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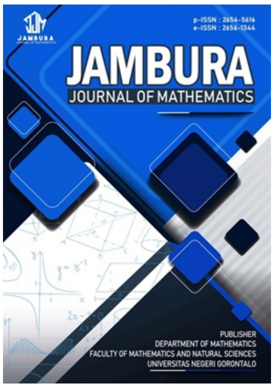

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Research Article

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The Hamiltonian and Hypohamiltonian of Generalized Petersen Graph ($GP_{n,9}$)

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ABSTRACT. *The study of Hamiltonian and Hypohamiltonian properties in the generalized Petersen graph $GP_{n,k}$ is interesting due to the unique structure and characteristics of these graphs. The method employed in this study involves searching for Hamiltonian cycles within the generalized Petersen graph $GP_{n,9}$. Not all of $GP_{n,9}$ graphs are Hamiltonian. For certain values of n, if the graph does not contain a Hamiltonian cycle, then one vertex should be removed from the graph to become Hamiltonian or neither. This research specifically investigates the Hypohamiltonian property of $GP_{n,9}$. The results show that for $\equiv 3\ (mod\ 19)$ and $n \equiv 5\ (mod 19)$, $GP_{n,9}$ is Hamiltonian. Meanwhile, for $n \equiv 0\ (mod\ 19)$, $GP_{n,9}$ is Hypohamiltonian. Furthermore, $n \equiv 1\ (mod\ 19)$, $n \equiv 2\ (mod\ 19)$, $n \equiv 4\ (mod\ 19)$ $GP_{n,9}$ is neither Hamiltonian nor Hypohamiltonian.*

1. Introduction

The Generalized Petersen graphs have been studied extensively due to their interesting properties and applications in various fields, including computer science, network analysis, combinatorics, and study about the type of genetic process, namely parthenogenesis [1].

Mathematicians have explored the structural properties, connectivity, chromatic properties, and other characteristics of Generalized Petersen graphs, making them an important area of research in graph theory. They provide valuable insights into the broader class of graphs and their properties, contributing to the advancement of mathematical knowledge and its applications. According to [2] graph theory succeeded in solving its first problem in 1973, namely the problem of the Koningsberg bridge in the city of Koningsberg. In Russia there is the Pregal river which flows around the island of Kneiphof and branches into two tributaries. This problem was solved by a Swiss mathematician named Leonhard Euler. Euler's arrangement speaks to this problem in a graph with four landmasses as vertices and seven bridges as edges.

According to [3] the Petersen graph is known as a regular graph of 3-degree at all its vertices and has been generalized. The Petersen graph is very popular to study because it is interesting, serves as an case of negation in different places and has different curiously properties. In [4] discussing the relationship between graph properties in Petersen graphs. Studying Hamiltonian cycles in the Petersen graph is particularly interesting due to the unique structure of the Petersen graph. Many paper publish about the Hamiltonian cycle and Hypohamiltonian graph such [5–7, 12–14]. All these paper just discussed about Hamiltonian graph or Hypohamiltonian graph. But, what about the graph that doesn't contain Hamiltonian cycle, is still possible to be Hamiltonian graph. That question became our motivation to study this paper.

The study about the Hamiltonian and Hypohamiltonian properties in graph becomes an interesting study because not all graphs have a Hamiltonian cycle, and when one vertex is removed, the graph will contain a Hamiltonian cycle called Hypohamiltonian. Finding Hamiltonian cycle in generalized Petersen graph $GP_{n,k}$ is interesting due to the unique structure and characteristics of these graphs. Many researcher have studied the Hamiltonian in graph and Hypohamiltonian properties of generalized Petersen graphs. The research was initiated by Danarto [8] by studying about the properties of the Hamiltonian and Hypohamiltonian of the generalized Petersen graphs $GP_{n,1}$ and $GP_{n,2}$. Furthermore, [14, 15] discussed about the Hamiltonian and Hypohamiltonian for Generalized Petersen Graph $GP_{n,6}$ and $GP_{n,7}$ consecutively. But, for $k \geq 9$ there is no paper discussed about the Hamilton and Hypohamiltonian of Generalized Petersen graph $GP_{n,k}$. In this paper, we characterize the generalized Petersen graph in such a way that it either has a Hamiltonian cycle, is hypohamiltonian, or is neither.

