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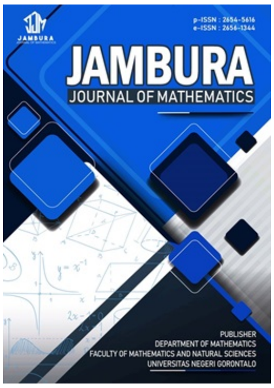
JAMBURA
JOURNAL OF MATHEMATICS

Volume 7, Issue 2, Pages 138–143, August 2025

Received 16 May 2025, Revised 1 July 2025, Accepted 4 July 2025, Published 8 July 2025

To Cite this Article : P. R. Hargandi, P. Purhadi, and A. Choiruddin,"Parameter Estimation and Hypothesis Testing of GTW Compound Correlated Bivariate Poisson Regression Model: A Theoretical Development", *Jambura J. Math*, vol. 7, no. 2, pp. 138–143, 2025, *https://doi.org/10.37905/jjom.v7i2.32712*

JOURNAL INFO • JAMBURA JOURNAL OF MATHEMATICS



Homepage	:	http://ejurnal.ung.ac.id/index.php/jjom/index
Journal Abbreviation	:	Jambura J. Math.
Frequency	:	Biannual (February and August)
Publication Language	:	English (preferable), Indonesia
DOI	:	https://doi.org/10.37905/jjom
Online ISSN	:	2656-1344
Editor-in-Chief	:	Hasan S. Panigoro
Publisher	:	Department of Mathematics, Universitas Negeri Gorontalo
Country	:	Indonesia
OAI Address	:	http://ejurnal.ung.ac.id/index.php/jjom/oai
Google Scholar ID	:	iWLjgaUAAAAJ
Email	:	info.jjom@ung.ac.id

Jambura Journal of Mathematics, Volume 7, Issue 2, Pages 138–143, August 2025 https://doi.org/10.37905/jjom.v7i2.32712

Research Article

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Parameter Estimation and Hypothesis Testing of GTW Compound Correlated Bivariate Poisson Regression Model: A Theoretical Development

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ARTICLE HISTORY
Received 16 May 2025
Revised 1 July 2025
Accepted 4 July 2025
Published 8 July 2025

KEYWORDS
Spatio-temporal Heterogeneity
Bivariate Count Data
Overdispersion
BHHH Algorithm
CCBPR
GTWCCBPR
MLRT

ABSTRACT. *Each observation location and time possesses distinct characteristics, reflecting heterogeneity at every observation point, both spatially and temporally. This condition renders the Compound Correlated Bivariate Poisson Regression (CCBPR) model inadequate for representing data dynamics that exhibit spatial and temporal heterogeneity. To address this limitation, the Geographically and Temporally Weighted Compound Correlated Bivariate Poisson Regression (GTWCCBPR) model is employed, which allows parameter variation across locations and time periods. This model also incorporates the exposure variable as a weighting factor to adjust for differences in risk across observational units. This study aims to estimate the parameters of the GTWCCBPR model using the Maximum Likelihood Estimation (MLE) approach. Due to the complex structure of the model, the log-likelihood function does not yield a closed-form solution. Therefore, parameter estimation is performed using the iterative Berndt-Hall-Hall-Hausman (BHHH) algorithm. Subsequently, hypothesis testing is conducted to evaluate the parameter similarity between the global model (CCBPR) and the spatiotemporal model (GTWCCBPR), as well as to assess the significance of each predictor variable. Simultaneous testing is carried out using the Maximum Likelihood Ratio Test (MLRT), while partial testing is conducted using the Z-test. The scope of this study is limited to theoretical formulation and methodological development, without empirical or simulation-based validation. Future research may extend this work by applying the GTWCCBPR model to practical datasets exhibiting spatio-temporal heterogeneity, particularly in areas such as public health (e.g., maternal and postneonatal mortality), epidemiology, or regional planning.*

1. Introduction

Poisson regression is a widely used statistical approach for modeling the relationship between count response variables and a set of explanatory variables. In this model, the response variable is assumed to follow a Poisson distribution, with the fundamental assumption of equidispersion, where the mean and variance are equal [1]. However, in many empirical applications, this assumption is often violated due to the presence of overdispersion, a condition in which the variance exceeds the mean. Such violations can lead to inefficient parameter estimates and invalid hypothesis testing results [2]. Furthermore, in situations involving two correlated response variables, more complex approaches are required to capture dependency structures and fluctuating variances appropriately [3].

To address overdispersion and simultaneously model two correlated response variables, the Compound Correlated Bivariate Poisson Regression (CCBPR) model has been developed. This model is an extension of bivariate Poisson regression, incorporating shared latent components to capture interdependence between responses and unexplained variance [4]. To enhance the model's flexibility in dealing with overdispersion, the Generalized Inverse Gaussian (GIG) distribution is employed as a mixing component. The GIG distribution has proven effective in handling highly skewed and heavy-tailed count data, offering more adaptive and flexible dispersion control [5, 6]. Nevertheless, a critical limitation of the CCBPR model is its assumption of globally homogeneous regression parameters across space and time. In reality, geographical characteristics, socioeconomic contexts, temporal dynamics, and policy interventions may induce spatial-temporal heterogeneity [7, 8]. Ignoring this heterogeneity may lead to biased parameter estimates and misleading interpretations.

