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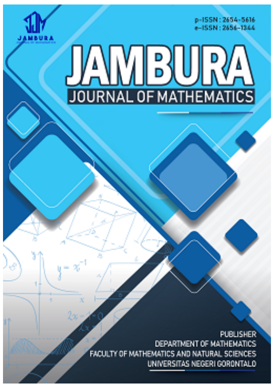

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Research Article

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The *k*- Tribonacci Matrix and the Pascal Matrix

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ABSTRACT. *This article discusses the relationship between the k-Tribonacci matrix $\mathcal{T}_n(k)$ and the Pascal matrix P_n, by first constructing the k-Tribonacci matrix and then looking for its inverse. From the inverse k-Tribonacci matrix, unique characteristics can be constructed so that general shapes can be constructed, and then from the relationship of the k-Tribonacci matrix $\mathcal{T}_n(k)$ and the Pascal matrix P_n obtain a new matrix, i.e. $\mathcal{U}_n(k)$. Furthermore, a factor is derived from the relationship of the k-Tribonacci matrix $\mathcal{T}_n(k)$ and the Pascal matrix P_n i.e. $P_n = \mathcal{T}_n(k)\mathcal{U}_n(k)$.*

1. Introduction

The tribonacci sequence is one of the generalizations of the Fibonacci sequence which each subsequent tribe is derived by summing up the three previous tribes beginning with the tribes 0, 0 and 1. Then, the tribes of the tribonacci sequence are called the tribonacci numbers and are denoted as $T_n, \forall n \epsilon \mathbb{N}$. In his book Koshy [1] gives the form of the tribonacci l sequence as follows:

$$0,\ 0,\ 1,\ 1,\ 2,\ 4,\ 7,\ 13,\ 24,\ 44,\ 81,\ 149,\ 274,\ \cdots\ .$$

The tribonacci sequence was originally studied by Feinbreg [2] in 1963, after 760 years of Fibonacci sequences introduced by Leonardo Pisano. Early in 1964 the tribonacci sequence had not so much appealed to some authors as the Fibonacci sequence, but with the passage of the tribonacci ranks undergoing development, it is evident that some authors have studied and developed a row of tribonacci marks in different contexts in various articles. Lather and Kumar [3] discuss the generalization of the tribonacci squances, Ramirez [4] obtain the relationship between the tribonacci numbers and the Pascal triangle, Kizilaslan [5] gives the form algebra of the generalized tribonacci matrix.

On the other side there is the Pascal's triangle, the numbers on the Pascal's triangle are binomial coefficients arranged in triangular form and combinatorically each element of the Pascal's triangle is coefficients in the form of simplification of the form of the sum of two numbers expressed as $(x+y)^n$, $\forall n \epsilon \mathbb{N}$. Pascal's triangle can be represented as a square matrix, with each element being a number on Pascal's triangle so that this matrix is called Pascal's matrix. Brawer and Pirovino [6] provide a form of matrix representation of the Pascal's triangle and discusses the algebraic form of the Pascal's matrix. Pascal matrix is an adjoint operator of the diferensial operator of translation [7].

Matrix factorizations discussed in several articles include the relationship between the matrix of the tribonacci and the Pascal matrix [8], the Pascal matrix with tetranacci matrix [9, 10], the first type of Stirling matrix with the tetranacci matrix [11], the second type of Stirling matrix with the tribonacci matrix [12], and the second type of Stirling matrix with the *k*-Fibonacci matrix [13]. They construct a new matrix so as to reveal the relationship of these matrices.

In contrast to the discussion in research [8–13], in this paper, we construct a $k-$tribonacci matrix is a generalization of the tribonacci matrix and discusses the relation between the $k-$tribonacci matrix and Pascal's matrix. From the relation of the $k-$tribonacci matrix and the Pascal matrix by using algebraic calculations, the definition of a new matrix is matrix $\mathcal{U}_n(k)$ Furthermore, from the definition of the a new matrix, there are several relation expressed in the theorem.