2. Preliminaries

2.1. Graph Theory

Graphs are a branch of mathematics that are widely used in everyday life, both for simple and complex things. According to [3, 9] and [10], graphs are defined as follows:

Definition 1. *The graph G is a pair of sets $(V(G), E(G))$ where $V(G)$ is a non-empty and finite set of objects called vertex, and $E(G)$ is a set of unordered pairs of different vertices in $V(G)$ are called edges. The set of vertices in G is denoted by $V(G)$ and the*

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set of edges is denoted by $E(G)$. Whereas the number of elements in $V(G)$ is called the order of G and is denoted by $p(G)$ and the number of elements in $E(G)$ is called the measure and is denoted by $q(G)$.

Definition 2. *Edge $e = (u, v)$ is called to connect vertex u and v if $e = (u, v)$ is an edge in the graph G, then u and $v$are called to be directly connected (adjacent). u and e, v and e are called to be directly related (incidents). Edge e is denoted by $e = uv$.*

2.2. Costum Graph

According to [2, 5] and [13] the following are some special graphs, which are discussed, cubic graphs, Petersen graphs, and generalized Petersen graphs.

Definition 3. *A cubic graph is a graph where every vertex v has 3-degree, or is often called a regular graph of 3-degree.* See Figure 1 for illustration.

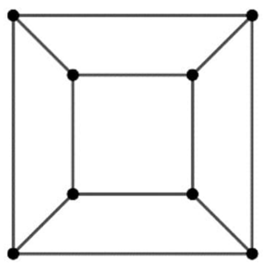

Figure 1. Cubic graph

Definition 4. *The Petersen graph is a regular graph of 3-degree. In a Petersen graph all the vertices have 3-degree so the Petersen graph is called a cubic graph with ten vertices and fifteen edges. The Petersen graph is vertex-transitive and edge-transitive, and is symmetrical.* See Figure 2 for illustration.

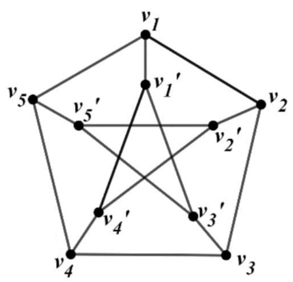

Figure 2. The Petersen graph

Definition 5. *The Generalized Petersen graph is denoted by $GP_{n,k}$ where n and k are positive integers with $2 \leq 2k < n$. It is a graph with*

$$V(GP_{n,k}) = \{u_0, u_1, \ldots, u_{\{n-1\}}, v_0, v_1, \ldots, v_{\{n-1\}}\},$$
$$E(GP_{n,k}) = \{u_iu_{i+1}, v_iv_{i+k}, u_iv_i \mid i = 0, 1, \ldots, n-1\},$$

where the addition in the index $(i+1)$ and $(i+k)$ is modulo n. The Generalized Petersen graph $GP_{n,k}$ has three types of edges, namely outer edges, inner edges, and spoke. The outer edge connects vertices u_i and u_{i+1}. The inner edge connects vertices v_i and v_{i+k} while the spoke connects vertices u_i and v_i.

2.3. Properties of Hamiltonians and Hypohamiltonians

According to [14], here are the definitions of Hamiltonian graph and Hypohamiltonian graph:

Definition 6. *A Hamiltonian graph is a graph that has a Hamiltonian cycle. A Hamiltonian path is a path $\forall$ point (V) traversed in graph G exactly only once, where the origin vertex $v_0 \neq end\ vertex\ v_n$, while a closed Hamiltonian path is a path $\forall$ point (V) traversed in graph G exactly only once, where the origin vertex $v_0 \neq end\ vertex\ v_n$.*

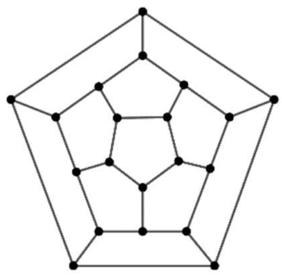

Figure 3. The Hamiltonian graph

Definition 7. *A graph G is called Hypohamiltonian if the graph G is not Hamiltonian, but if one vertex v is removed, then the subgraph $G - v$ is Hamiltonian. A graph G is said to be Hypohamiltonian if it satisfies the following definition:*

1. *A graph G is called Hypohamiltonian if it is not Hamiltonian,*
2. *If one vertex is removed from graph G, it will form a Hamiltonian cycle so that it is Hypohamiltonian.*

An example of a Hypohamiltonian graph can be seen in Figure 4.