Several studies have extended the CCBPR framework through various approaches. Salby [9] implemented the CCBPR model using the GIG distribution and included exposure variables, with parameter estimation conducted through the Maximum Likelihood Estimation (MLE) method via the iterative Berndt–Hall–Hall–Hausman (BHHH) algorithm. While effective in addressing overdispersion and response correlation, this approach remains global in nature and does not incorporate spatial or temporal heterogeneity. In contrast, Safarida [10] proposed the Geographically Weighted Compound Correlated Bivariate Poisson Regression (GWCCBPR) model, which allows for spatially varying parameters, but still lacks an explicit temporal component. Consequently, no existing model comprehensively

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Homepage : http://ejurnal.ung.ac.id/index.php/jjom/index / E-ISSN : 2656-1344

addresses bivariate correlation, overdispersion, and spatial-temporal heterogeneity simultaneously. Similarly, Fotheringham et al. [11] and Huang et al. [12] introduced geographically and temporally weighted regression (GTWR) for capturing local variation, although their models are not tailored for multivariate count data with overdispersion.

As a solution to these limitations, this study proposes the Geographically and Temporally Weighted Compound Correlated Bivariate Poisson Regression (GTWCCBPR) model. This model extends the CCBPR framework by incorporating spatial and temporal weights based on geographic proximity and observation time, enabling locally varying regression parameters. Exposure variables are also considered as risk adjusters to account for differences across observational units [13]. The GTWCCBPR model is designed to accommodate bivariate response dependency, correct for overdispersion, and capture spatial and temporal heterogeneity concurrently. Thus, it offers a methodological advancement for analyzing complex and dynamic bivariate count data and has potential applications across various location- and time-based phenomena.

This study specifically aims to estimate parameters and conduct hypothesis testing within the GTWCCBPR framework. Although the development presented in this study is theoretical, the proposed GTWCCBPR model is designed with practical applicability in mind, particularly for modeling real-world data characterized by spatial and temporal count processes. Potential applications include public health surveillance (e.g., maternal and neonatal mortality), regional economic disparity, and environmental risk analysis. Parameter estimation is performed using the Maximum Likelihood Estimation (MLE) method, optimized through the Berndt–Hall–Hall–Hausman (BHHH) algorithm due to the lack of closed-form solutions for the log-likelihood function. Hypothesis testing is conducted to assess the equivalence of parameters between the global CCBPR model and the spatial-temporal GTWCCBPR model, as well as to evaluate the significance of individual predictor variables. Simultaneous testing is carried out using the Maximum Likelihood Ratio Test (MLRT), while partial tests are conducted using Z-tests.

2. Model

2.1. Compound Correlated Bivariate Poisson Regression Models

The Compound Correlated Bivariate Poisson Regression (CCBPR) model extends the Bivariate Poisson Regression (BPR) model by introducing a compound Poisson structure. This extension allows the model to effectively handle overdispersion and correlation in count data. The CCBPR model is an advancement of the Bivariate Negative Binomial model [6], offering greater flexibility for modeling count-based outcomes with excessive variance. The probability mass function (PMF) of the CCBP distribution is defined as follows [5]:

$$(Y_1, Y_2) \sim CCBP\left(\lambda_1\left(q_1\right), \lambda_2\left(q_2\right), \lambda_0\left(q_0\right), \phi\right) \quad (1)$$

$$P\left(y_1, y_2\right) = \frac{1}{K_\gamma\left(v\right)} \sum_{a=0}^{s} \psi_1 \psi_2 K_{y_1+y_2-a+\gamma}\left(z\right) \quad (2)$$

where:

$$\psi_1 = \frac{{\lambda_0}^a\left(q_0\right)}{a!},$$

$$\psi_2 = \prod_{k=1}^{2} \frac{{\lambda_k}^{y_k-a}\left(q_k\right)}{\left(y_k-a\right)!}\left(\frac{v}{z}\right)^{y_1+y_2-a+\gamma},$$

$$z = \sqrt{v\left(v+2\left(\lambda_1\left(q_1\right)+\lambda_2\left(q_2\right)+\lambda_0\left(q_0\right)\right)\right)},$$

$$v = \sqrt{\phi^2+1}-1,\ -\infty<\gamma<\infty,$$

$$s = \min\left(y_1, y_2\right).$$

The terms K_γ and $K_{y_1+y_2-a+\gamma}$ are modified Bessel functions of the third type.

Let y_{ki} is the kth response variable for ith observation. Given a random sample $\left(Y_{1i}, Y_{2i}\right) \sim CCBP\left(\lambda_{ki}\left(q_{ki}\right), \lambda_{ki}\left(q_{ki}\right), \lambda_0\left(q_0\right), \phi\right)$, where $i = 1, 2, \cdots, n$ and $k = 1, 2$, the CCBPR model is expressed as:

$$\ln\frac{E(Y_{ki})}{q_{ki}} = {\mathbf{x}_i}^T \boldsymbol{\beta}_k, \quad (3)$$

where $E\left(Y_{ki}\right) = \lambda_{ki}\left(q_{ki}\right) = q_{ki}\exp\left({\mathbf{x}_i}^T\boldsymbol{\beta}_k\right)$, q_{ki} denotes the exposure variable from response variables for each observation, ${\mathbf{x}_i}^T = \begin{bmatrix} 1 & X_{1i} & X_{2i} & \ldots & X_{pi} \end{bmatrix}_{1\times(p+1)}$ represents the row vector of predictor variables, and $\boldsymbol{\beta}_k = \begin{bmatrix} \beta_{k0} & \beta_{k1} & \beta_{k2} & \ldots & \beta_{kp} \end{bmatrix}^T$ is the vector of regression coefficients for the k-th response.

