RESEARCH ARTICLE • OPEN ACCESS

Stability Analysis and Numerical Simulation of Prey-Mesopredator-Apex Predator Dynamic Model with Supplementary Food for Apex Predator

Resmawan Resmawan, Rizka Putri Handayani, Binti Mualifatul Rosydah, and Fahma Mu’jizatil Qur’ani

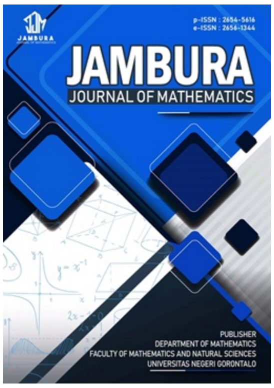
JAMBURA
JOURNAL OF MATHEMATICS

Volume 7, Issue 2, Pages 156–165, August 2025

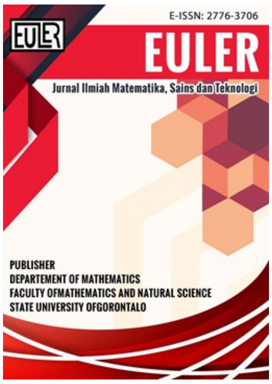
Received 27 March 2025, Revised 27 June 2025, Accepted 1 July 2025, Published 17 August 2025

To Cite this Article : R. Resmawan, R. P. Handayani, B. M. Rosydah, and F. M. Qur’ani,“Stability Analysis and Numerical Simulation of Prey-Mesopredator-Apex Predator Dynamic Model with Supplementary Food for Apex Predator”, *Jambura J. Math*, vol. 7, no. 2, pp. 156–165, 2025, *https://doi.org/10.37905/jjom.v7i2.31345*

JOURNAL INFO • JAMBURA JOURNAL OF MATHEMATICS



Homepage : http://ejurnal.ung.ac.id/index.php/jjom/index
Journal Abbreviation : Jambura J. Math.
Frequency : Biannual (February and August)
Publication Language : English (preferable), Indonesia
DOI : https://doi.org/10.37905/jjom
Online ISSN : 2656-1344
Editor-in-Chief : Hasan S. Panigoro
Publisher : Department of Mathematics, Universitas Negeri Gorontalo
Country : Indonesia
OAI Address : http://ejurnal.ung.ac.id/index.php/jjom/oai
Google Scholar ID : iWLjgaUAAAAJ
Email : info.jjom@ung.ac.id



Jambura Journal of Mathematics, Volume 7, Issue 2, Pages 156–165, August 2025 https://doi.org/10.37905/jjom.v7i2.31345

Research Article

Check for updates

Stability Analysis and Numerical Simulation of Prey-Mesopredator-Apex Predator Dynamic Model with Supplementary Food for Apex Predator

Resmawan Resmawan[1,2,*], **Rizka Putri Handayani**[1], **Binti Mualifatul Rosydah**[1,3], **Fahma Mu'jizatil Qur'ani**[1]

[1]*Department of Mathematics, Universitas Brawijaya, Malang 65145, Indonesia*
[2]*Department of Mathematics, Universitas Negeri Gorontalo, Bone Bolango 96554, Indonesia*
[3]*Study Program of Safety Engineering, Politeknik Perkapalan Negeri Surabaya, Surabaya 60111, Indonesia*

ARTICLE HISTORY
Received 27 March 2025
Revised 27 June 2025
Accepted 1 July 2025
Published 17 August 2025

KEYWORDS
Three-species model
Prey-predator model
Local stability
Equilibrium point
Mesopredator
Apex predator
Supplementary food

ABSTRACT. *This study formulates and mathematically analyzes a three-species dynamic model involving prey, mesopredator, and apex predator, considering the presence of supplementary food available only to the apex predator. The model is expressed as a three-dimensional nonlinear differential equation system and analyzed by proving the existence and uniqueness of solutions, positivity, and solution limitations to ensure mathematical validity in the biological domain. Furthermore, we study the local stability of the six equilibrium points of the system using the eigenvalue approach and the Routh-Hurwitz criterion. We perform numerical simulations and find that the stability of the system is highly sensitive to the parameters of predation efficiency and the capacity to utilize additional food. In addition, species extinction, dominance, or long-term coexistence also occur. The model shows how the relationships between different species and the support from external energy sources can change the community structure and affect whether predator species survive.*

1. Introduction

Predator-prey models have become an important basic approach in understanding the population dynamics of species in an ecosystem. Since it was first introduced in a simple form by Lotka and Volterra [1, 2], the model has been continuously developed to realistically describe ecological processes. Model development continues to be carried out with various more realistic ecological considerations, such as the limited amount of resources, nonlinear predation patterns, the existence of time delays in population response, and inter-species relationships in the ecosystem [3, 4]. These developments have given special attention to the treatment of predators, given their central role in regulating the balance of biological communities. Several studies have shown that changing the number of predators in mathematical models can have a big effect on how stable the system is. For example, proportional harvesting of predators [5], poisoning of predators [6], age structure, and intraspecific competition [7], anti-predator behavior [8], and the presence of intraspecific competition between predators [9] are known to change equilibrium points and cause transitions between species extinction, dominance, or coexistence. To keep going in this direction of development, the model needs to be expanded to include more species, not just predators and prey, but also more complex hierarchical structures, like interactions between prey, mesopredators, and apex predators.

The three-species model can illustrate more complex ecological dynamics, such as the impact of time delays and harvesting on the stability and bifurcation of population systems. Studies on these systems indicate that alterations in time delay or harvesting intensity can induce population fluctuations or shifts in stability direction [10]. Such structures are crucial in examining community stability, the dominance of certain species, and the extinction of others. Recent research has developed models that integrate several species by examining the impact of prey's fear of predators and harvesting efforts on the ecosystem. This complicates habits and impacts stability [11]. Researchers have developed dynamic methodologies employing nonlinear capture function forms, such as Monod-Haldane and Holling type II, incorporating temporal delays to more accurately represent predator-prey interactions and enhance the system's resemblance to a true ecosystem [12].

The heightened focus on ecological complexity has resulted in the incorporation of external environmental factors, such as supplementary food availability, as a significant element in models. Three-species models featuring two predators and one prey have been created to investigate the impact of supplemental food on system dynamics. Numerous recent research studies have investigated the dynamics of predator-prey systems by incorporating supplemental food, toxicity effects, and numerous time delays, which have been demonstrated to induce bifurcations and alter system stability [13, 14]. In addition, Arora [15] showed that the use of the Beddington-DeAngelis response function in a two-predator model can strengthen the survival of

---

[*] Corresponding Author.

Email : *resmawan@student.ub.ac.id* (R. Resmawan)

Homepage : http://ejurnal.ung.ac.id/index.php/jjom/index / E-ISSN : 2656-1344

predator species if supplementary food is available. Meanwhile, Kumar [16] added two discrete time delays and found that time delays in interaction as well as supplementary feeding can cause bifurcation transitions and affect the stability of the system. On the other hand, Mulugeta [17] combined toxicity and harvesting aspects in a one-prey, two-predator model and showed that the combination significantly affects the continuity of species coexistence. Most recently, Jana and Panja [18] showed that the extinction risk of species can be reduced if there is an adequate supply of supplementary food to predators. However, most previous studies assumed that supplemental food is equally available to all predators. Studies that explicitly analyze the influence of selective foraging, i.e., available to only one of the predators in a three-species system, are very limited. In addition, not many studies have addressed how changes in biomass conversion parameters and the availability of supplemental energy can affect the direction of system stability in communities with this tiered structure.

This study proposes to address the deficiency by formulating a three-species predator-prey model comprising a prey species, a mesopredator, and an apex predator, with supplementary food exclusively accessible to the apex predator. Mesopredators are intermediate-level predators that consume prey, however they may also fall victim to higher-level predators. On the other hand, the apex predator, at the top of the food chain, has no natural enemies. The construction of the model takes into account more realistic assumptions, such as the availability of supplementary food for the apex predator, which will reduce predation pressure on mesopredators and prey. This work intends to develop a three-species predator-prey mathematical model incorporating selective supplementary food, examine the stability of the system's equilibrium points, and investigate the long-term dynamics of the system by numerical simulations. This research aims to advance mathematical ecology theory and enhance species and ecosystem management strategies through quantitative methods.

