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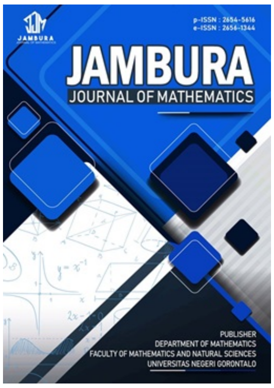

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Research Article

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Identify Solutions to Systems of Linear Latin for Square Equations over Maxmin-ω

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ABSTRACT. *Maxmin-ω algebra is a mathematical system that generalizes maxmin algebra by introducing the parameter ω ($0 < \omega \leq 1$), which regulates the algebraic operations to enhance its applicability in optimization and decision-making processes. When $\omega = 1$, the system corresponds to the max operation, whereas for ω approaching 0, it behaves like the min operation. This research investigates the solution characteristics of a linear equation system in maxmin-ω algebra, specifically $A \otimes_\omega x = b$, where A is a Latin square matrix. Understanding these solutions is crucial for determining the conditions of existence and uniqueness, which will ultimately influence the development of more efficient solution methods for various applications. Furthermore, the study analyzes the impact of the value of ω and the matrix permutation structure on the solutions of the system. This study employs an analytical approach utilizing maxmin-ω algebra theory to determine solution existence and assess the impact of ω variations in linear equations with Latin square matrices. The results reveal that the solution existence heavily depends on the composition of matrix A and the vector b. We show that in specific cases where the matrix A is a Latin square and the vector $\mathbf{b}$ satisfies certain constraints, the system has a unique solution in both the max-plus ($\omega = 1$) and min-plus ($\omega = \frac{1}{n}$) approaches. Moreover, column permutations of A do not affect the existence of solutions. However, row and element permutations alter the system structure, meaning solutions are not always guaranteed.*

1. Introduction

Max-plus algebra has been extensively used in applied mathematics for solving a wide range of problems, particularly in optimization and synchronization models. This algebra operates on the set of real numbers extended by negative infinity, with the basic operation of *maximum* ($\oplus$) replacing conventional addition, and *addition* ($\otimes$) replacing conventional multiplication. For example, given two scalars a and b, their max-plus sum is defined as

$$a \oplus b = \max(a, b),$$

while their max-plus product is given by

$$a \otimes b = a + b.$$

Linear equation systems in max-plus algebra have a distinctive form where addition is replaced by the maximum operation, while multiplication is replaced by regular addition [1]. These systems are often expressed as $A \otimes \mathbf{x} = \mathbf{b}$, where A is a coefficient matrix, $\mathbf{x}$ is a variable vector, and b is a result vector. Solving such systems involves finding the maximum values from combinations of matrix and vector elements.

Max-plus algebra linear systems are frequently applied to synchronization models, such as production systems [2, 3], and schedule optimization in transportation networks [4, 5]. In railway networks, the delay in one train's departure can affect the schedules of other trains. Max-plus algebra facilitates schedule analysis and optimization by systematically and efficiently considering such delays.

Despite its advantages, max-plus algebra has limitations in handling systems that require both maximum and minimum operations simultaneously. To address this limitation, maxmin-ω algebra extends maxmin algebra by incorporating a parameter ω ($0 < \omega \leq 1$), allowing a smooth transition between max and min operations. When $\omega = 1$, the maximum operation is used, while as ω approaches zero, the operation approaches the minimum. This property makes maxmin-ω algebra relevant for solving problems requiring the combination of both operations simultaneously.

Latin square matrices, which ensure distinct elements in each row and column, have been utilized in various mathematical applications, including combinatorics, cryptography, and optimization. In the context of maxmin-ω algebra, their structured properties provide a unique framework for analyzing linear systems. A *Latin square* matrix is a square matrix in which each row and column contains distinct elements [6, 7]. These matrices are significant not only in combinatorics theory but also have practical applications in various fields such as cryptography [8], experimental design [9], and data communication [10].

Linear systems in max-plus algebra can be solved using methods described in [11]. The solution of maxmin-ω linear systems adopts a similar approach, which is then extended to cover more general max-min-plus linear systems, resulting in a

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more flexible maxmin-ω system. Literature related to such systems generally focuses on two extremes, namely min-plus and max-plus, as previously discussed in [12]. However, research on maxmin-ω algebra remains limited. Specifically, the influence of the permutation structure of *Latin square* matrices on the existence of solutions in maxmin-ω algebra has not been thoroughly explored. The unique properties of *Latin square* matrices present both opportunities and challenges for understanding how the parameter ω affects the existence of solutions for these systems.