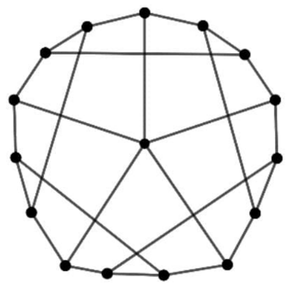

Figure 4. The Hypohamiltonian graph

3. Results and Discussion

In this section, we provide the existence of the Hamiltonian cycle and the validity of the Hamiltonian and Hypohamiltonian properties on the generalized Petersen graph $GP_{n,9}$ for $n = 19,\ 20,\ 21,\ 22,\ 23,\ 24$. The first step is to prove that the validity of the Hamiltonian properties on the Petersen graph which has been discussed in [15]. This will be proven by the following theorem:

Theorem 1. *The Petersen graph is not a Hamiltonian.*

Proof. By Definition 4, the Petersen graph is a regular graph of 3-degree. In a Petersen graph all the vertices have 3-degree so the Petersen graph is called a cubic graph with ten vertices and fifteen edges. The Petersen graph is vertex-transitive and edge-transitive, and is symmetrical. Now, It will be shown that the Petersen graph G is not Hamiltonian by using the edge-transitive Petersen graph property in Definition 5. Given the Petersen graph G, we have set of edges $A,\ B$, and C as follow :

$$A = \{v_1v_2,\ v_2v_3,\ v_3v_4,\ v_4v_5, v_5v_1\},$$
$$B = \{v_1v_1',\ v_2v_2',\ v_3v_3',\ v_4v_4',\ v_5v_5'\},$$
$$C = \{v_1'v_3',\ v_3'v_5',\ v_5'v_2',\ v_2'v_4',\ v_4'v_1\},$$

where $A,\ B$, and C are subset of $E(G)$ and A is a set of outer edges, B is a set of spokes, and C is a set of inner edges. Suppose that H is a cycle of the Petersen graph and H is a Hamiltonian then H must use an even number of sides of B, so that H has two or four sides because the maximum side that B has is five. Furthermore, by Definition 5, the Petersen graph is edge transitive and is symmetrical, it can be assumed that $v_1\ v_1' \in E(H)$, so one of v_1v_2 or $v_5v_1 \in E(H)$ and it can also be assumed that $v_1v_2 \in E(H)$. Then, based on Definition 7, it is known that the Petersen graph is a cubic graph, so $v_5v_1 \notin E(H)$. Therefore, v_4v_5 and $v_5v_5' \in E(H)$. If H using two edges of B, namely that v_1v_1' and $v_5v_1 \in E(H)$, then $v_2v_3,\ v_3v_4 \in E(H)$. However, this situation requires that the assumption that H is Hamiltonian is false. So, the Petersen graph is not Hamiltonian. □

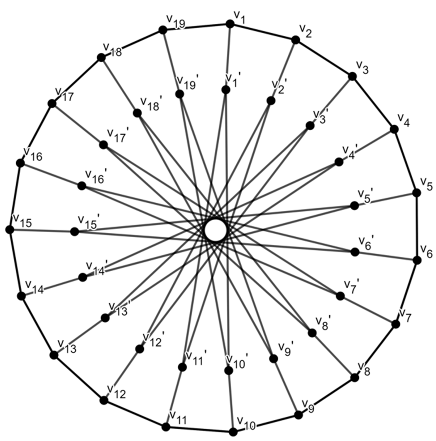

Figure 5. Generalized Petersen graph $GP_{19,9}$

The next step is to prove the validity of the Hamiltonian and Hypohamiltonian of the Petersen graph in general $GP_{n,9}$ with $n = 19,\ 20,\ 21,\ 22,\ 23,\ 24$. There will be three possibilities that apply to the generalized Petersen graph of the $GP_{n,9}$, that is Hamiltonian, Hypohamiltonian, and neither. According to Definition 1–Definition 5, for the $GP_{n,9}$, the value n that satisfy $2 \leq 2k < n$ are more than or equal 19.

Figure 5 illustrated the Generalized Petersen graph $GP_{19,9}$ with 38 vertices and 76 edges. Based on Figure 5, there is no Hamiltonian cycle found in $GP_{19,9}$, so it is not Hamiltonian. We ilustrated the Generalized Petersen graph $GP_{20,9}$ in Figure 6.

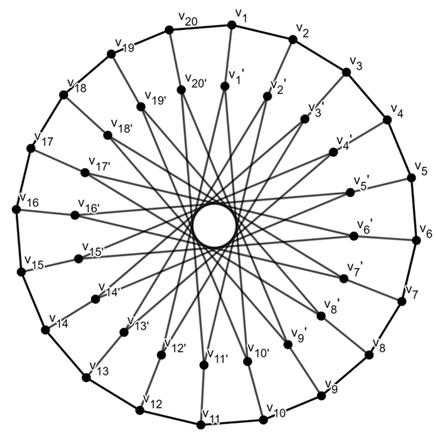

Figure 6. Generalized Petersen graph $GP_{20,9}$

The Generalized Petersen graph $GP_{20,9}$ in Figure 6 consist of 40 vertices and 80 edges. There is no Hamiltonian cycle was found in graph $GP_{20,9}$, so it is not Hamiltonian.

