b)$.
$\varphi(X) = X + \mathbf{v} = (x_1 + a, x_2 + b)$, with $X = (x_1, x_2)$.
If $b = 0$, then for any $X(x_1, x_2)$,

$$\begin{aligned} \tau_{\mathbf{v}}(X) &= (x_1, x_2) + (a, 0) \\ &= (x_1 + a, x_2) \\ &= \left(2\left(\frac{a}{2}\right) - (-x_1), x_2\right) \\ &= \Omega_{x=\frac{a}{2}}(-x_1, x_2) \\ &= \Omega_{x=\frac{a}{2}}(\Omega_{x=0}(X)) \\ &= \Omega_{x=\frac{a}{2}} \circ \Omega_{x=0}(X). \end{aligned}$$

In similar way, we can show that $\tau_{(0,b)} = \Omega_{y=\frac{b}{2}} \circ \Omega_{y=0}$. If $a \neq 0$ and $b \neq 0$, then

$$\tau_{(a,b)} = \tau_{(a,0)} \circ \tau_{(0,b)} = \Omega_{x=\frac{a}{2}} \circ \Omega_{x=0}\Omega_{y=\frac{b}{2}} \circ \Omega_{y=0}.$$

□

Theorem 9. *If φ is an isometry leaving the origin fixed, then φ is a motion.*

We determine all isometries of taxicab plane geometry in the following theorem. The proof of the theorem shows the structure of the isometry group and how each isometry breaks down into more basic isometries: reflections in lines parallel to x-axis, y-axis, $y = x$, or $y = -x$.

Theorem 10. *Every isometry is a motion.*

Proof. Let φ be any isometry. We consider several cases.

1. Case 1. $\varphi(O) = O$. Then in this case $\varphi \in I_O$ and thus a motion.
2. Case 2. $\varphi(P) = P$ for some $P \neq O$. Then $\tau_{-P} \circ \varphi \circ \tau_P$ is an isometry leaving O fixed and thus it is motion by Case 1.
3. Case 3. φ has no fixed-point. Let $P = \varphi(O)$. Then, $\psi = \tau_{-P} \circ \varphi$ is an isometry. Since $\psi(O) = \tau_{-P} \circ \varphi(O) = \tau_{-P}(P) = O$, then $\psi \in I_O$ and therefore a motion.

Consequently, $\varphi = \tau_P \circ \psi$ is also a motion. □

Let $\mathrm{Fix}_P(2)$ denotes the group of isometries leaving a point P fixed. The above proof shows that $\mathrm{Fix}_P(2)$ is a conjugate of $\mathrm{Fix}_O(2)$, because $\mathrm{Fix}_P(2) = \tau_{-P}\mathrm{Fix}_O(2)\,\tau_P$. Therefore, for any X and Y,

$$\mathrm{Fix}_X(2) \cong \mathrm{Fix}_Y(2).$$

Corollary 2. *For every point X on the plane, the group of isometries leaving X fixed, $\mathrm{Fix}_X(2)$, is isomorphic to dihedral group D_4:*

$$\mathrm{Fix}_X(2) \cong D_4.$$

Note that Case 2 can be handled as part of Case 3. Consequently, of all isometries, $\mathrm{Isom}_{d_T}(2)$ preserving taxicab distance, is a semidirect-product of group T of all translation and the symmetry group $\mathrm{Fix}_O(2)$, as it is mentioned in [4].

Corollary 3. *$\mathrm{Isom}_{d_T}(2)$ is a semidirect-product of T and $\mathrm{Fix}_O(2)$.*

4. Conclusion

The main finding of this work is that the way isometries in taxicab geometry a circle's vertices uniquely identifies them. This result paves the way to utilizing combinatorial reasoning to identify all isometries in taxicab plane geometry.

Additionally, we were able to delve farther into the algebraic structure of $\mathrm{Isom}_{d_T}(2)$. We show that $\mathrm{Isom}_{d_T}(2)$ a semidirect product of T and $\mathrm{Fix}_O(2)$. We further show that for every point on the plane, every group of isometries leave the point fixed is isomorphic to dihedral group.

Author Contributions. **O. Neswan** as the main author, provides and develops ideas and concepts, as well as outlines the proof of key results. **H. Sumartono** as the second author, provides details of the proofs. All authors discussed the results and contributed to the final manuscript.

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