This research aims to analyze the solution structure of linear equation systems using maxmin-ω algebra, specifically when the coefficient matrix is a Latin square. By integrating maxmin-ω algebra with Latin square properties, this study explores the impact of ω variations and matrix permutations on the existence of solutions, addressing gaps in previous research.

2. Preliminaries

This section introduces the fundamental notations, definitions, and propositions that serve as the basis for the discussions in subsequent sections.

In this paper, it is consistently assumed that $m, n \geq 1$ be integers, and define the index sets as $[m] := \{1, \ldots, m\}$ and $[n] := \{1, \ldots, n\}$. An n-tuple $(a_1, a_2, \ldots, a_n)$ is called a permutation of $[n]$ if its elements are a rearrangement of $\{1, 2, \ldots, n\}$. If elements are allowed to repeat, it is referred to as a multiset, with $|S|$ denoting the cardinality, or the number of elements, in the set S. For $0 \leq k \leq |S|$, the notation $\mathcal{P}(S, k)$ denotes the set of all subsets of S with size k.

In max-plus algebra, the set $\mathbb{R}_{\max}$ represents the set of real numbers extended with the element ε, which corresponds to $-\infty$. The fundamental operations in max-plus algebra are defined for $a, b \in \mathbb{R}_{\max}$ as:

$$a \oplus b = \max\{a, b\}, \quad a \otimes b = a + b. \tag{1}$$

This algebra forms a semiring $(\mathbb{R}_{\max}, \oplus, \otimes)$ where ε serves as the identity element for $\oplus$, and 0 as the identity element for $\otimes$.

The set of $m \times n$ matrices with entries in $\mathbb{R}_{\max}$ is denoted as $\mathbb{R}_{\max}^{m\times n}$, while vectors with m entries in $\mathbb{R}_{\max}$ are represented as $\mathbb{R}_{\max}^{m}$.

For a matrix $A \in \mathbb{R}_{\max}^{m\times n}$, the entry at row i and column j is denoted by $A(i, j)$. Matrix addition and multiplication in max-plus algebra are defined as:

$$[A \oplus B](i, j) = \max\{A(i, j), B(i, j)\}, \tag{2}$$

$$[A \otimes C](i, j) = \max_{1\leq k\leq n} \{A(i, k) + C(k, j)\}, \tag{3}$$

$$[\alpha \otimes A](i, j) = \alpha + A(i, j), \quad \text{for} \quad \alpha \in \mathbb{R}_{\max}. \tag{4}$$

As the dual of max-plus algebra, min-plus algebra forms a semiring $(\mathbb{R}_{\min}, \oplus', \otimes)$, where the fundamental operations are defined as:

$$a \oplus' b = \min\{a, b\}, \quad a \otimes b = a + b. \tag{5}$$

In this system, the identity element for the operation $\oplus'$ is given by $\varepsilon = +\infty$.

2.1. *Maxmin-ω Algebra*

The maxmin-ω algebra is a generalization of the max-plus algebra introduced by incorporating the parameter ω, where $0 < \omega \leq 1$. This parameter enables the operation to flexibly transition between the maximum operation ($\omega = 1$) and the minimum operation ($\omega \approx 0$). Given a multiset of real numbers $S = \{s_1, s_2, \ldots, s_n\}$, the maxmin-$\omega$ operation

$$\bigoplus_\omega S \tag{6}$$

yields the $\lceil \omega n \rceil$-th smallest element of S for $0 < \omega \leq 1$. It can be said that (6) is a generalization of the $\oplus$ and $\oplus'$ operations [12].