Figure 7 ilustrated the Generalized Petersen graph $GP_{21,9}$ with 42 vertices and 84 edges.

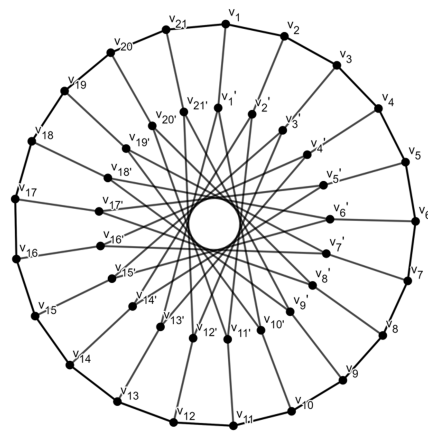

Figure 7. Generalized Petersen graph $GP_{21,9}$

Based on Figure 7, there is no Hamiltonian cycle was found in $GP_{21,9}$ so it is not Hamiltonian.

Figure 8 illustrated the Generalized Petersen graph $GP_{22,9}$ with 44 vertices and 88 edges. Based on Figure 8, there is a Hamiltonian cycle in $GP_{22,9}$ as follows

$$H = \{v_1, v_1', v_{10}', v_{10}, v_{11}, v_{11}', v_{20}', v_{20}, v_{21}, v_{21}', v_8', v_8, v_9, v_9', v_{18}', v_{18}, v_{19}, v_{19}', v_6', v_6, v_7, v_7', v_{16}', v_{16}, v_{17}, v_{17}', v_4', v_4, v_5, v_5', v_{14}', v_{14}, v_{15}, v_{15}', v_2', v_2, v_3, v_3', v_{12}', v_{12}, v_{13}, v_{13}', v_{22}, v_{22}', v_{22}, v_1\}.$$

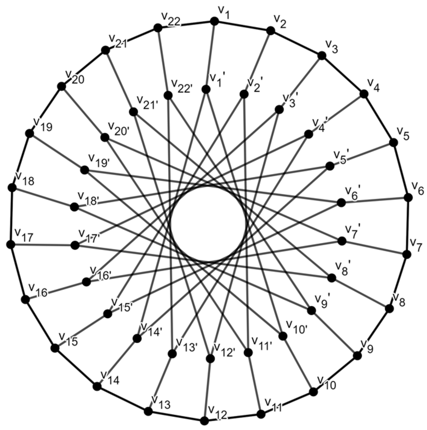

Figure 8. Generalized Petersen graph $GP_{22,9}$

So, we conclude that $GP_{22,9}$ is Hamiltonian.

Figure 9 show that Generalized Petersen graph $GP_{23,9}$ with 46 vertices and 92 edges. Based on Figure 9, there is no Hamil-

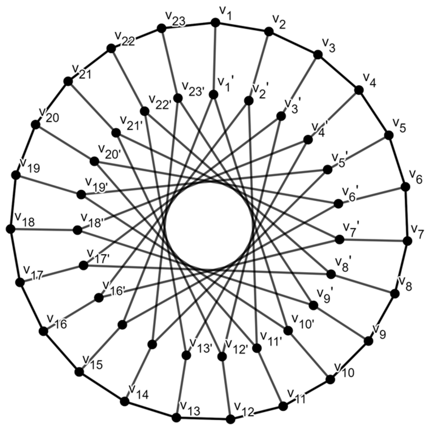

Figure 9. Generalized Petersen graph $GP_{23,9}$

tonian cycle was found in $GP_{23,9}$, so it is not Hamiltonian.

Figure 10 illustrated the Generalized Petersen graph $GP_{24,9}$ with 48 vertices and 96 edges. Based on Figure 10, there

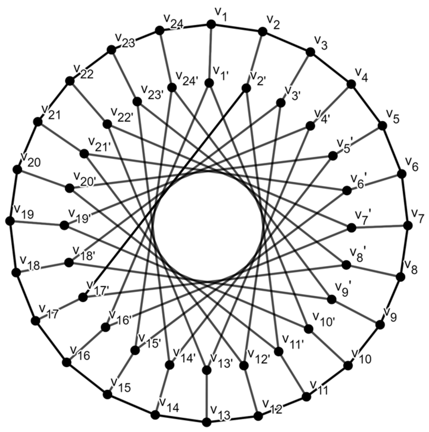

Figure 10. Generalized Petersen graph $GP_{24,9}$

is a Hamiltonian cycle as follows:

$$H = \{v_1, v_1', v_{10}', v_{10}, v_{11}, v_{11}', v_{20}', v_{20}, v_{21}, v_{21}', v_6', v_6, v_7, v_7', v_{16}', v_{16}, v_{17}, v_{17}', v_2', v_2, v_3, v_3', v_{12}', v_{12}, v_{13}, v_{13}', v_{22}', v_{22}, v_{23}, v_{23}', v_8', v_8, v_9, v_9', v_{18}', v_{18}, v_{19}, v_{19}', v_4', v_4, v_5, v_5', v_{14}', v_{14}, v_{15}, v_{15}', v_{24}', v_{24}, v_1\}.$$