2. Methods

The method used in this research is a literature study which refers to several articles discussing the *k*-tribonacci matrix and the Pascal matrix. The following are the steps taken in this research:

1. Construct and generalizes the form of the *k*-tribonacci matrix $\mathcal{T}_n(k)$
2. Construct and generalizes the form of the inverse *k*-tribonacci matrix $\mathcal{T}_n^{-1}(k)$
3. Determine the result of multiplying the inverse *k*-tribonacci matrix $\mathcal{T}_n^{-1}(k)$ and the Pascal P_n obtain a new matrix. Then define the new matrix, namely matrix $\mathcal{U}_n(k)$.
4. Show that the Pascal matrix P_n is the product of the *k*-tribonacci matrix $\mathcal{T}_n(k)$ and the matrix $\mathcal{U}_n(k)$, so that $P_n = \mathcal{T}_n(k)\ \mathcal{U}_n(k)$ is obtained.
5. Construct and define the inverse matrix $\mathcal{U}_n^{-1}(k)$.
6. Show that the *k*-tribonacci matrix $\mathcal{T}_n(k)$ is the product of the Pascal matrix P_n and the inverse matrix $\mathcal{U}_n^{-1}(k)$, so that $\mathcal{T}_n(k) = P_n\mathcal{U}_n^{-1}(k)$ is obtained.

The following are the basic theories used in this research. Definition 1, Theorem 1, and Theorem 2 refer to [14] used to prove the result in this research.

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Definition 1. If A is $n \times n$ matrix, and if a matrix B of the same ordo can be obtained such that $AB = BA = I$, then A is Invertible and B the invers of A.

Theorem 1. *If A and B are $n \times n$ matrices, then* $\det(A \cdot B) = \det(A) \cdot \det(B)$.

Proof. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, then

$\det(A) = a_{11}a_{22} - a_{12}a_{21}$ and $\det(B) = b_{11}b_{22} - b_{12}b_{21}$.

Next, it is shown that:

$$\begin{aligned} A \cdot B &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix} \end{aligned}$$

so that

$$\begin{aligned} \det(A \cdot B) &= (a_{11}b_{11} + a_{12}b_{21})(a_{21}b_{12} + a_{22}b_{22}) \\ &\quad - (a_{11}b_{12} + a_{12}b_{22})(a_{21}b_{11} + a_{22}b_{21}) \\ &= a_{11}a_{22}b_{11}b_{22} + a_{12}a_{21}b_{12}b_{21} - a_{11}a_{22}b_{12}b_{21} \\ &\quad - a_{12}a_{21}b_{11}b_{22} \\ &= (a_{11}a_{22} - a_{12}a_{21})(b_{11}b_{22} - b_{12}b_{21}) \\ &= \det(A) \cdot \det(B). \end{aligned}$$

□

Theorem 2. *A is a $n \times n$ matrix invertible if and only if* $\det(A) \neq 0$.

Proof. It will prove that if A invertible, then $\det(A) \neq 0$, and if $\det(A) \neq 0$, then A invertible.

(i) $(\Rightarrow)$ If A invertible, then $AA^{-1} = 1$. If the determinant is taken, then

$$\det\left(AA^{-1}\right) = \det(I) = 1 \tag{1}$$

According to Theorem 1, we obtained

$$\det\left(AA^{-1}\right) = \det(A) \cdot \det\left(A^{-1}\right).$$

From equation (1) we obtained $\det(A) \cdot \det\left(A^{-1}\right) = 1$, thus $\det(A) \neq 0$.

(ii) If $det(A) \neq 0$, then

$$A\,(adj\ A) = \det(A)\,.\,I \tag{2}$$

Dividing both sides with $det(A)$ for equation (2), we obtained

$$A\left(\frac{1}{\det(A)} adj\ A\right) = I \tag{3}$$

From equation (3), we obtained $AA^{-1} = 1$, thus A invertible.

□

3. Results and Discussion

Pascal's triangle are binomial coefficients arranged in triangular form. The binomial coefficients present in the Pascal triangle are the result of the sum of the sum of two numbers namely $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$ for n positive integers. In addition, in his book Bona [15] mentioned that combinatorically every element on the Pascal triangle line can be expressed as

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

with $0 \leq k \leq n$ and usually the number pattern is presented as shown in Figure 1.

$$\begin{array}{ccccccccccc} & & & & & \binom{0}{0} & & & & & \\ & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\ & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\ & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\ & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\ \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \\ \vdots & & & & & \vdots & & & & & \vdots \\ \binom{n}{0} & \binom{n}{1} & \cdots & \binom{n}{k-1} & & \cdots & & \binom{n}{k} & \cdots & & \binom{n}{n} \end{array}$$

