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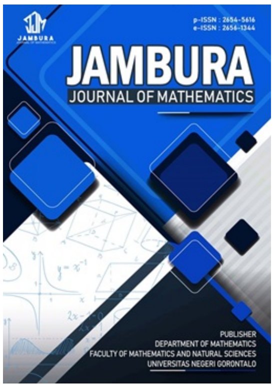
JAMBURA JOURNAL OF MATHEMATICS

Volume 8, Issue 1, Pages 1–6, February 2026

Received 18 August 2025, Revised 9 September 2025, Accepted 28 September 2025, Published 4 January 2026

To Cite this Article : A. F. Adebisi, T. A. Ojurongbe, and K. A. Okunola,"Application of the Laguerre Perturbed Galerkin Analysis Method for Solving Higher-Order Integro-Differential Equations", *Jambura J. Math*, vol. 8, no. 1, pp. 1–6, 2026, *https://doi.org/10.37905/jjom.v8i1.34001*

JOURNAL INFO • JAMBURA JOURNAL OF MATHEMATICS



Homepage	:	http://ejurnal.ung.ac.id/index.php/jjom/index
Journal Abbreviation	:	Jambura J. Math.
Frequency	:	Biannual (February and August)
Publication Language	:	English (preferable), Indonesia
DOI	:	https://doi.org/10.37905/jjom
Online ISSN	:	2656-1344
Editor-in-Chief	:	Hasan S. Panigoro
Publisher	:	Department of Mathematics, Universitas Negeri Gorontalo
Country	:	Indonesia
OAI Address	:	http://ejurnal.ung.ac.id/index.php/jjom/oai
Google Scholar ID	:	iWLjgaUAAAAJ
Email	:	info.jjom@ung.ac.id

Jambura Journal of Mathematics, Volume 8, Issue 1, Pages 1–6, February 2026 https://doi.org/10.37905/jjom.v8i1.34001

Research Article

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Application of the Laguerre Perturbed Galerkin Analysis Method for Solving Higher- Order Integro-Differential Equations

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ARTICLE HISTORY
Received 18 August 2025
Revised 9 September 2025
Accepted 28 September 2025
Published 4 January 2026

KEYWORDS
Laguerre Perturbed Galerkin method
Laguerre polynomials
Chebyshev polynomials
Galerkin approach
Residual minimization

ABSTRACT. *This study presents the development and implementation of a novel numerical method, the Laguerre Perturbed Galerkin (LPG) method, for solving higher-order integro-differential equations. The method leverages the advantages of Laguerre polynomials as basis functions while incorporating Chebyshev polynomials as perturbation terms to enhance both accuracy and efficiency. In the LPG method, the solution is approximated using Laguerre polynomials of degree N, with the residual error minimized via the Galerkin approach. Chebyshev polynomials are introduced as perturbation terms to further refine the solution. The residual is systematically reduced to a system of $(N+1)$ equations, which is then solved to determine the unknown coefficients of the approximating Laguerre polynomials. Comparative analyses demonstrate that the LPG method achieves superior accuracy and faster convergence rates compared to existing techniques, particularly for higher-order integro-differential equations. The findings contribute to the advancement of numerical methods in this domain, providing a powerful computational tool for scientists and engineers.*

1. Introduction

Integro-differential equations, which integrate both integral and differential operators, serve as crucial mathematical tools for modeling various complex phenomena in fields such as physics, engineering, biology, economics, and the social sciences [1, 2]. These equations naturally arise in systems exhibiting both local and nonlocal interactions, making them invaluable for understanding dynamic processes. The study of integro-differential equations dates back to the pioneering work of mathematicians such as Volterra, Fredholm, and Hammerstein, who laid the foundation for their theoretical analysis and numerical approximation [3–5]. Over time, researchers have made significant strides in investigating their properties, stability, and computational solutions [6]. Due to the inherent difficulty in obtaining exact analytical solutions, particularly for nonlinear and higher-order equations, various numerical methods have been developed to address these challenges [7].

Several prominent numerical techniques have been employed to solve integro-differential equations effectively. Adebisi et al. [8] utilized the Gauss-Legendre formula to develop a Legendre polynomial-based collocation method, demonstrating high accuracy at lower values of N. Olayiwola et al. [9] applied the variational iteration method (VIM), a modified Lagrange multiplier approach, to solve different types of integro-differential equations, yielding highly accurate results. Similarly, Taiwo et al. [10] implemented the Galerkin method using orthogonal polynomials to solve Volterra and Fredholm integro-differential equations, leveraging polynomial basis functions for improved approximations. Additional contributions include Aduroja et al. [11], who explored collocation approximation methods for Volterra integro-differential equations using polynomial basis functions, demonstrating reliability and effectiveness. Uwaheren et al. [12] successfully applied the Legendre-Galerkin method to fractional-order Fredholm integro-differential equations, showing rapid convergence at lower degrees of approximant N. Adebisi et al. [13] investigated linear fractional integro-differential equations, employing Galerkin and perturbed collocation analysis for numerical approximations.