Proposition 1. [13] *Let $S \subseteq \mathbb{R}$ be a multiset with $|S| = n$, and define $p = \lceil \omega n \rceil$. Then*

$$\bigoplus_\omega S = \bigoplus_{P\in\mathcal{P}([n],p)}{}' \left\{ \bigoplus P \right\}. \tag{7}$$

2.2. *Maxmin-ω Linear Equation System*

The maxmin-ω linear equation system is a mathematical model where conventional operations are replaced by the maxmin-ω operation. The form of this system is:

$$A \otimes_\omega \mathbf{x} = \mathbf{b}, \tag{8}$$

where $A \in R^{m\times n}$ is the coefficient matrix, $\mathbf{x} \in R^n$ is the solution vector, and $\mathbf{b} \in R^m$ is the result vector. The system of equations for the algebraic representation of the maxmin-ω operation in *conjunctive normal form* (CNF) and *disjunctive normal form* (DNF) is as follows [12]:

In CNF:

$$\bigoplus_{P\in\mathcal{P}([n],p)}{}' \left\{ \bigoplus_{j\in P} A(i, j) + x_j \right\} = b_i \quad \text{for } i \in [m]. \tag{9}$$

In DNF:

$$\bigoplus_{P\in\mathcal{P}([n],n+1-p)} \left\{ \bigoplus_{j\in P}{}' (A(i, j) + x_j) \right\} = b_i \quad \text{for } i \in [m]. \tag{10}$$

The normalized form of this system is expressed as:

$$A^* \otimes_\omega \mathbf{x} = \mathbf{0}, \tag{11}$$

where A^* is the normalized matrix [12], defined as:

$$A^*(i, \cdot) = -b_i \otimes A(i, \cdot). \tag{12}$$

Next, a proposed solution for the maxmin-ω system is expressed as:

$$\bar{x_k} = -\bigoplus_\omega A(., k), \quad \forall k \in [n]. \tag{13}$$

This solution represents a value satisfying the equation $A \otimes_\omega \mathbf{x} = \mathbf{b}$.

Definition 1. [12] Given a normalized maxmin-ω linear equation system, the solution set of this system is defined as:

$$S(A,\omega) = \{x \in \mathbb{R}^n \mid A \otimes_\omega x = \mathbf{0}\}. \tag{14}$$

A solution x is considered entirely active if every column of A^* contains at least one active element. An active element $A^*(i,j)$ satisfies:

$$A^*(i,j) + x_j = 0. \tag{15}$$

Definition 2. [12] Let $A \in \mathbb{R}^{m\times n}$. The principal order matrix $\bar{A}$ is defined as follows for $\forall i,j,k \in [n]$:

$$\bar{A}(i,j) = k \text{ if and only if } \bigoplus_{\frac{k}{m}} A(.,j) = A(i,j). \tag{16}$$

$\bar{A}$ is referred to as the principal order matrix associated with A, where $\bar{A}(i,j) = k$ implies $A(i,j)$ is the k-th smallest element of $A(.,j)$. Since elements in $A(.,j)$ are distinct, we have:

$$\{\bar{A}(1,j), \bar{A}(2,j), \dots, \bar{A}(n,j)\} = [n]. \tag{17}$$

Replacing A with $\bar{A}$ does not alter the solution index set of the linear system of eq. (8). However, it facilitates additional constraints for the solution indices.

Proposition 2. *[12] Consider the normalized linear system of eq. (8), where $A \in \mathbb{R}^{m\times n}$ is a matrix with distinct columns. If a tuple $(i_1,\dots,i_n) \in I(\bar{A},\omega)$, the following properties hold:*
$\{i_1, i_2, \dots, i_n\} = [m]$.
ii. $mp \le \sum_{k\in[n]} \bar{A}(i_k,k) \le mp + n - m$, *where* $p = \lceil \omega n \rceil$.

When the matrix A^* contains columns with duplicate values, the principal order matrix $\bar{A}$ can be constructed through an alternative method. Initially, duplicate values in each column of A^* are eliminated, resulting in a unique set denoted as col_j. Then, $\text{idx}_j(A,r)$ is defined as the set of row indices where the value r occurs in the j-th column of A. Mathematically, it is expressed as:

$$\text{idx}_j(A,r) = \{i \in [m] \mid A(i,j) = r\}. \tag{18}$$

The principal order matrix $\bar{A}$ is constructed by determining the cumulative count of elements smaller than $A(i,j)$ within the same column. This method establishes the relative ranking of elements within each column and is mathematically formulated as:

$$\bar{A}(i,j) = 1 + \sum_{l=1}^{k-1} |\text{idx}_j(A, \bigoplus_{\frac{l}{|\text{col}_j|}} \text{col}_j)| \quad \text{if} \quad \bigoplus_{\frac{k}{|\text{col}_j|}} \text{col}_j = A(i,j), \tag{19}$$

for $1 \le k \le |\text{col}_j|, i \in [m]$ and $j \in [n]$.