Figure 1. Pascal's triangle pattern

Pascal's triangle can be represented in the square matrix form of the lower triangular matrix with each element being a number on Pascal's triangle, so this matrix is called Pascal's matrix and is denoted by P_n, $\forall n \epsilon \mathbb{N}$. In general the form of Pascal P_n matrix can be expressed in the following matrix form,

$$P_n = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 \\ 1 & 3 & 3 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & p_{n3} & p_{n4} & \cdots & p_{nn} \end{bmatrix} \tag{4}$$

with

$$p_{n1} = \begin{pmatrix} n-1 \\ 0 \end{pmatrix}, \quad p_{n2} = \begin{pmatrix} n-1 \\ 1 \end{pmatrix}, \quad p_{n3} = \begin{pmatrix} n-1 \\ 2 \end{pmatrix}$$
$$p_{n4} = \begin{pmatrix} n-1 \\ 3 \end{pmatrix}, \quad p_{nn} = \begin{pmatrix} n-1 \\ n-1 \end{pmatrix}$$

Then, Sabeth et al. [8] describes the form of a Pascal matrix of $n \times n, \forall n \epsilon \mathbb{N}$ with each element of the Pascal matrix $P_n = [p_{ij}]$, $\forall i, j = 1, 2, 3, \cdots, n$ can be expressed and defined as

$$p_{ij} = \begin{cases} \begin{pmatrix} i-1 \\ j-1 \end{pmatrix} & , \text{ for } \quad i \geq j, \\ 0 & , \text{ for } \quad i < j. \end{cases} \tag{5}$$

Furthermore, gives the general form of the inverse of the Pascal matrix of $n \times n, \forall n \epsilon \mathbb{N}$ with each element of the inverse matrix Pascal $P_n^{-1} = \left[p_{ij}^{'}\right], \forall i, j = 1, 2, 3, \cdots, n$ can be expressed and defined as

$$p'_{ij} = \begin{cases} (-1)^{i+j}\begin{pmatrix} i-1 \\ j-1 \end{pmatrix} & \text{, for } \quad i \geq j, \\ 0 & \text{, for } \quad i < j. \end{cases} \tag{6}$$

The $k-$tribonacci sequence to one of the generalizations of the tribonacci sequence with the three original tribes is equal to the tribonacci sequence. In contrast to the Fibonacci sequence to, the $k-$tribonacci sequence to the still deserted devotees to be developed in the field of modern mathematics. Lather and Kumar [3] provide a recursive form of the $k-$tribonacci sequence to which is denoted by $T_{k,n}$ as.

$$T_{k,n} = \begin{cases} 0 & \text{, for } \quad n = 0,\ 1; \\ 1 & \text{, for } \quad n = 2; \\ kT_{k,n-1} + T_{k,n-2} + T_{k,n-3} & \text{, for } \quad n \geq 3. \end{cases} \tag{7}$$

Using the equation (7), the following is shown the form of the $k-$tribonacci number into Table 1.

Table 1. $k-$tribonacci numbers

n	0	1	2	3	4	5	6
$T_{k,n}$	0	0	1	k	k^2+1	k^3+2k+1	k^4+3k^2+2k+1
$T_{1,n}$	0	0	1	1	2	4	7
$T_{2,n}$	0	0	1	2	5	13	33
$T_{3,n}$	0	0	1	3	10	34	115
$T_{4,n}$	0	0	1	4	17	73	313
$T_{5,n}$	0	0	1	5	26	136	711
$T_{6,n}$	0	0	1	6	37	229	1417
$\vdots$				$\vdots$			$\vdots$