The structure of this study is as follows. In the second section, a three-species prey-prey mathematical model consisting of prey species, mesopredator, and apex predator with its assumptions will be presented. The third section will discuss the existence and uniqueness of solutions, nonnegativity and boundedness solution, existence of equilibrium points, stability of equilibrium point, numerical simulation, discussion and conclusion.

2. Model

This study develops a predator-prey model involving three populations by referring to the approach proposed by Jana and Panja [18]. In a previous study, Jana and Panja [18] analyzed a three-species model assuming an supplementary food supply for the predator at a constant rate of A :

$$\begin{aligned}\frac{dx_1}{d\tau} &= x_1\left(r-kx_1\right)-ax_1x_2-Abx_1x_3,\\ \frac{dx_2}{d\tau} &= cx_1x_2-dx_2-q_1E_1x_2^2,\\ \frac{dx_3}{d\tau} &= ex_1x_3+fx_2x_3+g\left(1-A\right)x_3-mx_3-q_2E_2x_3^2.\end{aligned} \tag{1}$$

Furthermore, we modified the model by incorporating more realistic assumptions about the availability of supplementary food for predators (A). In this case, if A is assumed to be an supplementary food for the apex predator (x_3), then there is a reduction in predation pressure on the other two populations by $(1-A)$. Moreover, we assume natural mortality in the prey population, resulting in the following model:

$$\begin{aligned}\frac{dx_1}{d\tau} &= x_1\left(r-kx_1\right)-ax_1x_2-\left(1-A\right)bx_1x_3-d_1x_1,\\ \frac{dx_2}{d\tau} &= cx_1x_2-\left(1-A\right)hx_2x_3-d_2x_2-q_1x_2^2,\\ \frac{dx_3}{d\tau} &= ex_1x_3+fx_2x_3+gAx_3-d_3x_3-q_2x_3^2,\end{aligned} \tag{2}$$

with parameters:

r = intrinsic growth rate of prey,
k = carrying capacity of prey,
a = predation rate of prey by mesopredators,
b = predation rate of prey by apex predators,
A = constant supply of supplementary food for apex predators,
c = biomass conversion rate of mesopredators as they feed on prey,
d_1 = natural mortality rate of prey,
d_2 = natural mortality rate of mesopredators,
d_3 = natural mortality rate of apex predator,
h = predation rate of mesopredators by apex predators,
e = biomass conversion rate of apex predators as they feed on prey,
f = biomass conversion rate of apex predators due to feeding on mesopredators,
g = growth rate of apex predator due to supplementary food availability,
q_1 = mortality rate of mesopredator due to intraspecific competition,
q_2 = mortality rate of apex predator due to intraspecific competition.

Thus, a model formulation is obtained involving three variables: prey population density (x_1), mesopredator population density (x_2), and apex predator population density (x_3) at time τ, as shown in eq. (2). The model formulation in eq. (2) is converted into a dimensionless model by using dimensionless variables and parameters. Suppose:

$$\begin{aligned}&x=\frac{1}{\zeta}x_1,\ \ y_1=\frac{1}{\omega}x_2,\ \ y_2=\frac{1}{\gamma}x_3\ ,\ \ \tau=t_0t,\ \ \alpha=\frac{c}{k},\\ &\beta=\frac{e}{k},\ \ \eta=\frac{h}{b},\ \ \theta=\frac{f}{a},\ \ \kappa_1=\frac{q_1}{a},\ \ \kappa_2=\frac{q_2}{b},\ \ \sigma=\frac{g}{r},\\ &\delta_1=\frac{d_1}{r},\ \ \delta_2=\frac{d_2}{r},\ \ \delta_3=\frac{d_3}{r},\end{aligned}$$

then the dimensionless model in eq. (3) is obtained:

$$\begin{aligned}\frac{dx}{dt} &= x\left(1-x\right)-xy_1-\left(1-A\right)xy_2-\delta_1x,\\ \frac{dy_1}{dt} &= \alpha xy_1-\left(1-A\right)\eta y_1y_2-\delta_2y_1-\kappa_1y_1^2,\\ \frac{dy_2}{dt} &= \beta xy_2+\theta y_1y_2+\sigma Ay_2-\delta_3y_2-\kappa_2y_2^2,\end{aligned} \tag{3}$$

with $x\left(0\right)>0,\ y_1\left(0\right)>0$, and $y_2\left(0\right)>0$.

3. Results and Discussion

3.1. *Existence and Uniqueness of Solutions*

Theorem 1. *System (3) with initial value* $V(0) = (x(0), y_1(0), y_2(0)) \in \mathrm{I}$, *has a unique solution* $V(t) \in \mathrm{I}$ *with* $\mathrm{I} = \left\{V = (x, y_1, y_2) \in R_+^3\right\}$.

Proof. We will show the existence and uniqueness of the solution of system (3) in $\mathrm{I} \times [0, T]$ with $T < \infty$. Suppose $V = (x, y_1, y_2)$ and $\overline{V} = (\overline{x}, \overline{y}_1, \overline{y}_2)$, with the mapping $U(V) = (U_1(V), U_2(V), U_3(V))$ and

$$\begin{aligned}
U_1(V) &= x(1-x) - xy_1 - (1-A)xy_2 - \delta_1 x,\\
U_2(V) &= \alpha xy_1 - (1-A)\eta y_1 y_2 - \delta_2 y_1 - \kappa_1 y_1^2,\\
U_3(V) &= \beta xy_2 + \theta y_1 y_2 + \sigma A y_2 - \delta_3 y_2 - \kappa_2 y_2^2.
\end{aligned}$$

For every V, $\overline{V} \in \mathrm{I}$, we have

$$\begin{aligned}
\left\|U(V) - U(\overline{V})\right\| &= \left|U_1(V) - U_1(\overline{V})\right| + \left|U_2(V) - U_2(\overline{V})\right|\\
&\quad + \left|U_3(V) - U_3(\overline{V})\right|\\
&= \big|\left[x(1-x) - xy_1 - (1-A)xy_2 - \delta_1 x\right]\\
&\quad - \left[\overline{x}(1-\overline{x}) - \overline{xy}_1 - (1-A)\overline{xy}_2 - \delta_1\overline{x}\right]\big|\\
&\quad + \big|\left[\alpha xy_1 - (1-A)\eta y_1 y_2 - \delta_2 y_1 - \kappa_1 y_1^2\right]\\
&\quad - \left[\alpha\overline{xy}_1 - (1-A)\eta\overline{y}_1\overline{y}_2 - \delta_2\overline{y}_1 - \kappa_1\overline{y}_1^2\right]\big|\\
&\quad + \big|\left[\beta xy_2 + \theta y_1 y_2 + \sigma A y_2 - \delta_3 y_2 - \kappa_2 y_2^2\right]\\
&\quad - \left[\beta\overline{xy}_2 + \theta\overline{y}_1\overline{y}_2 + \sigma A\overline{y}_2 - \delta_3\overline{y}_2 - \kappa_2\overline{y}_2^2\right]\big|\\
&= \big|\big((1-x) - \overline{y}_1 - (1-A)\overline{y}_2 - \delta_1\big)(x - \overline{x})\\
&\quad - x(y_1 - \overline{y}_1) - (1-A)x(y_2 - \overline{y}_2)\big|\\
&\quad + \big|\alpha\overline{y}_1(x - \overline{x}) + \big(\alpha x - (1-A)\eta\overline{y}_2 - \delta_2\big)\\
&\quad (y_1 - \overline{y}_1) - (1-A)\eta y_1(y_2 - \overline{y}_2)\\
&\quad - \kappa_1(y_1^2 - \overline{y}_1^2)\big|\\
&\quad + \big|\beta\overline{y}_2(x - \overline{x}) + \theta\overline{y}_2(y_1 - \overline{y}_1)\\
&\quad + \big(\beta x + \theta y_1 + \sigma A - \delta_3\big)(y_2 - \overline{y}_2)\\
&\quad - \kappa_2(y_2^2 - \overline{y}_2^2)\big|\\
&\leq \big|\big((1-x) - \overline{y}_1 - (1-A)\overline{y}_2 - \delta_1\big)(x - \overline{x})\big|\\
&\quad + |x(y_1 - \overline{y}_1)| + |(1-A)x(y_2 - \overline{y}_2)|\\
&\quad + |\alpha\overline{y}_1(x - \overline{x})| + \big|\big(\alpha x - (1-A)\eta\overline{y}_2 - \delta_2\big)\\
&\quad (y_1 - \overline{y}_1)\big| + |(1-A)\eta y_1(y_2 - \overline{y}_2)|\\
&\quad + \left|\kappa_1(y_1^2 - \overline{y}_1^2)\right| + |\beta\overline{y}_2(x - \overline{x})|\\
&\quad + |\theta\overline{y}_2(y_1 - \overline{y}_1)| + \big|\big(\beta x + \theta y_1 + \sigma A - \delta_3\big)\\
&\quad (y_2 - \overline{y}_2)\big| + \left|\kappa_2(y_2^2 - \overline{y}_2^2)\right|\\
&= \big(\big((1-x) - \overline{y}_1 - (1-A)\overline{y}_2 - \delta_1\big) + \alpha\overline{y}_1\\
&\quad + \beta\overline{y}_2\big)|x - \overline{x}| + (x + (\alpha x - (1-A)\eta\overline{y}_2\\
&\quad - \delta_2) + \theta\overline{y}_2 + \kappa_1|y_1 + \overline{y}_1|)|y_1 - \overline{y}_1|\\
&\quad + ((1-A)x + (1-A)\eta y_1\\
&\quad + (\beta x + \theta y_1 + \sigma A - \delta_3) + \kappa_2|y_2 + \overline{y}_2|)\\
&\qquad |y_2 - \overline{y}_2|.
\end{aligned}$$