2.2. Geographically and Temporally Weighted Compound Correlated Bivariate Poisson Regression Models

The GTWCCBPR model extends the CCBPR model by incorporating spatial and temporal variation into parameter estimation. This model is designed to handle overdispersed and correlated bivariate count data, where regression coefficients vary locally according to spatial coordinates (u_i, v_i) and observation time (t_i). The joint distribution of response variables (Y_1, Y_2) is modeled using a CCBPR structure, with intensity functions expressed as exposure-adjusted covariates and time–location-specific parameters. The joint probability function of the GTWCCBPR model for the response pair $(Y_{1il},\ Y_{2il})$, where $i = 1, 2, \ldots, n$ and $l = 1, 2, \ldots, L$, is defined as follows:

$$P\left(Y_{1il} = y_{1il}, Y_{2il} = y_{2il}\right) = \frac{1}{K_{\gamma l}\left(v\right)} \sum_{a=0}^{s} \varphi_1 \varphi_2 K_{y_{1il}+y_{2il}-a+\gamma}\left(z_{il}\right), \quad (4)$$

where:

$$\varphi_1 = \frac{{\lambda_0}^a\left(q_0\right)}{a!},$$

$$\varphi_2 = \prod_{k=1}^{2} \frac{q_{kil}\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{kLi}\right)^{y_{kil}-a}}{\left(y_{kil}-a\right)!}\left(\frac{v}{z_{il}}\right)^{y_{1il}+y_{2il}-a+\gamma},$$

$$z_{il} = \sqrt{v\left(v+2\left(\sum_{k=1}^{2} q_{kil}\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{kLi}\right)+\lambda_0\left(q_0\right)\right)\right)},$$

$$v = \sqrt{\phi^2+1}-1, -\infty<\gamma<\infty,\ s = \min\left(y_{1il}, y_{2il}\right),$$

$$\boldsymbol{\beta}_{kLi} = \boldsymbol{\beta}_{kl}\left(u_i,\ v_i,\ t_i\right); k = 1, 2.$$

The terms $K_{\gamma l}(v)$ and $K_{y_{1il}+y_{2il}-a+\gamma}(z)$ denote modifed Bessel function of the third type.

The GTWCCBPR model with exposure can generally be written:

$$E\left(Y_{kil}\right) = \lambda_{kil}\left(q_{kil}, \mathbf{x}_{il}\right) \quad (5)$$

$$= q_{kil}\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{kl}\left(u_i,v_i,t_i\right)\right),\ k=1,2,$$

where

$$\mathbf{x}_{il} = \begin{bmatrix}1 & x_{1il} & x_{2il} & \cdots & x_{pil}\end{bmatrix}^T,$$
$$\boldsymbol{\beta}_{kl}(u_i,v_i,t_i) = [\beta_{k0l}(u_i,v_i,t_i) \quad \beta_{k1l}(u_i,v_i,t_i) \quad \cdots \quad \beta_{kpl}(u_i,v_i,t_i)]^T.$$

Here, $\mathbf{x}_{il}$ and $\boldsymbol{\beta}_{kl}\left(u_i,\ v_i,\ t_i\right)$ denote the predictor vector and the spatiotemporally varying regression coefficients, respectively, consistent with the notation defined earlier. This model enables a flexible and localized analysis of spatial-temporal patterns, allowing for more accurate inference in datasets exhibiting geographic and temporal variation.

3. Results and Discussion

3.1. Parameter Estimation of GTWCCBPR

The GTWCCBPR model is an extension of the CCBPR model that incorporates both spatial and temporal heterogeneity in bivariate count data. It allows parameter estimates to vary across locations and time periods by applying a spatial-temporal weighting scheme. As a result, the model produces locally varying estimates for each spatio-temporal observation unit.

Parameter estimation in the GTWCCBPR model is performed using the Maximum Likelihood Estimation (MLE). The key parameters to be estimated include β_{kLi}, ϕ, and λ_0. The likelihood function is derived from the join probability density function (PDF) of Y_1 and Y_2, as specified in eq. (4). The parameters to be estimated in the GTWCCBPR model can be represented in matrix form as:

$$\boldsymbol{\theta}_{i^*L,GTWCCBPR} = \begin{bmatrix}\boldsymbol{\beta}_{1Li^*}^T & \boldsymbol{\beta}_{2Li^*}^T & \phi & \lambda_0\end{bmatrix}^T. \tag{6}$$

Applying the natural logarithm to the probability function in eq. (4), the ln probability for the i^* observation is obtained as:

$$\ln P\left(y_{1il},y_{2il}\middle|\boldsymbol{\theta}_{i^*L,GTWCCBPR}\right) = \ln\left[\frac{1}{K_{\gamma l}\left(v\right)}\sum_{a=0}^{s}\varphi_1\varphi_2^*\mathrm{K}_{\mathrm{y_{1il}+y_{2il}-a+\gamma}}\left(z_{i^*l}\right)\right] \tag{7}$$

where:

$$\varphi_2^* = \prod_{k=1}^{2}\frac{q_{kil}\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{kLi^*}\right)^{y_{kil}-a}}{\left(y_{kil}-a\right)!}\left(\frac{v}{z_{i^*l}}\right)^{y_{1il}+y_{2il}-a+\gamma},$$

$$z_{i^*l} = \sqrt{v\left(v+2\left(\sum_{k=1}^{2}q_{kil}\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{kLi^*}\right)+\lambda_0\left(q_0\right)\right)\right)}.$$