In this papper will be discussed some of the results obtained, including: defining the $k-$tribonacci matrix, the characteristics of the inverse $k-$tribonacci matrix and the relation between the $k-$tribonacci matrix and the Pascal matrix. Similar to the tribonacci sequence, the $k-$tribonacci sequences can also be represented in square matrices. With the same idea to define the tribonacci matrix by examining the idea of Sabeth et al. [8], the $k-$tribonacci sequence to be represented in the form of a lower triangular matrix with each element being the $k-$tribonacci number to and the main diagonal 1, so this matrix is called the $k-$tribonacci matrix to and denoted by $\mathcal{T}_n(k)$, $\forall n \epsilon \mathbb{N}$. In general, the shape of the $k-$tribonacci matrix can be expressed in the following matrix form

$$\mathcal{T}_n(k) = \begin{bmatrix} T_{k,2} & 0 & 0 & 0 & \cdots & 0 \\ T_{k,3} & T_{k,2} & 0 & 0 & \cdots & 0 \\ T_{k,4} & T_{k,3} & T_{k,2} & 0 & \cdots & 0 \\ T_{k,5} & T_{k,4} & T_{k,3} & T_{k,2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ T_{k,n} & T_{k,n-1} & T_{k,n-2} & T_{k,n-3} & \cdots & T_{k,2} \end{bmatrix}.$$

Then we get the general form of the $k-$tribonacci matrix to the order of $n \times n$, $\forall n \epsilon \mathbb{N}$ with each element of the $k-$tribonacci matrix to $\mathcal{T}_n^{-1}(k) = \left[t'_{ij}\right]$, $\forall i, j = 1,\ 2,\ 3,\ \cdots,\ n$ can be expressed and defined as

$$t_{i,j} = \begin{cases} T_{k,i-j+2} & \text{, for } \quad i \geq j, \\ 0 & \text{, for } \quad i < j. \end{cases} \tag{8}$$

Because the determinant of the $k-$tribonacci matrix $\mathcal{T}_n(k) \neq 0$, the $k-$tribonacci matrix $\mathcal{T}_n(k)$ has an invers. So, by calculating the pattern characteristics obtained from the inverse $k-$tribonacci matrix notated $\mathcal{T}_n^{-1}(k)$. Characteristics obtained from the inverse $k-$tribonacci matrix $\mathcal{T}_n^{-1}(k)$ ie each column contains a value $1, -k, -1, -1$ and the next value is 0 as seen in the following matrix,

$$\mathcal{T}_5^{-1}(k) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -k & 1 & 0 & 0 & 0 \\ -1 & -k & 1 & 0 & 0 \\ -1 & -1 & -k & 1 & 0 \\ 0 & -1 & -1 & -k & 1 \end{bmatrix}.$$

Then we obtain the general form of the inverse $k-$tribonacci matrix to $n \times n$, $\forall n \epsilon \mathbb{N}$ with each element of the inverse triblon matrix to $\mathcal{T}_n^{-1}(k) = \left[t'_{ij}\right]$, $\forall i, j = 1,\ 2,\ 3,\ \cdots, n$ can be expressed and defined as

$$t'_{ij} = \begin{cases} 1 & \text{, for } \quad i = j, \\ -k & \text{, for } \quad i-1 = j, \\ -1 & \text{, for } \quad i-3 \leq j \leq i-2, \\ 0 & \text{, for } \quad \text{else}. \end{cases}$$

Using the same idea from Sabeth et al. [8], a relationship between the $k-$tribonacci matrix to and the Pascal matrix will be established. Then, from the relationship of the two matrices the definition of a new matrix is obtained, the new matrix is the matrix $\mathcal{U}_n(k)$ defined as the lower triangular matrix. The element element of the matrix $\mathcal{U}_n(k)$ is obtained from the matrix product $\mathcal{T}_n^{-1}(k)$ with the matrix P_n.

Let $n = 2$, obtained elements for the matrix $\mathcal{U}_2(k)$ as follows:

$$\mathcal{T}_2^{-1}(k) \cdot P_2 = \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -k+1 & 1 \end{bmatrix}.$$

Let $n = 3$, obtained elements for the matrix $\mathcal{U}_3(k)$ as follows:

$$\mathcal{T}_3^{-1}(k) \cdot P_3 = \begin{bmatrix} 1 & 0 & 0 \\ -k & 1 & 0 \\ -1 & -k & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -k+1 & 1 & 0 \\ -k & 2-k & 1 \end{bmatrix}.$$

and suppose using $n = 5$, the element element for the matrix $\mathcal{U}_5(k)$ is obtained as follows:

$$\mathcal{T}_5^{-1}(k) \cdot P_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -k & 1 & 0 & 0 & 0 \\ -1 & -k & 1 & 0 & 0 \\ -1 & -1 & -k & 1 & 0 \\ 0 & -1 & -1 & -k & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -k+1 & 1 & 0 & 0 & 0 \\ -k & 2-k & 1 & 0 & 0 \\ -1-k & 2-2k & 3-k & 1 & 0 \\ -1-k & 1-3k & 5-3k & 4-k & 1 \end{bmatrix}. \tag{9}$$