Furthermore, suppose a positive constant M with $M = \max x, y_1, y_2, \overline{x}, \overline{y}_1, \overline{y}_2$, $\forall t \geq 0$, so that it is obtained

$$\begin{aligned}
\left\|U(V) - U\left(\overline{V}\right)\right\| &\leq (((1-M) - M - (1-A)M - \delta_1)\\
&\quad + \alpha M + \beta M)|x - \overline{x}|\\
&\quad + (M + (\alpha M - (1-A)\eta M - \delta_2) + \theta M\\
&\quad + 2\kappa_1 M)|y_1 - \overline{y}_1| + ((1-A)M\\
&\quad + (1-A)\eta M + (\beta M + \theta M + \sigma A - \delta_3)\\
&\quad + 2\kappa_2 M)|y_2 - \overline{y}_2|\\
&= W_1|x - \overline{x}| + W_2|y_1 - \overline{y}_1| + W_3|y_2 - \overline{y}_2|,
\end{aligned}$$

with

$$\begin{aligned}
W_1 &= 1 - \delta_1 + (\alpha + \beta + A - 3)M,\\
W_2 &= (1 + \alpha + \theta + \eta A + 2\kappa_1 - \eta)M - \delta_2,\\
W_3 &= (1 + \eta + \beta + \theta + 2\kappa_2 - A - \eta A)M + \sigma A - \delta_3.
\end{aligned}$$

By choosing a positive constant $W = \max(W_1, W_2, W_3)$, obtained

$$\begin{aligned}
\left\|U(V) - U\left(\overline{V}\right)\right\| &= W(|x - \overline{x}| + |y_1 - \overline{y}_1| + |y_2 - \overline{y}_2|)\\
&\leq W\left\|V - \overline{V}\right\|.
\end{aligned}$$

According to the Lipschitz condition [19], the function $U(V)$ satisfies the Lipschitz condition criteria so that there is a unique solution $V(t)$ of system (3) with initial values $V(0) = (x(0), y_1(0), y_2(0)) \in \mathrm{I}$. Therefore, the solution of system (3) exists and is unique. ☐

3.2. *Nonnegativity and Boundedness Solution*

This section demonstrates the nonnegativity and boundedness of the solution of system (3). Since the variables used in system (3) represent the population density and are limited by the environment's carrying capacity, the system's solution must be nonnegative and bounded. The solution of system (3) is guaranteed to be positive and finite based on Theorem 2 and Theorem 3.

Theorem 2. *The solution of system (3) is positive provided that* $x(0), y_1(0), y_2(0) \in \mathbb{R}_+^3$.

Proof. It will be proved that if $x(0) \geq 0$, $y_1(0) \geq 0$ and $y_2(0) \geq 0$, then $x(t) \geq 0$, $y_1(t) \geq 0$ and $y_2(t) \geq 0$ for every $t > 0$. Condition $x(0) = 0$ will result in $\frac{dx}{dt} = 0$ which indicates that there is no change in population size x. Next review the condition $x(0) > 0$. Suppose there is t_T with $0 < t < t_T$ such that $x(t) \geq 0$, $x(t_T) = 0$ and $x(t) < 0$ for $t > t_T$. Based on the system (3), the condition $x(t_T) = 0$ resulting in $\frac{dx}{dt} = 0$. This contradicts the statement $x(t) < 0$ for $t > t_T$, so the supposition is false. Thus, $x(t) \geq 0$ for every t. In the same way it can be shown that the condition $y_1(0) \geq 0$ will result in $y_1(t) \geq 0$ for every $t > 0$, and the condition $y_2(0) \geq 0$ will result in $y_2(t) \geq 0$ for every $t > 0$. ☐

Theorem 3. *All solutions of system (3) with initial values* $(x(0), y_1(0), y_2(0)) \in \mathbb{R}_+^3$ *are bounded to the interval* $I =$

$$\left\{(x,y_1,y_2)\in R_+^3 : 0<B\le \tfrac{(1+\sigma-\delta_1)^2}{4\sigma}+\epsilon,\ \epsilon>0\right\}.$$

Proof. Define a function

$$B=c_1x+c_2y_1+c_3y_2.$$

Derivatives B against t, obtain

$$\begin{aligned}
\frac{dB}{dt}&=c_1\frac{dx}{dt}+c_2\frac{dy_1}{dt}+c_3\frac{dy_2}{dt}\\
&=c_1\left[x\left(1-x\right)-xy_1-\left(1-A\right)xy_2-\delta_1x\right]\\
&\quad+c_2\left[\alpha xy_1-\left(1-A\right)\eta y_1y_2-\delta_2y_1-\kappa_1y_1^2\right]\\
&\quad+c_3\left[\beta xy_2+\theta y_1y_2+\sigma Ay_2-\delta_3y_2-\kappa_2y_2^2\right]\\
&=c_1x\left(1-x\right)-c_1xy_1-c_1\left(1-A\right)xy_2-c_1\delta_1x+c_2\alpha xy_1\\
&\quad-c_2\left(1-A\right)\eta y_1y_2-c_2\delta_2y_1-c_2\kappa_1y_1^2+c_3\beta xy_2\\
&\quad+c_3\theta y_1y_2+\sigma Ay_2-\delta_3y_2-\kappa_2y_2^2\\
&=c_1\left(1-\delta_1\right)x-c_1x^2-c_2\delta_2y_1-c_2\kappa_1y_1^2+\left(\sigma A-\delta_3\right)y_2\\
&\quad-\kappa_2y_2^2-\left(c_1-c_2\alpha\right)xy_1-\left(c_1-c_1A-c_3\beta\right)xy_2\\
&\quad-\left(c_2\eta-c_2\eta A-c_3\theta\right)y_1y_2.
\end{aligned}$$

Suppose a positive constant σ. It will be shown that

$$\frac{dB}{dt}+\sigma B\le K,\ K=\text{constants}$$

By completing the left side, we get

$$\begin{aligned}
\frac{dB}{dt}+\sigma B&=c_1\left(1-\delta_1\right)x-c_1x^2-c_2\delta_2y_1-c_2\kappa_1y_1^2\\
&\quad+\left(\sigma A-\delta_3\right)y_2-\kappa_2y_2^2-\left(c_1-c_2\alpha\right)xy_1\\
&\quad-\left(c_1-c_1A-c_3\beta\right)xy_2-\left(c_2\eta-c_2\eta A-c_3\theta\right)y_1y_2\\
&\quad+c_1\sigma x+c_2\sigma y_1+c_3\sigma y_2.
\end{aligned}$$