Next, for each parameter in the GTWCCBPR model, the ln probability function in eq. (7) is partially differentiated and set equal to zero, resulting in a system of estimating equations defined as follows:

$$\frac{\partial \ln P\left(y_{1il},y_{2il}\middle|\boldsymbol{\theta}_{i^*L,GTWCCBPR}\right)}{\partial\boldsymbol{\beta}_{1Li^*}} = \frac{1}{A_{il}}\sum_{a=0}^{s}\left[\left(y_{1il}-a\right)T_1 + q_{1il}\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{1Li^*}\right)\ \left(T_2+T_3\right)\right], \tag{8}$$

$$\frac{\partial \ln P\left(y_{1il},y_{2il}\middle|\boldsymbol{\theta}_{i^*L,GTWCCBPR}\right)}{\partial\boldsymbol{\beta}_{2Li^*}} = \frac{1}{A_{il}}\sum_{a=0}^{s}\left[\left(y_{2il}-a\right)T_1 + q_{2il}\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{2Li^*}\right)\ \left(T_2+T_3\right)\right], \tag{9}$$

where:

$$A_{il} = \sum_{a=0}^{s}\varphi_1\varphi_2^*\,K_{\mathrm{y_{1il}+y_{2il}-a+\gamma}}\left(z_{i^*l}\right),$$
$$T_1 = \varphi_1\varphi_2^*\mathbf{x}_{il}K_{\mathrm{y_{1il}+y_{2il}-a+\gamma}}\left(z_{i^*l}\right),$$
$$T_2 = \varphi_1\varphi_2^*(-1)\left(\mathrm{y_{1il}+y_{2il}-a+\gamma}\right)\left(\frac{v}{z_{i^*l}}\right)^{-1}\frac{\left(v\right)^2\mathbf{x}_{il}}{{z_{i^*l}}^3}K_{\mathrm{y_{1il}+y_{2il}-a+\gamma}}\left(z_{i^*l}\right),$$
$$T_3 = \varphi_1\varphi_2^*\frac{v\mathbf{x}_{il}}{z_{i^*l}}\left(\frac{\left(\mathrm{y_{1il}+y_{2il}-a+\gamma}\right)}{\left(z_{i^*l}\right)}K_{\mathrm{y_{1il}+y_{2il}-a+\gamma}}\left(z_{i^*l}\right) - K_{\mathrm{y_{1il}+y_{2il}-a+\gamma+1}}\left(z_{i^*l}\right)\right).$$

In this study, the parameters ϕ and λ_0 in the GTWCCBPR model are retricted and assumed to adopt the estimates obtained form the CCBPR model. The resulting equations are presented as follows:

$$\frac{\partial \ln P\left(y_{1i},y_{2i}\middle|\boldsymbol{\theta}_{\mathrm{CCBPR}}\right)}{\partial\lambda_0} = \frac{1}{B_i}\sum_{a=0}^{s}\left[\left(\frac{a\lambda_0^{a-1}}{a!}\psi_{2i}K_{y_{1i}+y_{2i}-a+\gamma}\left(z_i\right)\right) + \left(\frac{v^2}{\left(z_i\right)^{3/2}}U_1\right) + \left(\left(\frac{v}{z_i}\right)U_2\right)\right], \tag{10}$$

$$\frac{\partial \ln P\left(y_{1i},y_{2i}\middle|\boldsymbol{\theta}_{\mathrm{CCBPR}}\right)}{\partial\phi} = \frac{1}{B_i}\sum_{a=0}^{s}\left[\left(U_1\frac{\left(\frac{\phi z_i^*}{\left(\phi^2+1\right)^{1/2}}-\frac{v}{2z_i}Q\right)}{\left(z_i\right)^{3/2}}\right) + \left(U_2\frac{1}{2z_i}Q\right)\right] - \left(\frac{v}{\gamma}-\frac{K_{\gamma+1}\left(v\right)}{K_\gamma\left(v\right)}\left(\frac{\phi}{\left(\phi^2+1\right)^{1/2}}\right)\right), \tag{11}$$

where:

$$B_i = \sum_{a=0}^{s}\psi_{1i}\psi_{2i}K_{y_{1i}+y_{2i}-a+\gamma}\left(z_i\right),$$
$$U_1 = \psi_{1i}\left(y_{1i}+y_{2i}-a+\gamma\right)\psi_{2i}\left(\frac{v}{z_i}\right)^{-1}K_{y_{1i}+y_{2i}-a+\gamma}\left(z_i\right),$$
$$U_2 = \psi_{1i}\psi_{2i}\left(\frac{\left(y_{1i}+y_{2i}-a+\gamma\right)}{z_i}K_{y_{1i}+y_{2i}-a+\gamma}\left(z_i\right) - K_{y_{1i}+y_{2i}-a+\gamma+1}\left(z_i\right)\right),$$
$$Q = \left(\frac{\phi\left(v+2\sum_{k=1}^{2}q_{ki}\exp\left(\mathbf{x}_i^T\boldsymbol{\beta}_k\right)+\lambda_0\right)}{\left(\phi^2+1\right)^{1/2}} + \frac{\phi v}{\left(\phi^2+1\right)^{1/2}}\right).$$