So $GP_{24,9}$ is Hamiltonian.

Based on the discussion above, we find the existence of a Hamiltonian cycle in $GP_{n,9}$ with $n \equiv 3 (mod\ 19)$ and $n \equiv 5 (mod\ 19)$, so that it is a Hamiltonian Graph. Furthermore, for $GP_{n,9}$ with $n \equiv 0\,(mod\ 19)\,, n \equiv 1\,(mod\ 19)\,, n \equiv 2 (mod\ 19), n \equiv 4 (mod\ 19)$ there is no Hamiltonian cycle was found so there are two possibilities that is Hypohamiltonian or neither.

The following Theorem 2 show that there is no Hamiltonian cycle in a graph $GP_{n,9}$ with $n \equiv 0\ (mod\ 19)$.

> **Theorem 2.** *Generalized Petersen graph $GP_{n,9}$ with $n \equiv 0 (mod\ 19)$ is not Hamiltonian.*

Proof. It will be show that the generalized Petersen graph $GP_{n,9}$ with $n \equiv 0 (mod\ 19)$ is not Hamiltonian. The cycle formed that passes through each vertex is as follows:

$$H = \{v_1, v_1', v_{10}', v_{19}', v_9', v_{18}', v_8', v_{17}', v_7', v_{16}', v_6', v_{15}', v_5', v_{14}', v_4', v_{13}', v_3', v_{12}', v_2', v_{11}', v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_1\}.$$

By the cycle in set H, it can be seen that the cycle does not pass through the vertices $v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9$ and v_{10}. So, $GP_{n,9}$ with $n \equiv 0\ (mod\ 19)$ is not Hamiltonian. □

The following Theorem 3 show that there is no Hamiltonian cycle in a graph $GP_{n,9}$ with $n \equiv 1\ (mod\ 19)$.

> **Theorem 3.** *Generalized Petersen graph $GP_{n,9}$ with $n \equiv 1 (mod\ 19)$ is not Hamiltonian.*

Proof. It will be proven that the generalized Petersen graph $GP_{n,9}$ with $n \equiv 1 (mod\ 19)$ is not Hamiltonian by showing the absence of Hamiltonian cycles. The cycle for the generalized Petersen graph $GP_{n,9}$ with $n \equiv 1\ (mod\ 19)$ is as follows:

$$H = \{v_1, v_1', v_{10}', v_{10}, v_{11}, v_{11}', v_2', v_2, v_3, v_3', v_{12}', v_{12}, v_{13}, v_{13}', v_4', v_4, v_5, v_5', v_{14}', v_{14}, v_{15}, v_{15}', v_6', v_6, v_7, v_7', v_{16}', v_{16}, v_{17}, v_{17}', v_8', v_8, v_9, v_9', v_{18}', v_{18}, v_{19}, v_{19}', v_9, v_{20}', v_1\}.$$

By the cycle in set H, it can be seen that the cycle does not passed through the vertices v_{19}' and v_{20}'. So $GP_{n,9}$ with $n \equiv 1 (mod\ 19)$ is not Hamiltonian. □

The following Theorem 4 show that there is no Hamilton cycle in a graph $GP_{n,9}$ with $n \equiv 2\ (mod\ 19)$.

> **Theorem 4.** *Generalized Petersen graph $GP_{n,9}$ with $n \equiv 2 (mod\ 19)$ is not Hamiltonian.*

Proof. It will be proven that the generalized Petersen graph $GP_{n,9}$ with $n \equiv 2(mod\ 19)$ is not Hamiltonian by showing the absence of Hamiltonian cycles. The cycle for the generalized Petersen graph $GP_{n,9}$ with $n \equiv 2(mod\ 19)$ is as follows:

$$H = \{v_1, v_1', v_{10}', v_{10}, v_{11}, v_{11}', v_2', v_2, v_3, v_3', v_{12}', v_{12}, v_{13}, v_{13}', v_4', v_4, v_5, v_5', v_{14}', v_{14}, v_{15}, v_{15}', v_6, v_6', v_6, v_7, v_7', v_{16}', v_{16}, v_{17}, v_{17}', v_8', v_8, v_9, v_9', v_{18}', v_{18}, v_{19}, v_{20}, v_{21}, v_1\}.$$

By the cycle in set H, it can be seen that the cycle does not passed through the vertices v_{19}', v_{20}' and v_{21}'. So $GP_{n,9}$ with $n \equiv 2\,(mod\ 19)$ is not Hamiltonian. □

The following Theorem 5 show that there is a Hamiltonian cycle in a graph $GP_{n,9}$ with $n \equiv 3\ (mod\ 19)$.