By choosing $c_1=1,\ c_2=\frac{1}{\alpha}, c_3=\frac{1-A}{\beta}$, we obtain

$$\begin{aligned}
\frac{dB}{dt}+\sigma B&=(1-\delta_1)x-x^2-\frac{\delta_2}{\alpha}y_1-\frac{\kappa_1}{\alpha}y_1^2+(\sigma A-\delta_3)y_2\\
&\quad-\kappa_2y_2^2-\left(1-\frac{\alpha}{\alpha}\right)xy_1-\left(1-A-\frac{(1-A)\beta}{\beta}\right)xy_2\\
&\quad-\left(\frac{\eta}{\alpha}-\frac{\eta A}{\alpha}-\frac{(1-A)\theta}{\beta}\right)y_1y_2+\sigma x+\frac{\sigma}{\alpha}y_1\\
&\quad+\frac{(1-A)\sigma}{\beta}y_2\\
&=(1-\delta_1)x-x^2-\frac{\delta_2}{\alpha}y_1-\frac{\kappa_1}{\alpha}y_1^2+(\sigma A-\delta_3)y_2-\kappa_2y_2^2\\
&\quad-\frac{(\eta\beta-\alpha\theta)(1-A)}{\alpha\beta}y_1y_2+\sigma x+\frac{\sigma}{\alpha}y_1+\frac{(1-A)\sigma}{\beta}y_2\\
&=(1-\delta_1)x+\sigma x-x^2-\left(\frac{\delta_2+\sigma}{\alpha}\right)y_1\\
&\quad+\left(\frac{\beta(\sigma A-\delta_3)+\sigma(1-A)}{\beta}\right)y_2\\
&\quad-\left(\frac{\kappa_1}{\alpha}y_1^2+\kappa_2y_2^2+\frac{(\eta\beta-\alpha\theta)(1-A)}{\alpha\beta}y_1y_2\right)\\
&=(1-\delta_1)x+\sigma x-x^2-\left(\frac{\delta_2+\sigma}{\alpha}\right)y_1\\
&\quad-\left(\frac{\beta\delta_3-\sigma(\beta A+1-A)}{\beta}\right)y_2\\
&\quad-\left(\frac{\kappa_1}{\alpha}y_1^2+\kappa_2y_2^2+\frac{(\eta\beta-\alpha\theta)(1-A)}{\alpha\beta}y_1y_2\right).
\end{aligned}$$

If $\sigma=\frac{\beta\delta_3}{\beta A+1-A}$ and $\eta\beta-\alpha\theta>0$, then

$$\begin{aligned}
\frac{dB}{dt}+\sigma B&\le(1-\delta_1)x+\sigma x-x^2\\
&=-x^2+(1+\sigma-\delta_1)x\\
&=-x^2+\frac{2x(1+\sigma-\delta_1)}{2}\\
&=-\left(x-\frac{1+\sigma-\delta_1}{2}\right)^2+\frac{(1+\sigma-\delta_1)^2}{4}\\
&\le\frac{(1+\sigma-\delta_1)^2}{4}.
\end{aligned}$$

Therefore, it is obtained

$$\frac{dB}{dt}+\sigma B\le\frac{\left(1+\sigma-\delta_1\right)^2}{4}.$$

Based on Gronwall's inequality [20], we obtain

$$\lim_{t\to\infty}B\le\frac{\left(1+\sigma-\delta_1\right)^2}{4\sigma}.$$

This shows that the solution $B(x,y_1,y_2)$ is on the interval I, with

$$I=\left\{(x,y_1,y_2)\in R_+^3 : 0<B\le\frac{\left(1+\sigma-\delta_1\right)^2}{4\sigma}+\epsilon,\ \epsilon>0\right\}.$$

□

3.3. Existence of Equilibrium Points

The equilibrium point of system (3) is obtained when

$$\frac{dx}{dt}=\frac{dy_1}{dt}=\frac{dy_2}{dt}=0$$

so, we obtain

$$x=0\ \vee\ x=1-y_1-\left(1-A\right)y_2-\delta_1, \tag{4}$$

$$y_1=0\ \vee\ y_1=\frac{\alpha x-\left(1-A\right)\eta y_2-\delta_2}{\kappa_1}, \tag{5}$$

$$y_2=0\ \vee\ y_2=\frac{\beta x+\theta y_1+\sigma A-\delta_3}{\kappa_2}. \tag{6}$$

From eq. (4), eq. (5), and eq. (6), six equilibrium points are obtained, namely

1. The trivial equilibrium point indicating the extinction of the entire population, $E_1\left(0,0,0\right)$ that always exists in R_+^3.
2. The predator-free equilibrium point, $E_2\left(1-\delta_1,0,0\right)$ which exists if $\delta_1<1$.
3. The prey and mesopredator-free equilibrium points, $E_3\left(0,0,\frac{\sigma A-\delta_3}{\kappa_2}\right)$, which exists if $\delta_3<\sigma A$.
4. The mesopredator-free equilibrium point, $E_4\left(x^*,0,y_2^*\right)$, with

$$x^*=\frac{\kappa_2-(\sigma A-\delta_3)(1-A)-\delta_1\kappa_2}{\kappa_2+\beta(1-A)}$$

$$y_2^* = \frac{\Lambda_1}{\kappa_2\left(\kappa_2 + \beta(1-A)\right)},$$

where

$$\begin{aligned}\Lambda_1 &= \beta\left(\kappa_2 - (\sigma A - \delta_3)(1-A) - \delta_1\kappa_2\right)\\ &\quad + (\sigma A - \delta_3)\left(\kappa_2 + \beta(1-A)\right).\end{aligned}$$

The equilibrium point E_4 exists if it meets the conditions:

$$\begin{aligned}&\text{(i)} && \kappa_2 > \frac{(\sigma A - \delta_3)(1-A)}{1-\delta_1},\\ &\text{(ii)} && \delta_3 < \sigma A,\\ &\text{(iii)} && A < 1.\end{aligned}$$

5. The apex predator-free equilibrium point, $E_5\left(x^*, y_1^*, 0\right)$, with

$$x^* = \frac{\kappa_1\left(1-\delta_1\right) + \delta_2}{\kappa_1 + \alpha} \text{ and } y_1^* = \frac{\alpha\left(1-\delta_1\right) - \delta_2}{\kappa_1 + \alpha}.$$

The equilibrium point E_5 exists if it meets the conditions:

$$\begin{aligned}&\text{(i)} && \delta_1 < 1,\\ &\text{(ii)} && \delta_2 < \alpha\left(1-\delta_1\right).\end{aligned}$$

6. The interior equilibrium points that indicate the existence of the entire population, $E_6\left(x^*, y_1^*, y_2^*\right)$, with

$$x^* = \frac{\delta_2 + (1-\delta_1)\kappa_1 + (\eta - \kappa_1)(1-A)y_2^*}{\kappa_1 + \alpha},$$

$$y_1^* = \frac{(\alpha\kappa_2 - \eta(1-A)\beta)x^* - \eta(1-A)(\sigma A - \delta_3) - \delta_2\kappa_2}{\kappa_1\kappa_2 + \eta\theta(1-A)},$$

$$y_2^* = \frac{\Lambda_2}{\Lambda_3},$$

where

$$\begin{aligned}\Lambda_2 &= \kappa_1(1-\delta_1)\big(\kappa_1\beta + \theta\alpha\big) - \delta_2\big(\kappa_1\beta + \theta\alpha\big)\\ &\quad + (\alpha + \kappa_1)\big(\kappa_1\sigma A - \theta\delta_2 - \kappa_1\delta_3\big),\\ \Lambda_3 &= \big(\alpha + \kappa_1\big)\big(\theta(1-A)\eta + \kappa_1\kappa_2\big)\\ &\quad + \kappa_1(1-A)\big(\kappa_1\beta + \theta\alpha\big) - (1-A)\eta\big(\kappa_1\beta + \theta\alpha\big).\end{aligned}$$