Subsequently, f_j represents the maximum occurrence frequency of any element in the j-th column of $\bar{A}$:

$$f_j = \max\{|\text{idx}_j(\bar{A},1)|, |\text{idx}_j(\bar{A},2)|, \dots, |\text{idx}_j(\bar{A},m)|\}. \tag{20}$$

Proposition 3. *For the normalized linear system of eq. (8) with $A \in \mathbb{R}^{m\times n}$, if $(i_1,\dots,i_n) \in I(\bar{A},\omega)$, the following statements are true:*
i. $\text{idx}_1(\bar{A}, \bar{A}(i_1,1)) \cup \dots \cup \text{idx}_n(\bar{A}, \bar{A}(i_n,n)) = [m]$.
ii. $mp + n - f \le \bar{A}(i_1,1) + \dots + \bar{A}(i_n,n) \le mp + n - m$, *where* $p = \lceil \omega n \rceil$ *and* $f = f_1 + \dots + f_n$.

2.3. *Latin Squares*

A square matrix L of order n is referred to as a *Latin square* if it contains n distinct elements in every row and column [14]. Mathematically:

$$L_{ij} \neq L_{ik} \quad \text{for } j \neq k \text{ and } i,j,k \in [n],$$

indicating that L_{ij} differs from L_{ik} if $j \neq k$, applying to each row i of the matrix L.

2.4. *Matrix Permutations*

A permutation is a bijective function on a set S. The set of all permutations of S is denoted by G_S and referred to as the symmetric group on S. For $S = \{1,2,\dots,n\}$, this group is written as G_n [15]. Permutations of a *Latin square* involve rearranging rows, columns, or elements while maintaining the rules of a *Latin square*.

Definition 3. Given a *Latin square* A_1, three types of permutations that can be applied to A_1 are defined as follows:

i. Row Permutation: swapping the rows of A_1 according to a;

$$PermRow(A_1,a) = A_2, \quad \text{where } A_2(i,\cdot) = A_1(a(i),\cdot) \quad \text{for every } i \in [n].$$

ii. Column Permutation: swapping the columns of A_1 according to a;

$$PermCol(A_1,a) = A_3, \quad \text{where } A_3(\cdot,i) = A_1(\cdot,a(i)) \quad \text{for every } i \in [n].$$

iii. Element Permutation: swapping the elements of A_1 according to a;

$$PermElem(A_1,a) = A_4, \quad \text{where } A_1(k,l) = i \Rightarrow A_4(k,l) = a(i) \text{ for every } i,k,l \in [n].$$

3. Results and Discussion

This chapter discusses the linear equation system $A \otimes_\omega \mathbf{x} = \mathbf{b}$, where A is a *Latin square*. Section 3.1 identifies the existence of solutions for the system, which is then re-evaluated in Section 3.2 by varying the vPermutasi adalah fungsi bijektif pada suatu himpunan S. Value of ω to determine specific properties. Subsequently, Section 3.3 analyzes the existence of solutions by permuting the *Latin square* matrix, while keeping the vector **b** unchanged.

3.1. Necessary Condition for a Maxmin-ω Linear System to Have a Solution

The linear equation corresponding to maxmin-ω in finding the vector $\mathbf{x} \in \mathbb{R}^n$ is:

$$A \otimes_\omega \mathbf{x} = \mathbf{b} \tag{21}$$

where $A \in \mathbb{R}^{n\times n}$ and $\mathbf{b} \in \mathbb{R}^n$. To ensure the existence of a solution to this linear system, certain conditions must be satisfied by the elements of the matrix A and the vector $\mathbf{b}$. The following is a theorem that provides the necessary condition for the existence of a solution to the maxmin-ω linear system.