Table 2. Element values for the $\mathcal{U}_5\left(k\right)$ matrix

Matrix element $\mathcal{U}_5\left(k\right)$	Matrix element value $\mathcal{U}_5\left(k\right)$
u_{11}	$(1)\binom{1-1}{1-1}=\binom{0}{0}$
u_{21}	$(-k)\binom{1-1}{1-1}+(1)\binom{2-1}{1-1}=-k\binom{0}{0}+\binom{1}{0}$
u_{22}	$(-k)\binom{1-1}{2-1}+(1)\binom{2-1}{2-1}=-k\left(0\right)+\binom{1}{1}$
u_{31}	$(-1)\binom{1-1}{1-1}+(-k)\binom{2-1}{1-1}+(1)\binom{3-1}{1-1}=-\binom{0}{0}-k\binom{1}{0}+\binom{2}{0}$
u_{32}	$(-1)\binom{1-1}{2-1}+(-k)\binom{2-1}{2-1}+(1)\binom{3-1}{2-1}=-1\left(0\right)-k\binom{1}{1}+\binom{2}{1}$
$\vdots$	$\vdots$
u_{ij}	$\binom{i-1}{j-1}-k\binom{i-2}{j-1}-\binom{i-3}{j-1}-\binom{i-4}{j-1}$

Thus, based on the matrix multiplication in equation (9) we have the element element for the matrix $\mathcal{U}_5\left(k\right)$ that is,

$$\mathcal{U}_5\left(k\right)=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -k+1 & 1 & 0 & 0 & 0 \\ -k & 2-k & 1 & 0 & 0 \\ -1-k & 2-2k & 3-k & 1 & 0 \\ -1-k & 1-3k & 5-3k & 4-k & 1 \end{bmatrix}. \tag{10}$$

Taking into account each element in the matrix (10) for $i \geq j$, all values of the matrix element are listed as shown in Table 2.

Furthermore, taking into account each algebraic step from Table 2 is obtained the general form of the matrix $\mathcal{U}_n\left(k\right)$ of $n\times n$ is defined as Definition 2.

Definition 2. For each $n\epsilon\mathbb{N}$, $\mathcal{U}_n\left(k\right)$ is a matrix $n\times n$ with each element $\mathcal{U}_n\left(k\right)=\left[u_{ij}\right]$, $\forall i,j=1,\ 2,\ 3,\ \cdots,n$ can be expressed as

$$u_{ij}=\binom{i-1}{j-1}-k\binom{i-2}{j-1}-\binom{i-3}{j-1}-\binom{i-4}{j-1} \tag{11}$$

and $\forall i<j$, $u_{ij}=0$.

Furthermore, using Maple 13 application obtained the inverse matrix $\mathcal{U}_4\left(k\right)$ follows.

$$\mathcal{U}_4^{-1}\left(k\right)=\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1+k & 1 & 0 & 0 \\ k^2-2k+2 & k-2 & 1 & 0 \\ k^3-3k^2+5k-3 & k^2-3k+4 & k-3 & 1 \end{bmatrix}. \tag{12}$$

Notice each element in the matrix (12) for $i\geq j$ all the values of the matrix element are listed as shown in the Table 3.

Further, taking into account each of the algebraic steps of Table 3 is obtained the general form of the matrix $\mathcal{U}_n^{-1}\left(k\right)$ of the n $\times$ n order defined as Definition 3.

Definition 3. For each $n\epsilon\mathbb{N}$, $\mathcal{U}_n^{-1}\left(k\right)$ is a matrix $n\times n$ with each element $\mathcal{U}_n^{-1}\left(k\right)=\left[u'_{ij}\right]$, $\forall i,j=1,\ 2,\ 3,\ \cdots,n$ can be expressed as

$$u'_{ij}=\sum_{t=j}^{i}\left(-1\right)^{i+j}\binom{i-1}{t-1}T_{k,t-j+2} \tag{13}$$

and $\forall i<j$, $u'_{ij}=0$.