The equilibrium point E_6 exists if it meets the conditions:

$$\begin{aligned}&\text{(i)} && \delta_1 < 1 \quad \text{and} \quad \kappa_1 < \eta,\\ &\text{(ii)} && \alpha\kappa_2 > \eta(1-A)\beta, \quad \sigma A < \delta_3, \quad \text{and}\\ & && \delta_2\kappa_2 < \left(\alpha\kappa_2 - \eta(1-A)\beta\right)x^* - \eta(1-A)(\sigma A - \delta_3),\\ &\text{(iii)} && (\sigma A - \delta_3)(\kappa_1 + \alpha) < \beta\left(\delta_2 + \kappa_1(1-\delta_1)\right) \quad \text{and}\\ & && \kappa_2(\kappa_1 + \alpha) > \beta(\eta - \kappa_1)(1-A).\end{aligned}$$

3.4. Stability of Equilibrium Point

The Jacobian matrix of the system (3) is defined

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2}\\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y_1} & \frac{\partial f_3}{\partial y_2}\end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} & J_{13}\\ J_{21} & J_{22} & J_{23}\\ J_{31} & J_{32} & J_{33}\end{bmatrix},$$

where

$$\begin{aligned}J_{11} &= 1 - 2x - y_1 - (1-A)y_2 - \delta_1,\\ J_{12} &= -x,\\ J_{13} &= -(1-A)x,\\ J_{21} &= \alpha y_1,\\ J_{22} &= \alpha x - (1-A)\eta y_2 - \delta_2 - 2\kappa_1 y_1,\\ J_{23} &= -(1-A)\eta y_1,\\ J_{31} &= \beta y_2,\\ J_{32} &= \theta y_2,\\ J_{33} &= \beta x + \theta y_1 + \sigma A - \delta_3 - 2\kappa_2 y_2.\end{aligned}$$

Theorem 4. *Trivial equilibrium point $E_1(0,0,0)$ is locally asymptotically stable if $\delta_1 > 1$ and $\delta_3 > \sigma A$.*

Proof. The Jacobian matrix for $E_1(0,0,0)$ is

$$J_{E_1} = \begin{bmatrix} 1-\delta_1 & 0 & 0\\ 0 & -\delta_2 & 0\\ 0 & 0 & \sigma A - \delta_3\end{bmatrix},$$

which results in the eigenvalue $\lambda_1 = 1 - \delta_1$, $\lambda_2 = -\delta_2$, and $\lambda_3 = \sigma A - \delta_3$. Since all parameters are positive, the trivial equilibrium point E_1 is locally asymptotically stable provided $\delta_1 > 1$ and $\delta_3 > \sigma A$. ☐

Theorem 5. *Predator-free equilibrium point $E_2\left(1-\delta_1, 0, 0\right)$ is locally asymptotically stable if $\delta_2 > \alpha\left(1-\delta_1\right)$ and $\delta_3 > \beta\left(1-\delta_1\right) + \sigma A$.*

Proof. The Jacobian matrix for $E_2\left(1-\delta_1, 0, 0\right)$ is

$$J_{E_2} = \begin{bmatrix} \delta_1 - 1 & 1-\delta_1 & -\left(1-A\right)\left(1-\delta_1\right)\\ 0 & \alpha\left(1-\delta_1\right) - \delta_2 & 0\\ 0 & 0 & \beta\left(1-\delta_1\right) + \sigma A - \delta_3\end{bmatrix},$$

which results in the eigenvalue $\lambda_1 = \delta_1 - 1$, $\lambda_2 = \alpha\left(1-\delta_1\right) - \delta_2$, and $\lambda_3 = \beta\left(1-\delta_1\right) + \sigma A - \delta_3$. Since the existence condition E_1 is $\delta_1 < 1$, then the predator-free equilibrium point E_1 is locally asymptotically stable provided $\delta_2 > \alpha\left(1-\delta_1\right)$ and $\delta_3 > \beta\left(1-\delta_1\right) + \sigma A$. ☐

Theorem 6. *Prey and mesopredator-free equilibrium points, $E_3\left(0, 0, \frac{\sigma A - \delta_3}{k_2}\right)$ is locally asymptotically stable if*

$$\kappa_2 < \min\left\{\frac{\left(1-A\right)\left(\sigma A - \delta_3\right)}{\left(1-\delta_1\right)}, \frac{\eta\left(1-A\right)\left(\sigma A - \delta_3\right)}{\delta_2}\right\}.$$

Proof. The Jacobian matrix for $E_3\left(0,0,\frac{\sigma A-\delta_3}{k_2}\right)$ is

$$J_{E_3}=\begin{bmatrix} j_{11} & 0 & 0 \\ 0 & j_{22} & 0 \\ \frac{\beta(\sigma A-\delta_3)}{k_2} & \frac{\theta(\sigma A-\delta_3)}{k_2} & \delta_3-\sigma A \end{bmatrix},$$

with

$$\begin{aligned} j_{11}&=1-\frac{(1-A)(\sigma A-\delta_3)}{k_2}-\delta_1,\\ j_{22}&=-\frac{(1-A)\eta(\sigma A-\delta_3)}{k_2}-\delta_2, \end{aligned}$$

which results in an eigenvalues:

$$\begin{aligned} \lambda_1&=\frac{(1-\delta_1)\kappa_2-(1-A)(\sigma A-\delta_3)}{\kappa_2},\\ \lambda_2&=\frac{-\eta(1-A)(\sigma A-\delta_3)+\delta_2\kappa_2}{\kappa_2},\\ \lambda_3&=\delta_3-\sigma A. \end{aligned}$$

Based on the existence conditions E_3, $\delta_3<\sigma A$, so it can be confirmed that $\lambda_3<0$. Thus, the equilibrium point E_3 is locally asymptotically stable if λ_1 and λ_2 is negative, i.e.:

$$\kappa_2<\frac{(1-A)\left(\sigma A-\delta_3\right)}{(1-\delta_1)}\text{ and }\kappa_2<\frac{\eta\left(1-A\right)\left(\sigma A-\delta_3\right)}{\delta_2},$$

or it can be written

$$\kappa_2<\min\left\{\frac{(1-A)\left(\sigma A-\delta_3\right)}{(1-\delta_1)},\frac{\eta\left(1-A\right)\left(\sigma A-\delta_3\right)}{\delta_2}\right\}.$$

☐

Theorem 7. *Suppose* $\mathcal{S}_1=\kappa_2-\left(\sigma A-\delta_3\right)\left(1-A\right)-\delta_1\kappa_2$, $\mathcal{S}_2=\beta\left(\kappa_2-\left(\sigma A-\delta_3\right)\left(1-A\right)-\delta_1\kappa_2\right)+\left(\sigma A-\delta_3\right)\left(\kappa_2+\beta\left(1-A\right)\right)$, $x^*=\frac{\mathcal{S}_1}{\kappa_2+\beta(1-A)}$ *and* $y_2^*=\frac{\mathcal{S}_2}{\kappa_2(\kappa_2+\beta(1-A))}$. *The mesopredator-free equilibrium point,* $E_4\left(x^*,0,y_2^*\right)$ *is locally asymptotically stable if satisfied:*

$$\begin{aligned} &\text{(i)}\quad \alpha<\frac{(1-A)\eta y_2^*+\delta_2}{x^*},\\ &\text{(ii)}\quad \kappa_2+(1-A)\beta>0. \end{aligned}$$

Proof. The Jacobian matrix for $E_4\left(x^*,0,y_2^*\right)$ is

$$J_{E_4}=\begin{bmatrix} -x^* & -x^* & -\left(1-A\right)x^* \\ 0 & J_{22} & 0 \\ \beta y_2^* & \theta y_2^* & -k_2y_2^* \end{bmatrix},$$

where $J_{22}=\alpha x^*-\left(1-A\right)\eta y_2^*-\delta_2$. It is easy to show the first eigenvalue, which is $\lambda_1=\alpha x^*-\left(1-A\right)\eta y_2^*-\delta_2<0$ if $\alpha<\frac{(1-A)\eta y_2^*+\delta_2}{x^*}$. Furthermore, the other two eigenvalues are obtained from the quadratic equation:

$$\lambda^2+\left(x^*+k_2y_2^*\right)\lambda+\left(k_2+\left(1-A\right)\beta\right)x^*y_2^*=0. \qquad (7)$$

Based on the quadratic Equation (7), it is clear that $x^*+k_2y_2^*>0$. Therefore, an eigenvalue can be shown to be negative if $\left(k_2+\left(1-A\right)\beta\right)>0$. Thus, E_4 is locally asymptotically stable if it satisfies the conditions:

$$\begin{aligned} &\text{(i)}\quad \alpha<\frac{(1-A)\eta y_2^*+\delta_2}{x^*},\\ &\text{(ii)}\quad \kappa_2+(1-A)\beta>0. \end{aligned}$$

☐

Theorem 8. *Suppose* $x^*=\frac{\kappa_1(1-\delta_1)+\delta_2}{\kappa_1+\alpha}$ and $y_1^*=\frac{\alpha(1-\delta_1)-\delta_2}{\kappa_1+\alpha}$. *Apex predator-free equilibrium point* $E_5\left(x^*,y_1^*,0\right)$ *is locally asymptotically stable if* $\delta_3>\beta x^*+\theta y_1^*+\sigma A$.