Further advancements include Mamun et al. [14] who solved eighth-order boundary value problems using VIM, establishing that its approximate solutions converge to exact solutions. Olayiwola et al. [15] developed an efficient numerical method using Legendre polynomials for initial-value problems of integro-differential equations. Uwaheren et al. [12] introduced Akbari-Ganjis method (AGM) for solving Volterra integro-differential difference equations (VIDDE) with Legendre polynomials as basis functions, confirming AGM's robustness in tackling VIDDE problems.

Moreover, Ogunrinde et al. [16] proposed a six-step linear multistep method combined with Newton-Cotes quadrature for third-order Fredholm integro-differential equations, ensuring consistency, stability, and convergence. Chandel et al. [17] applied Legendre wavelets to solve higher-order Volterra integro-differential equations, achieving results remarkably close to exact solutions. Adebisi et al. [18] further employed the Galerkin method with Chebyshev polynomials as basis functions, demon-

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Homepage : http://ejurnal.ung.ac.id/index.php/jjom/index / E-ISSN : 2656-1344

strating rapid convergence as N increased.

Recent developments also highlight the role of hybrid methods and higher-order schemes. Owolanke et al. [19] developed a hybrid two-step method for second-order initial value problems, while Owolanke et al. [20] proposed an eighth-order two-step Taylor series algorithm for second-order ODEs. Ogunbamike and Owolanke [21] examined convergence for moving mass problems on Winkler foundations, and Ogunbamike et al. [22] studied the dynamic response of cantilever beams on elastic foundations. Yakusak and Owolanke [23] introduced a linear multistep collocation method for second-order IVPs, while Owolanke et al. [24] advanced canonical basis interpolation techniques for solving IVPs.

Additionally, other notable contributions include Avudainayagam and Vani [25] who applied the wavelet-Galerkin method, El-Sayead and Abdel-Aziz [26] who compared wavelet-Galerkin with Adomian decomposition, Fakhar-Izadi and Dehghan [27] who proposed a pseudo-spectral Legendre Galerkin method in population dynamics, Mohammed [28] who applied least-square and Chebyshev methods, and McKhtary [29] who analyzed discrete Galerkin approaches for fractional integro-differential equations.

Building on these previous works, this study aims to introduce a novel numerical approach, the Laguerre Perturbed Galerkin (LPG) method, for solving higher-order integro-differential equations. The proposed method integrates Laguerre polynomials as basis functions with Chebyshev polynomials as perturbation terms to enhance accuracy and efficiency. Through comparative analysis and convergence investigations, this research seeks to advance numerical methodologies in solving complex integro-differential equations, thereby providing a valuable computational tool for scientific and engineering applications.

2. Laguerre Polynomials

Laguerre polynomials are set of orthogonal polynomials that have found applications in various fields of mathematics and physics. These polynomials are named after the French mathematician Edmond Laguerre and are solutions to Laguerre's differential equation. The Laguerre polynomial denoted as $L_r(x)$ are defined by the formula:

$$L_r(x) = \frac{e^x}{r!}\frac{d^r}{dx^r}\left(e^{-x}x^r\right).$$

Recursive Formula:
We know that

$$L_r(x) = \frac{e^x}{r!}\frac{d^r}{dx^r}\left(e^{-x}x^r\right). \tag{1}$$

Putting $r = 0, 1, 2, 3, 4, \ldots$ in succession eq. (1) we obtain:

$$\begin{aligned}
L_0(x) &= \frac{e^x}{0!}\left(e^{-x}x^0\right) = 1,\\
L_1(x) &= \frac{e^x}{1!}\frac{d}{dx}\left(e^{-x}x\right) = 1 - x,\\
L_2(x) &= \frac{e^x}{2!}\frac{d^2}{dx^2}\left(e^{-x}x^2\right) = \frac{1}{2!}(x^2 - 4x + 2),\\
L_3(x) &= \frac{e^x}{3!}\frac{d^3}{dx^3}\left(e^{-x}x^3\right) = \frac{1}{3!}(6 - 18x + 9x^2 - x^3),\\
L_4(x) &= \frac{e^x}{4!}\frac{d^4}{dx^4}\left(e^{-x}x^4\right) = \frac{1}{4!}(24 - 96x + 72x^2 - 16x^3 + x^4),\\
L_5(x) &= \frac{1}{5!}(120 - 600x + 600x^2 - 200x^3 + 25x^4 - x^5),\\
L_6(x) &= \frac{1}{6!}(720 - 4320x + 5400x^2 - 2400x^3 + 450x^4 - 36x^5\\
&\quad + x^6).
\end{aligned}$$

2.1. *Properties*

Orthogonality: Laguerre polynomial satisfy an orthogonality relation that makes them useful in various mathematical problems.