From defining the matrix $\mathcal{U}_n\left(k\right)$ in equation (11) can be derived the following Theorem.

Theorem 3. *For each $n\epsilon\mathbb{N}$ Pascal's matrix P_n defined in equation (5) and the $k-$tribonacci matrix $\mathcal{T}_n\left(k\right)$ defined in equation (6), there is matrix $\mathcal{U}_n\left(k\right)$ defined in equation (11), the Pascal matrix can be expressed $P_n=\mathcal{T}_n\left(k\right)\ \mathcal{U}_n\left(k\right)$.*

Proof. Since the $k-$tribonacci matrix $\mathcal{T}_n\left(k\right)$ has an inverse and $\det\left(\mathcal{T}_n\left(k\right)\right)\neq 0$, the $k-$tribonacci matrix $\mathcal{T}_n\left(k\right)$ is an invertible matrix. It will be proven that

$$\mathcal{U}_n\left(k\right)=\ \mathcal{T}_n^{-1}\left(k\right)\ \ P_n\ . \tag{14}$$

Let $\mathcal{T}_n^{-1}\left(k\right)$ be the inverse of the $k-$tribonacci matrix $\mathcal{T}_n\left(k\right)$ defined in equation (8). Consider the right-hand segment of the equation (14), since the matrix $\mathcal{T}_n^{-1}\left(k\right)$and P_n is the lower triangular matrix with the main diagonal of $\mathcal{T}_n^{-1}\left(k\right)$ and P_n is 1, then the multiplication of the matrix $\mathcal{T}_n^{-1}\left(k\right)$ with P_n also produces the main diagonal i.e. 1 and is the lower triangular matrix as well.

Then consider the left side of equation (14), if $i=j$ then $u_{ij}=1$ and if $i<j$ then $u_{ij}=0$.

$$\begin{aligned} u_{ij} &= t'_{ir}p_{rj} \\ &= t'_{i1}p_{1j}+t'_{i2}p_{2j}\ +t'_{i3}p_{3j}\ +\cdots+t'_{in}p_{nj} \\ &= \sum_{r=1}^{n}t'_{ir}p_{rj}\ \ . \end{aligned}$$

Thus, it is evident that $\mathcal{T}_n^{-1}(k)\ \ P_n\ =\mathcal{U}_n\left(k\right)$. Thus, by the matrix $\mathcal{U}_n\left(k\right)$ in equation (11), the Pascal matrix P_n in equation (5) and $k-$tribonacci matrix $\mathcal{T}_n\left(k\right)$ of equation (8) can be expressed by $P_n=\mathcal{T}_n\left(k\right)\ \ \mathcal{U}_n\left(k\right)$. □

Furthermore, from Theorem 3, it can be derived and stated as Corollary 1.

Table 3. Element values for the $\mathcal{U}_4^{-1}(k)$ matrix

Matrix element $\mathcal{U}_4^{-1}(k)$	Matrix element value $\mathcal{U}_4^{-1}(k)$
u'_{11}	$(-1)^{1+1}\binom{1-1}{1-1}T_{k,1-1+2}=\binom{0}{0}T_{k,2}$
u'_{21}	$(-1)^{3}\binom{2-1}{1-1}T_{k,1-1+2}+(-1)^{4}\binom{2-1}{2-1}T_{k,2-1+2}=-1\binom{1}{0}T_{k,2}+\binom{1}{1}T_{k,3}$
u'_{22}	$(-1)^{3}\binom{2-1}{1-1}T_{k,1-2+2}+(-1)^{4}\binom{2-1}{2-1}T_{k,2-2+2}=-1\binom{1}{0}T_{k,1}+\binom{1}{1}T_{k,2}$
$\vdots$	$\vdots$
u'_{ij}	$\sum_{t=j}^{i}(-1)^{i+j}\binom{i-1}{t-1}T_{k,t-j+2}$

Corollary 1. *For every* $1 \leq j \leq i$ *obtained*

$$\binom{i-1}{j-1}=\sum_{r=j}^{n}T_{k,i-r+2}M_1 \tag{15}$$

with

$$M_1=\left[\binom{r-1}{j-1}-k\binom{r-2}{j-1}-\binom{r-3}{j-1}-\binom{r-4}{j-1}\right].$$