Theorem 5. *Generalized Petersen graph $GP_{n,9}$ with $n \equiv 3\ (mod\ 19)$ is Hamiltonian.*

Proof. It will be proven that the generalized Petersen graph $GP_{n,9}$ with $n \equiv 3\,(mod\ 19)$ is Hamiltonian by showing the existence of Hamiltonian cycle. In the generalized Petersen graph $GP_{n,9}$ with $n \equiv 3\,(mod\ 19)$ there is a cycle that passed through all vertices exactly once. It will start from the start vertex v_1 and return to the end vertex v_1. So the cycle pattern is obtained as follows:

$$H = \{v_1, v_1', v_{10}', v_{10}, v_{11}, v_{11}', v_{20}', v_{20}, v_{21}, v_{21}', v_8', v_8, v_9, v_9', v_{18}', v_{18}, v_{19}, v_{19}', v_6', v_6, v_7, v_7, v_7', v_{16}', v_{16}, v_{17}, v_{17}', v_4', v_4, v_5, v_5', v_{14}', v_{14}, v_{15}, v_{15}', v_2', v_2, v_3, v_3', v_{12}', v_{12}, v_{13}, v_{13}', v_{22}', v_{22}, v_1\}.$$

The cycle obtained is a Hamiltonian cycle. So, it showed that $GP_{n,9}$ with $n \equiv 3\ (mod\ 19)$ is a Hamiltonian graph. □

The following Theorem 6 show that there is no Hamiltonian cycle in a graph $GP_{n,9}$ with $n \equiv 4\ (mod\ 19)$.

Theorem 6. *Generalized Petersen graph $GP_{n,9}$ with $n \equiv 4\,(mod\ 19)$ is not Hamiltonian.*

Proof. It will be proven that the generalized Petersen graph $GP_{n,9}$ with $n \equiv 4\,(mod\ 19)$ is not Hamiltonian by showing the absence of Hamiltonian cycles. The cycle for the generalized Petersen graph $GP_{n,9}$ with $n \equiv 4\,(mod\ 19)$ is as follows:

$$H = \{v_1, v_1', v_{10}', v_{19}', v_5', v_{14}', v_{23}', v_9', v_{18}', v_4', v_{13}', v_{22}', v_8', v_{17}', v_3', v_{12}', v_{21}', v_7', v_{16}', v_2', v_{11}', v_{20}', v_6', v_{15}', v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}\}.$$

By the cycle in set H, it can be seen that the cycle does not passed through the vertices $v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}$. So, $GP_{n,9}$ with $n \equiv 4\ (mod\ 19)$ is not Hamiltonian. □

The following Theorem 7 show that there is a Hamiltonian cycle in a graph $GP_{n,9}$ with $n \equiv 5\ (mod\ 19)$.

Theorem 7. *Generalized Petersen graph $GP_{n,9}$ with $n \equiv 5\ (mod\ 19)$ is Hamiltonian.*

Proof. It will be proven that the generalized Petersen graph $GP_{n,9}$ with $n \equiv 5\ (mod\ 19)$ is Hamiltonian by showing the existence of Hamiltonian cycle. In the generalized Petersen graph $GP_{n,9}$ with $n \equiv 5\ (mod\ 19)$ there is a cycle that passed through all vertices exactly once. It will start from the start vertex v_1 and return to the end vertex v_1. So the cycle pattern is obtained as follows:

$$H = \{v_1, v_1', v_{10}', v_{10}, v_{11}, v_{11}', v_{20}', v_{20}, v_{21}, v_{21}', v_6', v_6, v_7, v_7', v_{16}', v_{16}, v_{17}, v_{17}', v_2', v_2, v_3, v_3', v_{12}', v_{12}, v_{13}, v_{13}', v_{22}', v_{22}, v_{23}, v_{23}', v_8', v_8, v_9, v_9', v_{18}', v_{18}, v_{19}, v_{19}', v_4', v_4, v_5, v_5', v_{14}', v_{14}, v_{15}, v_{15}', v_{24}', v_{24}, v_1\}.$$

The cycle obtained is a Hamiltonian cycle. So, it proven $GP_{n,9}$ with $n \equiv 5\,(mod\ 19)$ is a Hamiltonian. □

Theorem 1−7 showed that the graphs $GP_{n,9}$ with $n \equiv 0(mod\ 19), n \equiv 1(mod\ 19), n \equiv 2(mod\ 19), n \equiv 3(mod\ 19), n \equiv 4(mod\ 19), n \equiv 5(mod\ 19)$ are not Hamiltonian, so there are two possibilities, that is Hypohamiltonian graph or neither. In this section, we will be prove the validity of Hypohamiltonian properties.