The likelihood function used to estimate the population parameters in the GTWCCBPR model is given as follows.

$$L\left(\boldsymbol{\theta}_{iL,GTWCCBPR}\right) = \prod_{l=1}^{L}\prod_{i=1}^{n}P\left(Y_{1il}=y_{1il},Y_{2il}=y_{2il}\right)$$

$$= \prod_{l=1}^{L}\prod_{i=1}^{n}\left(\frac{1}{K_{\gamma l}(v)}\sum_{a=0}^{s}\varphi_1\varphi_2 K_{y_{1il}+y_{2il}-a+\gamma}(z_{il})\right).$$

The natural logaritm is applied to the likelihood function. Then, the spatial and temporal weights $w_{ii^*l}\ (u_i, v_i, t_i)$ are are incorporated into the natural logarithm of likelihood function, resulting in the weighted ln likelihood formulation.

$$\begin{aligned}
&\ell\left(\boldsymbol{\theta}_{i^*L,GTWCCBPR}\right)\\
&= \sum_{l=1}^{L}\sum_{i=1}^{n}\ln(P\left(Y_{1il}=y_{1il}, Y_{2il}=y_{2il};\ \boldsymbol{\beta}_{1Li^*};\boldsymbol{\beta}_{2Li^*}\right))^{w_{ii^*l}}\\
&= -\sum_{l=1}^{L}\sum_{i=1}^{n} w_{ii^*l}\ln\left(K_{\gamma l}(v)\right) + \left(\sum_{l=1}^{L}\sum_{i=1}^{n} w_{ii^*l}\ln C_{i^*}\right),
\end{aligned} \tag{12}$$

where

$$C_{i^*} = \sum_{a=0}^{s}\varphi_1\varphi_2^{*}\mathrm{K}_{\mathrm{y_{1il}+y_{2il}-a+\gamma}}(z_{i^*l}).$$

Next, for each parameter in GTWCCBPR model in eq. (6), the ln likelihood function in eq. (12) is partially differentiated with respect to each parameter and set equal to zero. Let $\boldsymbol{\theta}_{i^*L,GTWCCBPR}$ denote the local parameter vector at location i^* and time L, comprising the regression coefficient $\boldsymbol{\beta}_{1Li^*}$, $\boldsymbol{\beta}_{2Li^*}$, the dispersion parameter ϕ, and the shared latent parameter λ_0. The local estimating function, or vector, is defined as:

$$\mathbf{g}\left(\boldsymbol{\theta}_{i^*L,GTWCCBPR}\right) = \begin{bmatrix} \frac{\partial\ell\left(\boldsymbol{\theta}_{i^*L,GTWCCBPR}\right)}{\partial\boldsymbol{\beta}_{1Li^*}} \\ \frac{\partial\ell\left(\boldsymbol{\theta}_{i^*L,GTWCCBPR}\right)}{\partial\boldsymbol{\beta}_{2Li^*}} \\ \frac{\partial\ell\left(\boldsymbol{\theta}_{i^*L,GTWCCBPR}\right)}{\partial\phi} \\ \frac{\partial\ell\left(\boldsymbol{\theta}_{i^*L,GTWCCBPR}\right)}{\partial\lambda_0} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{l=1}^{L}\sum_{i=1}^{n} w_{ii^*l}\frac{\partial\ln P\left(y_{1li},y_{2li}|\boldsymbol{\theta}_{i^*L,GTWCCBPR}\right)}{\partial\boldsymbol{\beta}_{1Li^*}} \\ \sum_{l=1}^{L}\sum_{i=1}^{n} w_{ii^*l}\frac{\partial\ln P\left(y_{1li},y_{2li}|\boldsymbol{\theta}_{i^*L,GTWCCBPR}\right)}{\partial\boldsymbol{\beta}_{2Li^*}} \\ \sum_{l=1}^{L}\sum_{i=1}^{n} w_{ii^*l}\frac{\partial\ln P\left(y_{1i},y_{2i}|\boldsymbol{\theta}_{\mathrm{CCBPR}}\right)}{\partial\phi} \\ \sum_{l=1}^{L}\sum_{i=1}^{n} w_{ii^*l}\frac{\partial\ln P\left(y_{1i},y_{2i}|\boldsymbol{\theta}_{\mathrm{CCBPR}}\right)}{\partial\lambda_0} \end{bmatrix}. \tag{13}$$

Each component of the vector $\mathbf{g}\left(\boldsymbol{\theta}_{i^*L,GTWCCBPR}\right)$ represents a weighted summation of the partial derivatives of the local log-likelihood contributions from all spatial units i over all time periods l, where the weight function w_{ii^*l} accounts for both spatial and temporal proximity to the target location i^* at time L. The expression for the partial derivatives $\frac{\partial\ln P\left(y_{1li},y_{2li}|\boldsymbol{\theta}_{i^*L,GTWCCBPR}\right)}{\partial\left(\boldsymbol{\theta}_{i^*L,GTWCCBPR}\right)}$ have been derived in eq. (8)–eq. (11).