Theorem 1. *Given the linear equation system $A \otimes_\omega x = b$, if $S(A,\omega) \neq \emptyset$, then*

$$b_i - b_j \geq \min\{A(i,\cdot) - A(j,\cdot)\}, \quad \forall i,j \in [n].$$

Proof. Suppose $S(A,\omega) \neq \emptyset$, meaning there exists at least one solution for the system of linear equations $A \otimes_\omega \mathbf{x} = \mathbf{b}$. Based on the maxmin-$\omega$ operations in Proposition 1, we have:

In CNF:

$$\bigoplus'_{P\in\mathcal{P}([n],p)} \left\{ \bigoplus_{j\in P} A(i,j) + x_j \right\} = b_i \quad \text{for } i \in [n].$$

and in DNF:

$$\bigoplus_{P\in\mathcal{P}([n],n+1-p)} \left\{ \bigoplus'_{j\in P} (A(i,j) + x_j) \right\} = b_i \quad \text{for } i \in [n].$$

Since $S(A,\omega) \neq \emptyset$, there exists a vector x that satisfies both equations above. For the same x, we obtain:

$$b_i = \bigoplus_{S_1\in\mathcal{P}([n],n+1-p)} \left\{ \bigoplus'_{k\in S_1} (A(i,k) + x_k) \right\} \quad \text{for } i \in [n],$$

$$b_j = \bigoplus'_{S_2\in\mathcal{P}([n],p)} \left\{ \bigoplus_{l\in S_2} (A(j,l) + x_l) \right\} \quad \text{for } j \in [n].$$

Based on the definition of the maxmin-ω operation, we know that:

$$b_i \geq \bigoplus'_{k\in S_1} \{A(i,k) + x_k\}, \quad \forall S_1 \in \mathcal{P}([n], n+1-p),$$

$$b_j \leq \bigoplus_{l\in S_2} \{A(j,l) + x_l\}, \quad \forall S_2 \in \mathcal{P}([n], p).$$

Recall that $A \leq B \rightarrow -A \geq -B$, so:

$$-b_j \geq \left(-\bigoplus_{l\in S_2} \{A(j,l) + x_l\} \right).$$

If b_i and $-b_j$ are added, we get:

$$b_i - b_j \geq \bigoplus'_{k\in S_1} \{A(i,k) + x_k\} + \left(-\bigoplus_{l\in S_2} \{A(j,l) + x_l\} \right),$$

$$b_i - b_j \geq \bigoplus'_{k\in S_1} \{A(i,k) + x_k\} + \bigoplus'_{l\in S_2} \{-A(j,l) - x_l\}.$$

Thus, for $l = k$:

$$b_i - b_j \geq \bigoplus'_{k\in[n]} \{A(i,k) - A(j,k)\}.$$

Therefore:

$$b_i - b_j \geq \min\{A(i,\cdot) - A(j,\cdot)\}.$$

Hence, it has been proven that if $S(A,\omega) \neq \emptyset$, then $b_i - b_j \geq \min\{A(i,\cdot) - A(j,\cdot)\}, \forall i,j \in [n]$. □

Example

Suppose we are given a normalized system of linear equations where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

and $\omega \in \{\frac{1}{3}, \frac{2}{3}, 1\}$.

According to Definition 2, we have $\bar{A} = A$. Furthermore, Proposition 2 states that if $\{i_1, i_2, i_3\} \in I(\bar{A},\omega)$, then

$$\{i_1, i_2, i_3\} = \{1,2,3\} \text{ and } \sum_{j=1}^{3} \bar{A}(i_j, j) = 9\omega. \tag{22}$$

For $\omega = \frac{1}{3}$, we obtain $\sum_{j=1}^{3} \bar{A}(i_j,j) = 3$. Since $\sum_{j=1}^{3} \bar{A}(i_j,j) = 3$, it follows that $\bar{A}(i_1,1) = 1$, $\bar{A}(i_2,2) = 1$, and $\bar{A}(i_3,3) = 1$.

Next, we verify whether $\bar{A} \otimes_{\frac{1}{3}} \bar{x} = 0$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \otimes_{\frac{1}{3}} \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Only one tuple satisfies eq. (22), namely $(1,2,3)$, which corresponds to the vector $\bar{x} = \begin{bmatrix} -1 & -1 & -1 \end{bmatrix}^T$.

Thus, based on Definition 1, the solution set of this system is as follows: $S(\bar{A}, \frac{1}{3}) = \left\{ \begin{bmatrix} -1 & -1 & -1 \end{bmatrix}^T \right\}$. Directly, for other values of ω, the solution can be obtained as follows:

$$S(\bar{A}, \frac{2}{3}) = \left\{ \begin{bmatrix} -1 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ -3 \end{bmatrix} \right\}, \; S(\bar{A}, 1) = \left\{ \begin{bmatrix} -3 \\ -3 \\ -3 \end{bmatrix} \right\}$$

thus, the solution condition

$$b_i - b_j \geq \min\{A(i,\cdot) - A(j,\cdot)\}, \quad \forall i,j \in [n] \text{ is satisfied.}$$

Furthermore, it will be shown that Theorem 1 does not hold equivalently (in both directions). This means that even if

$$b_i - b_j \geq \min\{A(i,\cdot) - A(j,\cdot)\}, \quad \forall i,j \in [n],$$

it does not necessarily imply that $S(A,\omega) \neq \emptyset$. Thus, the following Corollary is obtained.