Proof. The Jacobian matrix for $E_5\left(x^*,y_1^*,0\right)$ is

$$J_{E_5}=\begin{bmatrix} -x^* & -x^* & -\left(1-A\right)x^* \\ \alpha y_1^* & -k_1y_1^* & -\left(1-A\right)\eta y_1^* \\ 0 & 0 & J_{33} \end{bmatrix},$$

where $J_{33}=\beta x^*+\theta y_1^*+\sigma A-\delta_3$. It is easy to show the first eigenvalue, which is $\lambda_1=\beta x^*+\theta y_1^*+\sigma A-\delta_3<0$ if $\delta_3>\beta x^*+\theta y_1^*+\sigma A$. Furthermore, the other two eigenvalues are obtained from the quadratic equation:

$$\lambda^2+\left(x^*+k_1y_1^*\right)\lambda+\left(k_1+\alpha\right)x^*y_1^*=0. \qquad (8)$$

Since $x^*+k_1y_1^*>0$ and $k_1+\alpha>0$, then the other two eigenvalues are negative. Thus, E_5 is locally asymptotically stable if $\delta_3>\beta x^*+\theta y_1^*+\sigma A$. ☐

Theorem 9. *Interior equilibrium point* $E_6\left(x^*,y_1^*,y_2^*\right)$ *locally asymptotically stable under certain conditions.*

Proof. The Jacobian matrix for $E_6\left(x^*,y_1^*,y_2^*\right)$ is

$$J_{E_6}=\begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}, \qquad (9)$$

where $J_{11}=-x^*$, $J_{12}=-x^*$, $J_{13}=-\left(1-A\right)x^*$, $J_{21}=\alpha y_1^*$, $J_{22}=-\kappa_1y_1^*$, $J_{23}=-\left(1-A\right)\eta y_1^*$, $J_{31}=\beta y_2^*$, $J_{32}=\theta y_2^*$, and $J_{33}=-\kappa_2y_2^*$. Based on the Jacobian matrix (9), the characteristic equation is obtained

$$\lambda^3+\sigma_1\lambda^2+\sigma_2\lambda+\sigma_3=0, \qquad (10)$$

where

$$\begin{aligned} \sigma_1&=-\left(J_{11}+J_{22}+J_{33}\right)>0,\\ \sigma_2&=J_{11}J_{22}+J_{11}J_{33}+J_{22}J_{33}-J_{12}J_{21}-J_{13}J_{31}-J_{32}J_{23},\\ \sigma_3&=J_{11}J_{32}J_{23}+J_{12}J_{21}J_{33}+J_{13}J_{31}J_{22}-J_{11}J_{22}J_{33}\\ &\quad-J_{12}J_{23}J_{31}-J_{13}J_{21}J_{32}. \end{aligned}$$

Furthermore, the stability properties can be analyzed using the Ruth-Hurwitz criterion provided that it can be shown that

$\sigma_1, \sigma_2, \sigma_3$ is a positive coefficient. From the characteristic Equation (10), the Hurwitz matrix is obtained:

$$H = \begin{bmatrix} \sigma_1 & 1 & 0 \\ \sigma_3 & \sigma_2 & \sigma_1 \\ 0 & 0 & \sigma_3 \end{bmatrix}.$$

All roots of the characteristic equation are negative if the determinant of the submatrix H is positive. It will be shown that:

$$|H_1| = |\sigma_1| > 0,$$

$$|H_2| = \begin{vmatrix} \sigma_1 & 1 \\ \sigma_3 & \sigma_2 \end{vmatrix} > 0, \quad \text{and}$$

$$|H_3| = \begin{vmatrix} \sigma_1 & 1 & 0 \\ \sigma_3 & \sigma_2 & \sigma_1 \\ 0 & 0 & \sigma_3 \end{vmatrix} > 0.$$

Since $\sigma_1 > 0$, then $|H_1| > 0$. Furthermore, it can be shown that $|H_2| > 0$ and $|H_3| > 0$ if $\sigma_1\sigma_2 > \sigma_3$. Thus, the interior equilibrium point E_6 can be shown to be locally asymptotically stable by showing that σ_2, σ_3 is positive and $\sigma_1\sigma_2 > \sigma_3$. ☐

3.5. Numerical Simulation

This section provides numerical simulations to confirm and validate the analytical findings regarding the stability of the system's equilibrium points. Numerical simulations were obtained using the fourth order Runge-Kutta method. In the absence of empirical field data, the parameter values used in this model are determined based on ecological assumptions and biological definitions as explained in Section 2. The hypothetical parameter values outlined in Table 1 are utilized to qualitatively examine the model dynamics and provide meaningful insights into the system's behavior.

Table 1. Parameter value

Parameter	Value
A	0.30
α	0.95
η	1.80
β	0.10
θ	0.70
σ	0.30
δ_1	1.40
δ_2	0.70
δ_3	0.40
κ_1	0.10
κ_2	0.10

Using the parameter values in Table 1, we obtain the phase and time series portraits that describe the stability of the trivial equilibrium point $E_0\,(0,0,0)$, which is shown in Figure 1.

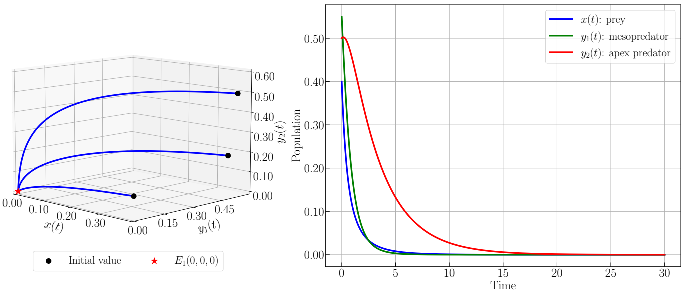


Figure 1. Phase portrait and time series diagram for $E_1\,(0,0,0)$ stable, indicating the extinction of the entire population

The simulation results in Figure 1 show that all trajectories of the three different initial conditions converge quickly to the trivial equilibrium point $E_0 = (0,\ 0,\ 0)$, where the three populations: prey (x), mesopredator (y_1), and apex predator (y_2), undergoes total extinction in a relatively short time. This signifies that E_0 is locally asymptotically stable. This phenomena illustrates the incapacity of all species to sustain their survival owing to feeble interspecies interactions, elevated death rates, or a discordance between reproductive and competitive factors. Consequently, the system experiences a total ecological collapse, marked by the permanent extinction of the whole population. This condition is mainly triggered by the high value of the parameter δ_1. This is directly related to the prey's natural mortality rate, which causes the prey population to decline dramatically in a short period of time. The loss of the prey population then has a direct impact on the extinction of the predator because there are not enough resources available to sustain its life.

Furthermore, if the parameter value δ_1 is decreased to $\delta_1 = 0.40$, while the other parameters are fixed as listed in Table 1, an illustration of the stability of the predator-free equilibrium point is obtained $E_2\,(0.6,0,0)$, as shown in Figure 2.