The first derivate of the ln likelihood function in equation does not yield a closed-form solution, therefore the MLE estimation must be obtained numerically. In this study, the BHHH iteration method is umployed for numerical estimation, as the Hessian matrix can be approximated without requiring computation the second derivates of the ln likelihood function $\ell\left(\boldsymbol{\theta}_{i^*L,GTWCCBPR}\right)$ [14, 15]. Unlike other optimization techniques such as Fisher Scoring and Newton-Raphson, which require computation of second-order derivatives, the BHHH algorithm relies only on the outer product of gradients. This approach simplifies the optimization process, improves numerical stability, and guarantees that each iteration increases the value of ln likelihood function, thereby enhancing convergence reliability.

These properties make BHHH particularly suitable for the GTWCCBPR model, where parameter estimation must be performed locally at multiple spatio-temporal locations. Since the likelihood equations do not have closed-form solutions, parameter estimation is conducted numerically through a two-phase process: an initialization step to generate starting values, followed by an iterative procedure applied independently at each space-time point using the gradient and the approximated Hessian.

The detailed estimation procedure is outlined out in the following steps: **Stage I** : Initial Parameter Estimation

- Step 1. Initialize iteration counter: $m = 0$
- Step 2. Compute initial estimate $\boldsymbol{\theta}^{(0)}_{i^*L,GTWCCBPR}$, where the initial parameter estimates are obtained form the fitted CCBPR model.
- Step 3. Form the initial parameter vector:

$$\boldsymbol{\theta}^{(0)}_{i^*L,GTWCCBPR} = \begin{bmatrix} \widehat{\boldsymbol{\beta}}^{T(0)}_{1Li^*} & \widehat{\boldsymbol{\beta}}^{T(0)}_{2Li^*} & \widehat{\phi}^{(0)} & \widehat{\lambda}^{(0)}_0 \end{bmatrix}^T.$$

- Step 4. Set the convergence tolerance $\varepsilon > 0$ (e.g., $\varepsilon = 10^{-5}$) and maximum iterations M (e.g., $M = 1000$).

Stage II : Iterative Estimation Using the BHHH Algorithm

- Step 1. Compute the gradient vector $\mathbf{g}\left(\boldsymbol{\theta}_{i^*L,GTWCCBPR}\right)$. The gradient vector is obtained by taking the partial derivatives of the local weighted ln likelihood function with respect to each parameter, as shown in eq. (13).
- Step 2. Construct the Hessian approximation as a negative definite matrix using the outer product of the gradient vector components.

$$\begin{aligned}
&\mathbf{H}\left(\boldsymbol{\theta}_{i^*L,GTWCCBPR}\right)\\
&= -\sum_{l=1}^{L}\sum_{i=1}^{n} w_{ii^*l}\,\mathbf{g}_{il}\left(\boldsymbol{\theta}_{i^*L,GTWCCBPR}\right)\mathbf{g}_{il}\left(\boldsymbol{\theta}_{i^*L,GTWCCBPR}\right)^T
\end{aligned}$$

  The term $\mathbf{g}_{il}\left(\boldsymbol{\theta}_{i^*L,GTWCCBPR}\right)$ denotes the gradient vector, whose components are defined in eq. (8)–eq. (11).
- Step 3. Substitute the initial parameter values into the elements of the gradient vector and the Hessian matrix.
- Step 4. Start the iteration at $m = 0$ with:

$$\begin{aligned}
\widehat{\boldsymbol{\theta}}^{(m+1)}_{i^*L,GTWCCBPR} &= \widehat{\boldsymbol{\theta}}^{(m)}_{i^*L,GTWCCBPR}\\
&- \mathbf{H}^{-1}\left(\widehat{\boldsymbol{\theta}}^{(m)}_{i^*L,GTWCCBPR}\right)\mathbf{g}\left(\widehat{\boldsymbol{\theta}}^{(m)}_{i^*L,GTWCCBPR}\right).
\end{aligned}$$

- Step 5. Stop the iteration when:

$$\left\|\widehat{\boldsymbol{\theta}}^{(m+1)}_{i^*L,GTWCCBPR} - \widehat{\boldsymbol{\theta}}^{(t)}_{i^*L,GTWCCBPR}\right\| \le \varepsilon.$$

  or if the maximum iterations *M* is reached. In this study, the convergence criterion is based on the relative change in the parameter estimates. The iteration stops when the change is smaller than a small positive threshold ε (e.g., $\varepsilon = 10^{-5}$), or when the number of iterations reaches a maximum limit

M (e.g., M = 1000). These values are commonly adopted in numerical optimization and can be adjusted depending on the model complexity, data size, or convergence behavior. To ensure numerical stability during the iterative process, the condition number of the Hessian approximation is monitored.

- Step 7. The final parameter vector is:

$$\begin{aligned}\widehat{\boldsymbol{\theta}}_{i^*L,GTWCCBPR} &= \widehat{\boldsymbol{\theta}}^{(M)}_{i^*L,GTWCCBPR}\\ &= \begin{bmatrix}\widehat{\boldsymbol{\beta}}^T_{1Li^*} & \widehat{\boldsymbol{\beta}}^T_{2Li^*} & \widehat{\phi} & \widehat{\lambda}_0\end{bmatrix}^T.\end{aligned}$$