Corollary 1. *Given the system of linear eq. (8), if*

$$b_i - b_j < \min\{A(i,\cdot) - A(j,\cdot)\}, \exists i,j \in [n],$$

then $S(A,\omega) = \emptyset$. In other words, the system of linear equations has no solution.

Example

Suppose we are given the system of linear eq. (21), where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \quad \text{and} \quad \omega \in \left\{\frac{1}{3}, \frac{2}{3}, 1\right\}.$$

Next, we attempt to find the solution of the system for each ω. Since the linear equations are not yet normalized, the first step is normalization. Based on eq. (12), we obtain

$$\mathbf{A}^* = \begin{bmatrix} 0 & 1 & 2 \\ -1 & -3 & -2 \\ -1 & 0 & -2 \end{bmatrix}.$$

The next step is to construct the principal ordering matrix, denoted as $\bar{\mathbf{A}}$, based on the obtained $\mathbf{A}^*$:

$$\bar{\mathbf{A}} = \begin{bmatrix} 3 & 3 & 3 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$

The following shows that the system has no solution for any ω.

From $\bar{\mathbf{A}}$, we obtain $f_1 = 2$, $f_2 = 1$, $f_3 = 2$, and $f = 5$. Based on Proposition 3, if $\{i_1, i_2, i_3\} \in I(\bar{\mathbf{A}}, \omega)$, then

$$\text{idx}_1(\bar{\mathbf{A}}, \bar{\mathbf{A}}(i_1,1)) \cup \text{idx}_2(\bar{\mathbf{A}}, \bar{\mathbf{A}}(i_2,2)) \cup \text{idx}_3(\bar{\mathbf{A}}, \bar{\mathbf{A}}(i_3,3)) = \{1,2,3\}$$

and

$$9\omega - 2 \leq \bar{\mathbf{A}}(i_1,1) + \bar{\mathbf{A}}(i_2,2) + \bar{\mathbf{A}}(i_3,3) \leq 9\omega.$$

For $\omega = \frac{1}{3}$, the conditions are

$$\text{idx}_1(\bar{\mathbf{A}}, \bar{\mathbf{A}}(i_1,1)) \cup \text{idx}_2(\bar{\mathbf{A}}, \bar{\mathbf{A}}(i_2,2)) \cup \text{idx}_3(\bar{\mathbf{A}}, \bar{\mathbf{A}}(i_3,3)) = \{1,2,3\}$$

and

$$1 \leq \bar{\mathbf{A}}(i_1,1) + \bar{\mathbf{A}}(i_2,2) + \bar{\mathbf{A}}(i_3,3) \leq 3.$$

Since no tuple satisfies these conditions, we conclude that $S(\mathbf{A}, \frac{1}{3}) = \emptyset$. Directly, it can be concluded that for each remaining value of ω, no solution satisfies the given conditions.

3.2. *The Correlation of the Value of ω with the Existence of Solutions in Linear Systems of Latin Square*

One of the main results to be discussed is that for a matrix A which is a *Latin square* and certain constraints on the values of b_i, this system of equations has at least one solution for every value of ω. Additionally, it will be shown that for certain values of ω, such as $\omega = 1$ and $\omega = \frac{1}{n}$, the solution is unique.

Theorem 2. *Given the system of linear eq. (8) where A is a Latin Square matrix with elements $1,2,\ldots,n$. If*

$$-1 \leq b_i - b_j \leq 1, \quad \forall i,j \in [n],$$

then $S(A,\omega) \neq \emptyset$ for every ω.

Proof. We will show that there exists at least one vector $x \in \mathbb{R}^n$ such that $A \otimes_\omega x = b$ for every $\omega \in (0,1]$ when $-1 \leq b_i - b_j \leq 1$, $\forall i,j \in [n]$.