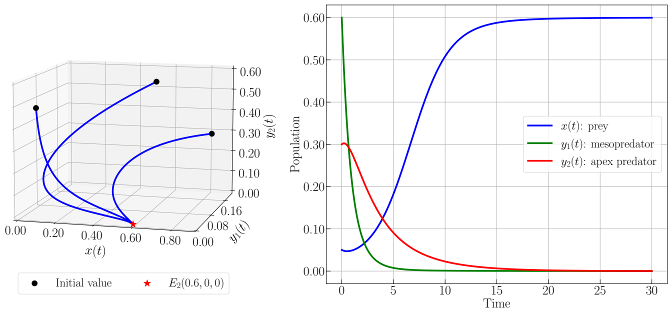


Figure 2. Phase portrait and time series diagram for $E_2\,(0.60,0,0)$ stable, indicating the extinction of predators

Figure 2 shows that all trajectories of the various initial conditions of the system converge to the predator-free equilibrium point $E_2\,(0.60,0,0)$, which is characterized by the extinction of mesopredators and apex predators, while the prey survives. This shows that E_2 is locally asymptotically stable. This situation illustrates the dominance of prey in the ecosystem due to the inability of predators, both mesopredators and apex predators, to maintain their populations. This is likely due to the lack of resources available to predators, so the system naturally supports prey survival but not predator population sustainability. The availability of supplementary food sources for apex predators is quite small $(A = 0.30)$ is in fact not strong enough to sustain the survival of predators.

Different conditions occur when the availability of supplementary food sources is increased to $A = 0.50$ accompanied by an increase in the growth rate of the apex predator due to the availability of supplementary food becomes $\sigma = 1.30$. The increase in resources from the availability of additional apex predator food causes a change in system stability towards the prey and mesopredator-free equilibrium points, $E_3\,(0, 0, 2.50)$ as shown in Figure 3.

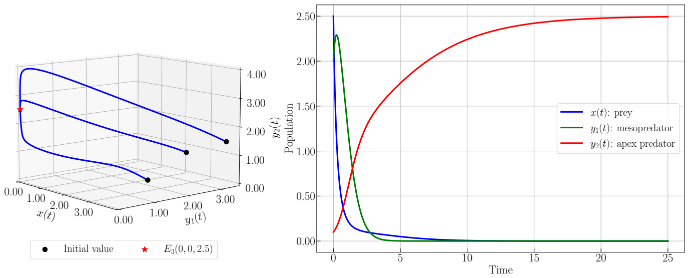


Figure 3. Phase portrait and time series diagram for $E_3\,(0, 0, 2.50)$ stable, indicating the extinction of prey and mesopredator

Figure 3 shows that all trajectories from various initial conditions of the system move towards the prey and mesopredator-free equilibrium points, $E_3\,(0, 0, 2.50)$, which is characterized by the extinction of prey and mesopredators, and the dominance of apex predators. This indicates that E_3 is locally asymptotically stable. This condition reflects the success of the apex predator in maintaining its population independently, without dependence on the presence of prey and mesopredators. This is possible due to an increase in the availability of sufficient supplementary food $(A = 0.50)$ and a significant increase in the efficiency of utilization of additional resources by apex predators $(\sigma = 1.30)$, which allowed the apex predator to thrive despite the extinction of the other two species.

Next, the condition is shown when the availability of supplementary food is reduced to $A = 0.40$, with other parameter values fixed, then the stability of the system will change towards the mesopredator-free equilibrium point $E_4\,(0.18, 0, 0.69)$ as shown in Figure 4.

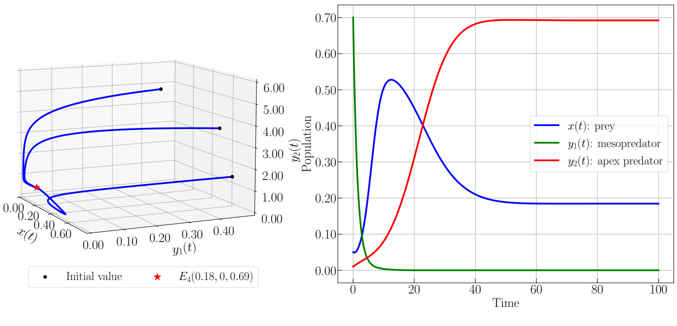


Figure 4. Phase portrait and time series diagram for $E_4\,(0.18, 0, 0.69)$ stable, indicating the extinction of mesopredator

Figure 4 shows that as the availability of supplementary food decreases, all system trajectories from different initial conditions lead to the mesopredator-free equilibrium point, $E_4\,(0.18, 0, 0.69)$. This shows that E_4 is locally asymptotically stable. This condition illustrates that only prey and apex predators are able to survive in the long term, while mesopredators are extinct due to competitive pressure and lack of adequate resources. Decrease in the value of A limits the contribution of supplementary food to the apex predator, but in this scenario it is still sufficient to sustain its survival, allowing the establishment of a balance between prey and apex predators without the presence of mesopredators. On the other hand, the reduction of supplementary food also increases the predation pressure of apex predators on mesopredators, which in turn accelerates their extinction. This finding confirms that the level of supplementary food availability is a key factor in shaping community structure and determining the stability of the ecological system of three interacting species.

Furthermore, it is shown that the dynamics that occur when mesopredators increase predation pressure on prey is characterized by an increase in predator biomass conversion due to preying on prey $(\alpha = 1.25)$. If this condition is accompanied by a significant decrease in the growth effectiveness of apex predators from supplementary food sources $(\sigma = 0.30)$, then the stability of the system will change towards the apex predator-free equilibrium point $E_5\,(0.57, 0.03, 0)$ as shown in Figure 5.

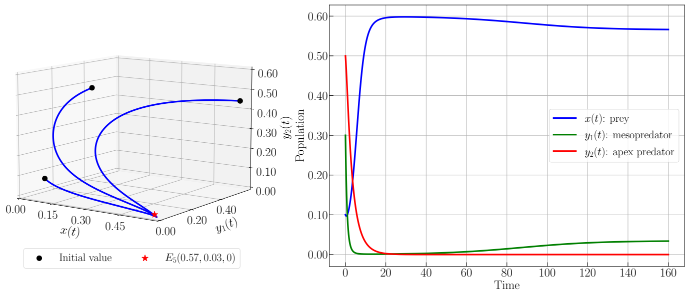


Figure 5. Phase portrait and time series diagram for $E_5\,(0.57, 0.03, 0)$ stable, indicating the extinction of apex-predator

Figure 5 shows that all system trajectories from various initial conditions lead to the apex predator-free equilibrium point, $E_5\,(0.57, 0.03, 0)$, which indicates that the equilibrium point is locally asymptotically stable. This dynamic reflects the success of mesopredators in maintaining their population through increasing the efficiency of biomass utilization from predation on prey $(\alpha = 1.25)$, while the apex predator's ability to survive decreases due to significantly reduced effectiveness in utilizing supplementary food sources $(\sigma = 0.30)$. Under these conditions, prey and mesopredators form a stable two-species interaction, while apex predators experience extinction because they do not obtain enough energy, either from supplementary food sources or through interactions with other species. This finding confirms that system stability is strongly influenced by the intensity of interspecies interactions and the availability of resources that support the survival of each population.

Furthermore, if the efficiency of biomass conversion by mesopredators due to prey eating increases significantly $(\alpha = 3.50)$, accompanied by an increase in the growth effectiveness of the apex predator due to the availability of supplementary food in sufficient quantities $(\sigma = 1.30)$, then the stability of the system will shift towards the interior equilibrium point $E_6\,(0.32, 0.06, 0.32)$, as shown in Figure 6.

Figure 6 shows that all system trajectories from various ini-

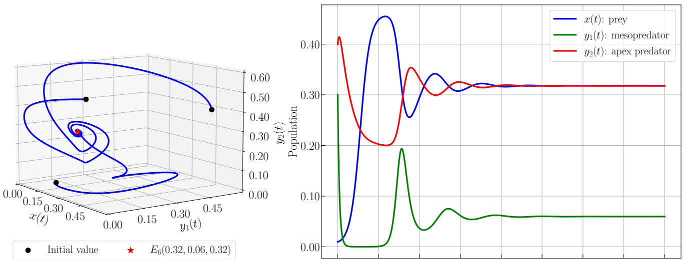


Figure 6. Phase portrait and time series diagram for $E_6\,(0.32, 0.06, 0.32)$ stable, indicating the existence of the entire population

tial conditions eventually converge to the interior equilibrium point, $E_6\,(0.32, 0.06, 0.32)$, which reflects the stable existence of the three species in the system: prey, mesopredator, and apex predator. This condition indicates that the equilibrium point is locally asymptotically stable, characterized by population fluctuations at an early time before the system reaches a stable state. The success of the system in reaching equilibrium is supported by an increase in the efficiency of prey predation by mesopredators $(\alpha = 3.50)$ as well as the high ability of apex predators to utilize supplementary food $(\sigma = 1.30)$. This situation allows for a balanced and sustainable tri-species interaction, where no species goes extinct and all populations survive in the long term. This finding confirms that the strength of interspecies interactions and resource availability are key factors in creating stability and diversity in ecological systems.