3.2. Hypothesis Testing of GTWCCBPR

Before evaluating the significance of predictor variables, a model similarity test is first conducted to compare the global model (CCBPR) with the proposed spatiotemporal model (GTWCCBPR). This test aims to assess whether the inclusion of spatial and temporal weights significantly improves model fit. In this theoretical development, hypothesis testing procedures such as Z-tests and MLRT are illustrated using a conventional significance level of $\alpha = 0.05$, although the value of α may be adjusted depending on the research context or desired level of inference precision. The hypotheses are formulated as follows:

$$\begin{aligned}&H_0 : \beta_{kjLi} = \beta_{kj};\ k = 1, 2;\ i = 1, 2, \ldots, n; j = 1, 2, \ldots, p,\\ &H_1 : \exists\beta_{kjLi} \neq \beta_{kj}.\end{aligned}$$

The test statistic is given by:

$$F = \frac{\frac{G^2_{CCBPR}}{df_1}}{\frac{G^2_{GTWCCBPR}}{df_2}},$$

where G^2_{CCBPR} is the deviance of the global CCBPR model, $G^2_{GTWCCBPR}$ is the deviance of the spatiotemporal GTWCCBPR model, and df_1 and df_2 denote the respective degrees of freedom for the CCBPR and GTWCCBPR models.

The deviance of the GTWCCBPR model, denoted $G^2_{GTWCCBPR}$, is obtained using the following test statistic based on the likelihood ratio:

$$\begin{aligned}G^2_{GTWCCBPR} &= -2\ln\left(\frac{L(\widehat{\boldsymbol{\omega}}_{GTWCCBPR})}{L(\widehat{\boldsymbol{\Omega}}_{GTWCCBPR})}\right)\\ &= 2\left(\ln L(\widehat{\boldsymbol{\Omega}}_{GTWCCBPR}) - \ln L(\widehat{\boldsymbol{\omega}}_{GTWCCBPR})\right),\end{aligned}$$

where $L\left(\widehat{\boldsymbol{\Omega}}_{GTWCCBPR}\right)$ denotes the ln likelihood under the full model specification that incorporates all predictor terms, and $L\left(\widehat{\omega}_{GTWCCBPR}\right)$ refers to the ln likelihood evaluated under the restricted model without predictors. ln likelihood function under the population, $l\left(\widehat{\boldsymbol{\Omega}}_{GTWCCBPR}\right) = \ln L\left(\widehat{\boldsymbol{\Omega}}_{GTWCCBPR}\right)$ is expressed as follows:

$$\begin{aligned}&l(\widehat{\boldsymbol{\Omega}}_{GTWCCBPR})\\ &= \sum_{l=1}^{L}\sum_{i=1}^{n}\ln\left(\frac{1}{K_{y_l}(\widehat{\upsilon})}\sum_{a=0}^{s}\widehat{\phi}_1\widehat{\phi}_2 K_{y_{1il}+y_{2il}-a+\gamma}(\widehat{Z}_{il})\right),\end{aligned}$$

where

$$\begin{aligned}\widehat{\phi}_1 &= \frac{\widehat{\lambda}^a_0(q_0)}{a!},\\ \widehat{\phi}_2 &= \prod_{k=1}^{2}\frac{q_{kil}\exp(\mathbf{x}^T_{il}\widehat{\boldsymbol{\beta}}_{kLi})^{y_{kil}-a}}{(y_{kil}-a)!}\left(\frac{\widehat{\upsilon}}{\widehat{Z}_{il}}\right)^{y_{1il}+y_{2il}-a+\gamma},\\ \widehat{Z}_{il} &= \sqrt{\widehat{\upsilon}\left(\widehat{\upsilon}+2\left(\sum_{k=1}^{2}q_{kil}\exp(\mathbf{x}^T_{il}\widehat{\boldsymbol{\beta}}_{kLi})\right)+\widehat{\lambda}_0\right)},\\ \widehat{\upsilon} &= \widehat{\phi}^2+1-1.\end{aligned}$$

The parameter estimation process under the null hypothesis H_0, based on the ln likelihood function $l\left(\widehat{\omega}_{GTWCCBPR}\right)$, follows the same procedure as in the full model. The log-likelihood function is partially differentiated with respect to each element in the parameter vector $\widehat{\omega}_{GTWCCBPR}$, and the resulting estimating equations are obtained by setting these derivatives equal to zero. Parameter estimation is then carried out numerically using the Berndt–Hall–Hall–Hausman (BHHH) iterative algorithm, without requiring the explicit computation of the second-order derivatives of the log-likelihood function.

$$\begin{aligned}&l(\widehat{\boldsymbol{\omega}}_{GTWCCBPR})\\ &= \sum_{l=1}^{L}\sum_{i=1}^{n}\ln\left(\frac{1}{K_{y_l}(\widehat{\upsilon}_\omega)}\sum_{a=0}^{s}\widehat{\phi}_{\omega 1}\widehat{\phi}_{\omega 2}K_{y_{1il}+y_{2il}-a+\gamma}(\widehat{Z}_{\omega il})\right),\end{aligned}$$

where

$$\begin{aligned}\widehat{\phi}_{\omega 1} &= \frac{\widehat{\lambda}^a_{0\omega}(q_0)}{a!},\\ \widehat{\phi}_{\omega 2} &= \prod_{k=1}^{2}\frac{q_{kil}\exp(\mathbf{x}^T_{il}\widehat{\boldsymbol{\beta}}_{0k\omega L})^{y_{kil}-a}}{(y_{kil}-a)!}\left(\frac{\widehat{\upsilon}_\omega}{\widehat{Z}_{\omega il}}\right)^{y_{1il}+y_{2il}-a+\gamma},\\ \widehat{Z}_{\omega il} &= \sqrt{\widehat{\upsilon}_\omega\left(\widehat{\upsilon}_\omega+2\left(\sum_{k=1}^{2}q_{kil}\exp(\mathbf{x}^T_{il}\widehat{\boldsymbol{\beta}}_{0k\omega L})\right)+\widehat{\lambda}_0\right)},\\ \widehat{\upsilon}_\omega &= \sqrt{\widehat{\phi}^2_\omega+1}-1.\end{aligned}$$

Criteria for rejection H_0 is $F > F_{\alpha;df_1;df_2}$.