Application: Laguerre polynomials are extensively used in quantum mechanics, probability theory and differential equations.

In summary, Laguerre polynomials are a powerful mathematical tool with wide-ranging application. Understanding their properties, recursive formula and the first few polynomials can provide insights into their utility in various mathematical and physical problems.

2.2. *Chebyshev and shifted Chebyshev polynomials*

Chebyshev polynomials are sequence of orthogonal polynomials which are related to De-Moivre's formula and which can be defined recursively. One usually distinguishes between Chebyshev polynomials of first kind which are denoted by T_r and Chebyshev polynomials of second kind which are denoted by U_r.

2.2.1. Chebyshev polynomials of first kind

Chebyshev polynomials of first kind $T_r(x)$ is defined as:

$$T_r(x) = \cos\left(r\cos^{-1}(x)\right), \quad -1 \le x \le 1. \tag{2}$$

Or equivalently,

$$T_r(x) = \cos(r\theta), \quad \text{where } \theta = \cos^{-1}(x).$$

The few Chebyshev polynomials of the first kind are:

r	$T_r(x)$
0	$T_0(x) = 1$
1	$T_1(x) = x$
2	$T_2(x) = 2x^2 - 1$
3	$T_3(x) = 4x^3 - 3x$
4	$T_4(x) = 8x^4 - 8x^2 + 1$
5	$T_5(x) = 16x^5 - 20x^3 + 5x$

2.2.2. The Shifted Chebyshev polynomials

For convenience and for the sake of problems that exist in intervals other than $-1 \le x \le 1$, $T_r(x)$ is in this subsection normalized to a general finite range $a \le x \le b$ as follows:

$$T^*_R(x) = \cos\left(R\cos^{-1}(x)\right), \quad -1 \le x \le 1, \tag{3}$$

and the recurrence relation is given by

$$T^*_{R+1}(x) = 2xT^*_R(x) - T^*_{R-1}(x), \quad R \ge 1,$$

where R is the degree of the polynomial.

In general, Chebyshev polynomial valid in $a \le x \le b$ is given as:

$$T_R^*(x) = \cos\left[R\cos^{-1}\left(\frac{2x-b-a}{b-a}\right)\right], \quad -1 \le x \le 1, \tag{4}$$

and the recurrence relation is given as:

$$T_{R+1}^*(x) = 2\left(\frac{2x-b-a}{b-a}\right)T_R^*(x) - T_{R-1}^*(x).$$

Few terms of the shifted Chebyshev polynomials valid in the interval $[0,1]$ are given below:

$$\begin{aligned}
T_0^*(x) &= 1,\\
T_1^*(x) &= 2x-1,\\
T_2^*(x) &= 8x^2-8x+1,\\
T_3^*(x) &= 32x^3-48x^2+18x-1,\\
T_4^*(x) &= 128x^4-256x^3+100x^2-32x+1,\\
T_5^*(x) &= 512x^5-1280x^4+1120x^3-400x^2+50x-1,\\
T_6^*(x) &= 2048x^6-6144x^5+6912x^4-5484x^3+840x^2\\
&\quad -72x+1,\\
T_7^*(x) &= 8192x^7-28672x^6+39424x^5-26990x^4+9408x^3\\
&\quad -1568x^2+98x-1,\\
T_8^*(x) &= 32765x^8-131072x^7+212992x^6-40224x^5\\
&\quad +84480x^4-21504x^3+26868x^2-128x+1.
\end{aligned}$$

3. Results and Discussion

3.1. Formulation of the Perturbed Galerkin Method

The methodology of the proposed study involves the perturbation of the Galerkin weighted residual method using Laguerre polynomials as the basis function and Chebyshev polynomials as perturbation terms.

Consider the integro-differential equation below:

$$z^{(n)}(x) = f(x) + \lambda\int_{a(x)}^{b(x)} K(x,t)z(t)\,dt, \quad z_i(0) = \phi_i(0), \tag{5}$$

where $a(x), b(x), f(x), \lambda$ and the kernel $K(x,t)$ are as previously defined. We adopt an approximate solution in the form:

$$z_N(x) = \sum_{k=0}^{N} a_k L_k(x) + P_n(x), \tag{6}$$

where $L_k(x)$ is the Laguerre polynomial and a_k, $k = 0,1,\dots,N$ are unknown constants to be determined.