The following Theorem 8 show that $GP_{n,9}$ with n≡0 (mod 19) is Hypohamiltonian. By Definition 7, a graph G is called hypohamiltonian if the graph G is not Hamiltonian, but if one vertex v is removed, then the subgraph $G - v$ is Hamiltonian.

Theorem 8. *Generalized Petersen graph $GP_{n,9}$ with $n \equiv 0\,(mod\ 19)$ is Hypohamiltonian.*

Proof. It will be proven that the generalized Petersen graph $GP_{n,9}$ with $n \equiv 0\,(mod\ 19)$ is not Hypohamiltonian. Based on Theorem 3, the generalized Petersen graph $GP_{n,9}$ with $n \equiv 0\,(mod\ 19)$ already satisfied the first condition of Definition 7. For the second condition, take any point on the inner vertex. Suppose that point v_{10}' is deleted, it will be proven that the generalized Petersen graph $GP_{n,9}$ with $n \equiv 0\,(mod\ 19)$ is Hypohamiltonian by showing the absence of Hamiltonian cycles.

$$H = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{19}', v_9', v_{18}', v_8', v_{17}', v_7', v_{16}', v_6', v_{15}', v_5', v_{14}', v_4', v_{13}', v_3', v_{12}', v_2', v_{11}', v_1', v_1\}.$$

$GP_{n,9} - v_{10}'$ there is exist the Hamiltonian cycle. So $GP_{n,9}$ with $n \equiv 0\,(mod\ 19)$ is Hypohamiltonian. □

The following Theorem 9−13 show that $GP_{n,9}$ with certain n is not Hypohamiltonian.

Theorem 9. *Generalized Petersen graph $GP_{n,9}$ with $n \equiv 1\,(mod\ 19)$ is not Hypohamiltonian.*

Proof. It will be proven that the generalized Petersen graph $GP_{n,9}$ with $n \equiv 1\,(mod\,19)$ is not Hypohamiltonian. Based on Theorem 4, the generalized Petersen graph $GP_{n,9}$ with $n \equiv 1\,(mod\,19)$ already satisfied the first condition. For the second condition, take any point on the inner vertex. Suppose that point v'_{20} is deleted, It will be proven that the generalized Petersen graph $GP_{n,9}$ with $n \equiv 1\,(mod\,19)$ is not Hypohamiltonian by showing the absence of Hamiltonian cycles.

$$H = \{v_1, v'_1, v'_{10}, v_{10}, v_{11}, v'_{11}, v'_2, v_2, v_3, v'_3, v'_{12}, v_{12}, v_{13}, v'_{13}, v'_4, v_4, v_5, v'_5, v'_{14}, v_{14}, v_{15}, v'_{15}, v'_6, v_6, v_7, v'_7, v'_{16}, v_{16}, v_{17}, v'_{17}, v'_8, v_8, v_9, v'_9, v'_{18}, v_{18}, v_{19}, v'_{19}, v'_8, v_8, v_9, v'_9, v'_{18}, v_{18}, v_{19}, v_{20}, v_1\}.$$

The cycle obtained is not Hamiltonian cycle so $GP_{n,9} - v'_{20}$ not Hamiltonian. Both condition are met, so it can be concluded that $GP_{n,9}$ with $n \equiv 1\,(mod\,19)$ is not Hypohamiltonian. □

Theorem 10. *Generalized Petersen graph $GP_{n,9}$ with $n \equiv 2\ (mod\,19)$ is not Hypohamiltonian.*

Proof. It will be proven that the generalized Petersen graph $GP_{n,9}$ with $n \equiv 2\,(mod\,19)$ is not Hypohamiltonian. Based on Theorem 5, the generalized Petersen graph $GP_{n,9}$ with $n \equiv 2\,(mod\,19)$ already satisfied the first condition of Definition 7. For the second condition, take any point on the inner vertex. Suppose that point v'_3 is removed, It will be proven that the generalized Petersen graph $GP_{n,9}$ with $n \equiv 2\,(mod\,19)$ is not Hypohamiltonian by showing the absence of Hamiltonian cycles.