Subsequently, the GTWCCBPR model is evaluated using the Maximum Likelihood Ratio Test (MLRT) to assess the statistical significance of the predictor variables [16]. This method supports both simultaneous testing, which examines the joint effect of all predictors, and partial testing, which evaluates the significance of each parameter individually. The overall model significance is tested under the following null and alternative hypotheses:

$$\begin{aligned}&H_0 : \beta_{k1Li} = \beta_{k2Li} = \cdots = \beta_{kpLi} = 0;\ \forall k = 1, 2;\ i = 1, 2, \ldots, n,\\ &H_1 : \exists\beta_{kjLi} \neq 0;\ \text{with } k = 1, 2;\ i = 1, 2, \ldots, n; j = 1, 2, \ldots, p.\end{aligned}$$

The test statistic for the MLRT is given by:

$$\begin{aligned}G^2 &= -2\ln\left(\frac{L\left(\widehat{\omega}_{GTWCCBPR}\right)}{L\left(\widehat{\boldsymbol{\Omega}}_{GTWCCBPR}\right)}\right)\\ &= 2\left(\ln L\left(\widehat{\boldsymbol{\Omega}}_{GTWCCBPR}\right) - \ln L\left(\widehat{\omega}_{GTWCCBPR}\right)\right).\end{aligned}$$

Criteria for rejection H_0 is $G^2 > \chi^2_{\alpha;p}$, where the degress of freedom are given $p = n\left((\boldsymbol{\Omega}_{GTWCCBPR}) - (\boldsymbol{\omega}_{GTWCCBPR})\right)$.

A partial hypothesis is conducted to determine which specific predictor variables significantly influence the model. The hypothesis for an individual parameter β_{kjLi} are:

$$H_0 : \beta_{kjLi} = 0,$$
$$H_1 : \beta_{kjLi} \neq 0; \text{ with } k = 1, 2; \ i = 1, 2, \ldots, n; j = 1, 2, \ldots, p.$$

The test statistics for individual parameters follows a Z-test formula:

$$Z = \frac{\widehat{\beta}_{kjLi}}{se\left(\widehat{\beta}_{kjLi}\right)},$$

where $se\left(\widehat{\beta}_{kjLi}\right) = \sqrt{\widehat{Var}\left(\widehat{\beta}_{kjLi}\right)}$. The variance of $\widehat{\beta}_{kjLi}$ is obtained from the main diagonal elements of the estimated variance-covariance matrix, computed as$\widehat{Cov}\left(\widehat{\boldsymbol{\theta}}\right) = -\widehat{\mathrm{E}}\left(\mathbf{H}^{-1}\left(\widehat{\boldsymbol{\theta}}\right)\right) = -\mathbf{H}^{-1}\left(\widehat{\boldsymbol{\theta}}\right)$. The null hypothesis H_0 is rejected when $|Z| > Z_{\alpha/2}$ with α denotes the significance level.

4. Conclusion

Parameter estimation for the Geographically and Temporally Weighted Compound Correlated Bivariate Poisson Regression (GTWCCBPR) model is carried out using the Maximum Likelihood Estimation (MLE) method. Due to the absence of a closed-form solution for the derived log-likelihood function, the estimation process is implemented through the iterative Berndt–Hall–Hall–Hausman (BHHH) optimization algorithm to ensure convergence and numerical stability. After parameter estimation, hypothesis testing is conducted to evaluate model similarity between the global CCBPR and the proposed GTWCCBPR models, as well as to assess the significance of each predictor variable. Simultaneous testing is performed using the Maximum Likelihood Ratio Test (MLRT), while partial testing is conducted using the Z-test. This study contributes to the theoretical development of spatiotemporal count regression models by offering a flexible and robust framework for analyzing correlated bivariate count data characterized by overdispersion and spatial-temporal heterogeneity. Future research may explore the empirical application of the GTWCCBPR model using real-world datasets to evaluate its practical performance. Although this study is theoretical, the proposed model is expected to be applicable in areas such as public health surveillance, regional development planning, and environmental risk modeling, where bivariate count data and spatio-temporal dynamics are prevalent.

Author Contributions. **Priyanka Ratulangi Hargandi:** conceptualization, methodology, formal analysis, writing-original draft preparation, writing-review and editing. **Purhadi Purhadi:** conceptualization, methodology, supervision, writing-review and editing. **Achmad Choiruddin:** conceptualization, methodology, supervision, writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Acknowledgement. The authors would like to express their sincere gratitude to the editors and reviewers for their thorough review, valuable insights, and constructive suggestions, which have greatly enhanced the clarity and quality of this paper.

Funding. This research received no external funding.

Conflict of interest. The authors declare no conflict of interest related to this article.

Data availability. Not applicable.

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