It is sufficient to consider each element of P_n *in the matrix multiplication* $P_n=\mathcal{T}_n(k)\ \mathcal{U}_n(k)$ *algebraically obtained as follows:*

$$\begin{aligned}p_{ij}&=t_{ir}u_{rj}\\&=t_{ij}u_{jj}+t_{ij+1}u_{j+1j}+t_{ij+2}u_{j+2j}+\cdots+t_{in}u_{nj}\\&=T_{k,i-j+2}u_{jj}+T_{k,i-j+1+2}u_{j+1j}+T_{k,i-j+2+2}u_{j+2j}\\&\quad+\cdots+T_{k,i-n+2}u_{nj}\\&=T_{k,i-j+2}u_{jj}+T_{k,i-j+3}u_{j+1j}+T_{k,i-j+4}u_{j+2j}\\&\quad+\cdots+T_{k,i-n+2}u_{nj}\\&=\sum_{r=j}^{n}T_{k,i-r+2}u_{rj}.\end{aligned} \tag{16}$$

Furthermore, using equation (16) is obtained

$$\begin{aligned}p_{ij}&=\sum_{r=j}^{n}T_{k,i-r+2}u_{rj}\,,\\\binom{i-1}{j-1}&=\sum_{r=j}^{n}T_{k,i-r+2}M_2\end{aligned}$$

with

$$M_2=\left[\binom{r-1}{j-1}-k\binom{r-2}{j-1}-\binom{r-3}{j-1}-\binom{r-4}{j-1}\right].$$

Thus, using equation (16) can be derived equation (15).

Furthermore, from defining the matrix $\mathcal{U}_n^{-1}(k)$ in equation (13) the following theorem can be derived.

Theorem 4. *For each* $n\in\mathbb{N}$ *Pascal matrix* P_n *defined in equation (5) and the* $k-$*tribonacci matrix* $\mathcal{T}_n(k)$ *defined in equation (6), there is matrix* $\mathcal{U}_n^{-1}(k)$ *defined in equation (13), the* $k-$*tribonacci matrix can be expressed* $\mathcal{T}_n(k)=P_n\mathcal{U}_n^{-1}(k)$.

Proof. Since the Pascal matrix P_n has an inverse and $\det(P_n)\neq 0$, the Pascal matrix P_n is an invertible matrix. It will be proven that

$$\mathcal{U}_n^{-1}(k)=P_n^{-1}\,\mathcal{T}_n(k)\,. \tag{17}$$

Let P_n^{-1} be the inverse of the Pascal matrix P_n defined in equation (6). Consider the right-hand segment of the equation (17), since the matrix P_n^{-1} and $\mathcal{T}_n(k)$ is the lower triangular matrix with the main diagonal of P_n^{-1} and $\mathcal{T}_n(k)$ is 1 , then the multiplication of the matrix P_n^{-1} with $\mathcal{T}_n(k)$ also produces the main diagonal ie 1 and is the lower triangular matrix as well.

Then consider the left side of equation (17). If $i=j$, then $u_{ij}=1$ and if $i<j$, then $u_{ij}=0$.

$$\begin{aligned}u'_{ij}&=p'_{ir}t_{rj}\\&=p'_{i1}t_{1j}+p'_{i2}t_{2j}+p'_{i3}t_{3j}+\cdots+p'_{in}t_{nj}\\&=\sum_{r=1}^{n}p'_{ir}t_{rj}.\end{aligned}$$

Thus, it is evident that $P_n^{-1}\mathcal{T}_n(k)=\mathcal{U}_n^{-1}(k)$. Thus, by the matrix $\mathcal{U}_n^{-1}(k)$ in equation (13), the Pascal matrix P_n in equation (5) and $k-$tribonacci matrix $\mathcal{T}_n(k)$ of equation (8) can be expressed by $\mathcal{T}_n(k)=P_n\mathcal{U}_n^{-1}(k)$. □

4. Conclusion

In this paper, we get the general form of the $k-$tribonacci matrix and define a new matrix $\mathcal{U}_n(k)$. In addition, we use the a new matrix obtained a factor of the relationship between the $k-$tribonacci matrix $\mathcal{T}_n(k)$ and the Pascal matrix P_n can be expressed as $P_n=\mathcal{T}_n(k)\ \mathcal{U}_n(k)$.

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