$$H = \{v_1, v'_1, v'_{13}, v_{13}, v_{12}, v'_{12}, v'_{21}, v_{21}, v_{20}, v'_{20}, v'_{11}, v_{11}, v_{10}, v'_{10}, v'_{19}, v_{19}, v_{18}, v'_{18}, v'_9, v_9, v_8, v'_8, v'_{17}, v_{17}, v_{16}, v'_{16}, v'_7, v_7, v_6, v'_6, v'_{15}, v_{15}, v_{14}, v'_{14}, v'_5, v_5, v_4, v'_4, v'_{13}, v_{13}, v_{14}, v'_{14}, v'_2, v_2, v_3, v_4, v'_4, v'_{13}, v'_1, v_1\}.$$

The cycle obtained is not Hamiltonian cycle so $GP_{n,9} - v'_3$ not Hamiltonian. Both condition are met, so it can be concluded that $GP_{n,9}$ with $n \equiv 2\,(mod\,19)$ is not Hypohamiltonian. □

Theorem 11. *Generalized Petersen graph $GP_{n,9}$ with $n \equiv 3\,(mod\,19)$ is not Hypohamiltonian.*

Proof. It will be proven that the Petersen graph is generalized $GP_{n,9}$ with $n \equiv 3\,(mod\,19)$ is not Hypohamiltonian. Based on Theorem 6, the generalized Petersen graph $GP_{n,9}$ with $n \equiv 3\,(mod\,19)$ is Hamiltonian. Thus, based on Definition 7, the generalized Petersen graph $GP_{n,9}$ does not satisfy the first condition. Therefore, the generalized Petersen graph $GP_{n,9}$ with $n \equiv 3\,(mod\,19)$ is not Hypohamiltonian. □

Theorem 12. *Generalized Petersen graph $GP_{n,9}$ with $n \equiv 4\,(mod\,19)$ is not Hypohamiltonian.*

Proof. It will be proven that the generalized Petersen graph $GP_{n,9}$ with $n \equiv 4\,(mod\,19)$ is not Hypohamiltonian. Based on Theorem 7, the generalized Petersen graph $GP_{n,9}$ with $n \equiv 4\,(mod\,19)$ already satisfied the first condition. For the second condition, take any point on the inner vertex. Suppose that point v_2 is deleted, it will be proven that the generalized Petersen graph $GP_{n,9}$ with $n \equiv 4\,(mod\,19)$ is not Hypohamiltonian by showing the absence of Hamiltonian cycles.

$$H = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}, v'_{23}, v'_9, v'_{18}, v'_4, v'_{13}, v'_{22}, v'_8, v'_{17}, v'_3, v'_{12}, v'_{21}, v'_7, v'_{16}, v_{16}, v_{15}, v'_{15}, v'_6, v'_{20}, v'_{11}, v_{11}, v_{10}, v'_{10}, v'_{19}, v'_5, v'_{14}, v_{14}, v_{15}, v'_{15}, v'_1, v_1\}.$$

Therefore, the generalized Petersen graph $GP_{n,9}$ with $n \equiv 4\,(mod\,19)$ is not Hypohamiltonian. □

Theorem 13. *Generalized Petersen graph $GP_{n,9}$ with $n \equiv 5\,(mod\,19)$ is not Hypohamiltonian.*

Proof. It will be proven that the Petersen graph is generalized $GP_{n,9}$ with $n \equiv 5\,(mod\,19)$ is not Hypohamiltonian. Based on Theorem 8, the generalized Petersen graph $GP_{n,9}$ with $n \equiv 5\,(mod\,19)$ is Hamiltonian. Thus, based on Definition 7, the generalized Petersen graph $GP_{n,9}$ does not satisfy the first condition. Therefore, the generalized Petersen graph $GP_{n,9}$ with $n \equiv 5\,(mod\,19)$ is not Hypohamiltonian. □

4. Conclusion

Based on the discussion above, the properties of the Hamiltonian and Hypohamiltonian on Petersen graphs and generalize Petersen graphs is not Hamiltonian, but Hypohamiltonian. Generalized Petersen graph $GP_{n,9}$ with $n \equiv 3(mod\ 19)$ and $n \equiv 5(mod\ 19)$ are a Hamiltonian. For Generalized Petersen graph $GP_{n,9}$ with $n \equiv 0(mod\ 19)$ is Hypohamiltonian. Meanwhile for $n \equiv 1(mod\ 19), n \equiv 2(mod\ 19), n \equiv 4(mod\ 19)$ are neither.

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