4. Conclusion

This study develops a three-species population dynamics model that includes prey, mesopredator, and apex predator, incorporating aspects of nonlinear interactions between species as well as the availability of supplementary food for the apex predator. Analytically, the model has been validated by proving the existence and uniqueness of the solution, as well as satisfying the properties of positivity and solution limitation. This guarantees that the model solution consistently resides inside a biologically relevant domain across time. The research indicates that the system possesses six equilibrium points, reflecting diverse ecological configurations, including total extinction, dominance of prey or apex predators, and coexistence among two or three species. The local stability of each equilibrium point is significantly influenced by specific characteristics, especially those concerning predation efficiency, resource availability, and predator growth rate.

Through numerical simulations, it is shown that changes in the parameters can cause significant shifts in the dynamics of the system. When predation efficiency is low or supplemental food is limited, the system tends to favor predator extinction or prey dominance. In contrast, an increase in the efficiency of mesopredator interactions with prey and an increase in the utilization of supplementary food sources by apex predators can maintain the existence of certain species or even create conditions for stable three-species coexistence. This study's findings affirm that ecological balance is influenced not only by species presence but also by the intensity of interspecies interactions and the availability of resources essential for population survival. These findings significantly enhance the advancement of ecosystem modeling and biodiversity management techniques that account for the intricacies of interconnections and system responses to environmental change.

Nevertheless, the current model assumes constant supplementary food supply and does not account for stochastic elements such as environmental noise, which may affect population dynamics in real ecosystems. These assumptions, while analytically tractable, limit the ecological realism of the model. Future research should consider extending the model by incorporating time-varying supplementary food and stochastic perturbations to better reflect natural variability and enhance predictive capacity. Such extensions would provide more robust insights into ecological stability and inform strategies for biodiversity conservation under uncertain environmental conditions.

Author Contributions. **R. Resmawan**: Software, conceptualization, formal analysis, writing—original draft preparation, writing—review and editing, visualization. **R. P. Handayani**: Conceptualization, writing—review and editing. **B. M. Rosydah**: Conceptualization, investigation, writing—original draft preparation. **F. M. Qur'ani**: Conceptualization, resources, writing—review and editing. All authors have read and agreed to the published version of the manuscript.

Acknowledgement. The authors wish to express their sincere gratitude to the the editor and anonymous reviewers for their meticulous evaluation, valuable comments, and constructive suggestions, which have substantially contributed to the enhancement of this paper.

Funding. This research was conducted without any external financial support.

Conflict of interest. The authors declare that there is no conflict of interest concerning this article.

Data availability. Not applicable.

References

[1] A. J. Lotka, *Elements of Physical Biology*. Baltimore, MD, USA: Williams and Wilkins, 1925.

[2] V. Volterra, *Variazioni e fluttuazioni del numero d'individui in specie animali conviventi*. Città di Castello, Italy: Società Anonima Tipografica "Leonardo da Vinci", 1927.

[3] R. P. Gupta, P. Chandra, and M. Banerjee, "Dynamical complexity of a prey-predator model with nonlinear predator harvesting," *Discrete Contin. Dyn. Syst. B*, vol. 20, no. 2, pp. 423–443, 2015, doi: 10.3934/dcdsb.2015.20.423.

[4] N. Pal *et al.*, "Stability and bifurcation analysis of a three-species food chain model with delay," *Int. J. Bifurcation Chaos*, vol. 25, no. 9, p. 1550123, 2015, doi: 10.1142/S0218127415501230.

[5] S. Maisaroh, R. Resmawan, and E. Rahmi, "Stability analysis of predator-prey model with disease infection in prey and proportional harvesting in predator," *Jambura J. Biomathematics*, vol. 1, no. 1, pp. 8–15, Jun. 2020, doi: 10.34312/jjbm.v1i1.5948.

[6] S. H. Arsyad, R. Resmawan, and N. Achmad, "Analysis of the Leslie-Gower predator-prey model with poison administration to predators," *J. Riset Apl. Mat.*, vol. 4, no. 1, pp. 1–16, Apr. 2020, doi: 10.26740/jram.v4n1.p1-16.

[7] H. S. Panigoro, E. Rahmi, and R. Resmawan, "Bifurcation analysis of a predator–prey model involving age structure, intraspecific competition, Michaelis–Menten type harvesting, and memory effect," *Front. Appl. Math. Stat.*, vol. 8, Art. no. 1077831, 2023, doi: 10.3389/fams.2022.1077831.

[8] I. Djakaria, M. B. Gaib, and R. Resmawan, "Analysis of the Rosenzweig-MacArthur predator-prey model with anti-predator behavior,"

CAUCHY J. Math. Pure Appl., vol. 6, no. 4, pp. 260–269, 2019, doi: 10.18860/ca.v6i4.11472.

[9] R. Resmawan *et al.*, "Dynamical analysis of a predator-prey model involving intraspecific competition in predator and prey protection," *JTAM J. Teori Apl. Mat.*, vol. 24, no. 4, pp. 706–723, 2019, doi: 10.31764/jtam.v8i3.22154.

[10] Z. Zhang and A. B. Wan, "Bifurcation analysis of a three-species ecological system with time delay and harvesting," *Adv. Differ. Equ.*, vol. 2017, Art. no. 342, 2017, doi: 10.1186/s13662-017-1393-x.

[11] A. Mondal, A. K. Pal, and G. P. Samanta, "Complex dynamics of two prey–one predator model together with fear effect and harvesting efforts in preys," *J. Comput. Math. Data Sci.*, vol. 6, p. 100071, Jan. 2023, doi: 10.1016/j.jcmds.2022.100071.

[12] H. J. Alsakaji, S. Kundu, and F. A. Rihan, "Delay differential model of one-predator two-prey system with Monod-Haldane and Holling type II functional responses," *Appl. Math. Comput.*, vol. 397, p. 125919, May 2021, doi: 10.1016/j.amc.2020.125919.

[13] A. Kumar, "Stability and bifurcation of a prey–predator system with additional food and two discrete delays," *Comput. Model. Eng. Sci.*, vol. 126, no. 2, pp. 505–547, 2021, doi: 10.32604/cmes.2021.013206.

[14] B. T. Mulugeta, "Bifurcation analysis of a one-prey and two-predators model with additional food and harvesting subject to toxicity," *Int. J. Bifurcation Chaos*, vol. 31, no. 6, 2021, doi: 10.1142/S0218127421500899.

[15] C. Arora, "Dynamics of one-prey and two-predator system highlighting the significance of additional food for predators with Beddington–DeAngelis functional response," *Differ. Equ. Dyn. Syst.*, vol. 30, no. 2, pp. 411–431, 2022, doi: 10.1007/s12591-018-0442-6.

[16] A. Kumar, "Stability and bifurcation of a prey–predator system with additional food and two discrete delays," *Comput. Model. Eng. Sci.*, vol. 126, no. 2, pp. 505–547, 2021, doi: 10.32604/cmes.2021.013206.

[17] B. T. Mulugeta, "Bifurcation analysis of a one-prey and two-predators model with additional food and harvesting subject to toxicity," *Int. J. Bifurcation Chaos*, vol. 31, no. 6, 2021, doi: 10.1142/S0218127421500899.

[18] D. K. Jana and P. Panja, "Effects of supplying additional food for a scavenger species in a prey–predator–scavenger model with quadratic harvesting," *Int. J. Model. Simul.*, vol. 43, no. 3, pp. 250–264, 2023, doi: 10.1080/02286203.2022.2065658.

[19] S. Lynch, *Dynamical Systems with Applications Using Mathematica®*. Cham, Switzerland: Springer Int. Publ., 2017, doi: 10.1007/978-3-319-61485-4.

[20] W. N. Li and W. Sheng, "Some Gronwall type inequalities on time scales," *J. Math. Inequal.*, vol. 4, no. 1, pp. 67–76, 2010, doi: 10.7153/jmi-04-08.