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix},$$

with $\omega = \frac{k}{n}$ where $k \in [n]$. Without loss of generality, assume that $a_{11} = a_{22} = \cdots = a_{nn} = k$. We will show that

$$\bigoplus\nolimits_\omega \{a_{l1} + (b_1 - b_l), \ldots, a_{ln} + (b_n - b_l)\} = k$$

for every $l \in [n]$. Since $b_i - b_j \geq -1$ for $i,j \in [n]$, we have

$$\bigoplus\nolimits_\omega \{a_{l1} + (b_1 - b_l), \ldots, a_{ln} + (b_n - b_l)\} \geq$$

$$\bigoplus\nolimits_\omega \{a_{l1} - 1, a_{l2} - 1, \ldots, a_{ll}, a_{ln} - 1\}.$$

Furthermore, from the condition $\{a_{l1}, \ldots, a_{ln}\} = [n]$ and $\omega = k/n$, it follows that

$$\bigoplus\nolimits_\omega \{a_{l1} - 1, a_{l2} - 1, \ldots, a_{ll}, a_{ln} - 1\} = k,$$

which implies

$$\bigoplus\nolimits_\omega \{a_{l1} + (b_1 - b_l), \ldots, a_{ln} + (b_n - b_l)\} \geq k.$$

For the case where $b_i - b_j \geq 1$, by similar reasoning, we obtain

$$\bigoplus\nolimits_\omega \{a_{l1} + (b_1 - b_l), \ldots, a_{ln} + (b_n - b_l)\} \leq k.$$

□

Furthermore, the following Corollary demonstrates that for certain cases, the solution to the system of linear equations is unique.

Corollary 2. *Given the system of linear eq. (8) where A is a Latin square matrix with elements $1,2,\ldots,n$. If $-1 \leq b_i - b_j \leq 1$ for all $i,j \in [n]$, then:*

$$S(A,\omega) \neq \emptyset \text{ for every } \omega.$$

Furthermore, for $\omega = 1$ and $\omega = \frac{1}{n}$, this system has a unique solution.

Proof. Consider the system of linear equations:

$$A \otimes_\omega \mathbf{x} = \mathbf{b},$$

where A is an $n \times n$ Latin square matrix, $\mathbf{b} \in \mathbb{R}^n$, and $\omega \in (0, 1]$. We will prove that for $\omega = 1$ and $\omega = \frac{1}{n}$, the system has a unique solution.

For $\omega = 1$, the operation $\otimes_\omega$ corresponds to the max-plus operation:

$$A \otimes_\omega \mathbf{x} = \mathbf{b} \Rightarrow \bigoplus_{P \in \mathcal{P}([n], n+1-p)} \left\{ \bigoplus_{j \in P}' (A(i,j) + x_j) \right\} = b_i$$

$$\forall i \in [n].$$

Since A is a Latin square matrix, each column $A(., j)$ contains distinct elements, ensuring that for every i, there is exactly one j that maximizes $A(i, j) + x_j$. This guarantees the existence of a solution x, and the distinct nature of the matrix elements ensures the solution is unique.

Similarly, for $\omega = \frac{1}{n}$, the operation $\otimes_\omega$ corresponds to the min-plus operation:

$$A \otimes_\omega \mathbf{x} = \mathbf{b} \Rightarrow \bigoplus_{P \in \mathcal{P}([n], p)}' \left\{ \bigoplus_{j \in P} A(i,j) + x_j \right\} = b_i \quad \forall i \in [n].$$

Due to the distinct elements in each column $A(., j)$, there is exactly one j that minimizes $A(i, j) + x_j$. As a result, the solution x is unique.

□

3.3. *The Influence of the Permutation of Latin Square Structure on the Solutions of Linear Systems*

In this discussion, we will review how the permutations of columns, rows, and elements of a *Latin square* affect the existence of solutions when the maxmin-ω method is applied. Given the system of linear eq. (8) in the form:

$$A_1 \otimes_\omega \mathbf{x} = \mathbf{b},$$

where A_1 is an $n \times n$ Latin square matrix, $\mathbf{x} \in \mathbb{R}^n$ is the solution vector, $\mathbf{b} \in \mathbb{R}^n$ is the right-hand side vector, and $\otimes_\omega$ is the maxmin-ω operation, which forms the primary focus of this research.

In this study, an analysis is conducted on the impact of transforming matrix A_1 through a permutation of its structure generated by the permutation function α. This permutation yields a new matrix A', defining a modified system of linear equations:

$$A' \otimes_\omega \mathbf{x}' = \mathbf{b}.$$

3.3.1. Row Permutation

Row permutation is a transformation that changes the positions of the rows of a matrix without altering the elements within the columns. Based on Definition 3 if $\alpha : [n] \to [n]$ is a permutation function, then applying a row permutation to matrix A_1 produces a new matrix A_2, defined as:

$$A_2(i, j) = A_1(\alpha(i), j), \quad \forall i, j \in [n].$$

This means that row i of A_1 is moved to row $\alpha(i)$ in A_2, while the elements in each column j remain in the same column. The permuted matrix A_2 is then used in the new system of linear equations:

$$A_2 \otimes_\omega \mathbf{x}' = \mathbf{b}.$$

Row permutation changes the positions of active elements in the rows, and as a result, the existence of a solution cannot always be guaranteed. The system loses a solution if the sufficient condition for the existence of a solution is no longer satisfied. The system obtains a new solution if the structure of matrix A_2 satisfies the sufficient condition that was previously not met.

3.3.2. Column Permutation

Column permutation is a transformation that changes the position of the columns of a matrix without altering the elements within each row. Based on Definition 3 if $a : [n] \to [n]$ is a permutation function, then the column permutation on matrix A_1 produces a new matrix A_3, with:

$$A_3(i, j) = A_1(i, a(j)), \quad \forall i, j \in [n].$$

In other words, column j of A_1 is moved to column $a(j)$ in A_3, while the elements in each row i remain in the same row. The resulting permuted matrix A_3 is then used in the new system of linear equations:

$$A_3 \otimes_\omega \mathbf{x}' = \mathbf{b}.$$

However, since $A_3(i, j) = A_1(i, a(j))$, the values of the elements in each row i remain the same, even though the positions of the columns change. Therefore, the column permutation on matrix A_1 does not alter the existence of the solution, as the active elements in the maxmin-ω operation remain unchanged. Thus, the initial solution from the system $A_1 \otimes_\omega \mathbf{x} = \mathbf{b}$ remains valid for the new system $A_3 \otimes_\omega \mathbf{x}' = \mathbf{b}$, with the solution permuted according to the function α.

3.3.3. Element Permutation

Element permutation is a transformation that swaps the values of the elements in matrix A_1 according to the rules of the permutation function α. Based on Definition 3 if an element $A_1(k, l) = i$, then this element is mapped to $\alpha(i)$, resulting in a new matrix A_4 with:

$$A_4(k, l) = \alpha(A_1(k, l)), \quad \forall k, l \in [n].$$

In other words, the value of each element $A_1(k, l)$ in the matrix A is changed to $\alpha(i)$, where $i = A_1(k, l)$, without altering the positions of the elements in the rows or columns. Thus, the structure of the matrix is preserved, but the values of the elements are modified according to the permutation function α. The resulting permuted matrix A_4 is then used in the new system of linear equations:

$$A_4 \otimes_\omega \mathbf{x} = \mathbf{b}.$$

In some cases, element permutation alters the positions of the active elements within the rows, which may disrupt the initial matrix structure. The system loses a solution if the sufficient condition for the existence of a solution is no longer satisfied. The system obtains a new solution if the structure of matrix A_4 satisfies a sufficient condition that was previously not met.

4. Conclusion

This study concludes that in maxmin-ω algebra, the existence of solutions for linear equation systems with Latin square coefficient matrices is primarily determined by the structural properties of the matrix. A solution exists if the differences between the elements of the result vector **b** remain within the minimum bounds set by the matrix elements. Variations in the parameter ω typically do not influence the existence of solutions, except in cases where vector b satisfies certain boundary conditions. For *Latin square* matrices with elements $1, 2, \ldots, n$, the existence and uniqueness of solutions are guaranteed when $\omega = 1$ (max-plus) or $\omega = \frac{1}{n}$ (min-plus), as long as the differences between vector **b**'s elements fall within the range $[-1, 1]$. The uniqueness of solutions relates to the properties of the *Latin square* matrix, where all elements in each row and column are unique. Additionally, column permutations do not alter the existence of solutions, as the solution depends on the set of elements within each row rather than their order. Conversely, row or element permutations may modify the system structure, potentially affecting both the existence and uniqueness of solutions. These findings emphasize the critical role of Latin square matrix properties in ensuring solution consistency within maxmin-ω algebra.

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