The perturbation term is defined as:

$$P_n(x) = \sum_{r=1}^{n} \tau_r T_{n-r+1}^*(x), \tag{7}$$

where $T^*(x)$ denotes the shifted Chebyshev polynomial basis function, and τ_r, $r = 1,2,\dots,n$ are the free tau parameters to be determined. Substituting eq. (7) into eq. (6), we obtain:

$$z_N(x) = \sum_{k=0}^{N} a_k L_k(x) + \sum_{r=1}^{n} \tau_r T_{n-r+1}^*(x). \tag{8}$$

Substituting eq. (8) into eq. (5) gives:

$$\begin{aligned}
z_N^{(n)}(x) &= \frac{d^n}{dx^n}\left[\sum_{k=0}^{N} a_k L_k(x) + \sum_{r=1}^{n} \tau_r T_{n-r+1}^*(x)\right]\\
&= f(x) + \lambda\int_{a(x)}^{b(x)} K(x,t)\left[\sum_{k=0}^{N} a_k L_k(t)\right.\\
&\quad \left. + \sum_{r=1}^{n} \tau_r T_{n-r+1}^*(t)\right] dt.
\end{aligned} \tag{9}$$

Hence, the residual $R(z,x)$ becomes:

$$\begin{aligned}
R(z,x) &= \frac{d^n}{dx^n}\left[\sum_{k=0}^{N} a_k L_k(x) + \sum_{r=1}^{n} \tau_r T_{n-r+1}^*(x)\right] - f(x)\\
&\quad - \lambda\int_{a(x)}^{b(x)} K(x,t)\left[\sum_{k=0}^{N} a_k L_k(t)\right.\\
&\quad \left. + \sum_{r=1}^{n} \tau_r T_{n-r+1}^*(t)\right] dt\\
&= 0
\end{aligned} \tag{10}$$

To determine the constant coefficients, we apply the Galerkin method by taking the inner product of the residual with the basis functions $L_k(x)$, for $k = 0,1,2,\dots,N$:

$$\int_g^h R(z,x)\cdot L_k(x)\,dx = 0, \; g \le x \le h, \; k = 0,1,2,\dots,N. \tag{11}$$

Eq. (11) simplifies to a system of $N+1$ linear algebraic equations in the $N+1$ unknowns a_k, $k = 0,1,\dots,N$. Solving this system yields the coefficients, which are then substituted into eq. (6) to obtain the approximate solution.

It is important to note that when the problem includes initial conditions, these conditions are applied first before using the Galerkin method to generate the remaining equations.

Example 1. Consider the integro-differential equation:

$$z''(x) = f(x) + \lambda\int_0^x (x-t)z(t)\,dt, \quad z(0) = z'(0) = 0. \tag{12}$$

We approximate the solution using Laguerre polynomials and Chebyshev polynomial perturbation as follows:

$$z_{10}(x) = \sum_{k=0}^{10} a_k L_k(x) + P_3(x), \tag{13}$$

where the perturbation term is:

$$P_3(x) = \sum_{r=1}^{3} \tau_r T_{3-r+1}^*(x) = \tau_1 T_3^*(x) + \tau_2 T_2^*(x) + \tau_3 T_1^*(x). \tag{14}$$

So the full approximation becomes:

$$z_{10}(x) = a_0L_0(x) + a_1L_1(x) + a_2L_2(x) + a_3L_3(x)$$

Table 1. Exact and Approximate Results of Example 2

x	Exact	PGM	Ogunrinde et al. (LMQ)	Error (PGM)	Error (LMQ)
0.08333	-0.9230352546	-0.9230352546	-0.9230324073	0.000e+00	1.00e-10
0.16666	-0.8564864815	-0.8564864815	-0.8564814810	0.000e+00	5.00e-10
0.24999	-0.7968818750	-0.7968818750	-0.7968749992	0.000e+00	8.00e-10
0.33332	-0.7407496296	-0.7407496296	-0.7407407395	0.000e+00	1.30e-09
0.41665	-0.6846179397	-0.6846179397	-0.6846064798	0.000e+00	1.70e-09
0.49998	-0.6250149998	-0.6250149998	-0.6249999978	0.000e+00	2.30e-09
0.58331	-0.5584690042	-0.5584690042	-0.5584490710	0.000e+00	3.20e-09
0.66664	-0.4815081474	-0.4815081474	-0.4814814774	0.000e+00	4.30e-09
0.74997	-0.3906606239	-0.3906606239	-0.3906249947	0.000e+00	5.50e-09
0.83330	-0.2824546280	-0.2824546280	-0.2824074009	0.000e+00	6.80e-09
0.91663	-0.1534183541	-0.1534183541	-0.1533564773	0.000e+00	8.40e-09

$$+ a_4L_4(x) + a_5L_5(x) + a_6L_6(x) + a_7L_7(x) + a_8L_8(x) + a_9L_9(x) + a_{10}L_{10}(x) + \tau_1T_3^*(x) + \tau_2T_2^*(x) + \tau_3T_1^*(x) \tag{15}$$

Substituting eq. (15) into eq. (12), we obtain the residual:

$$\begin{aligned} R(z,x) &= z_{10}''(x) - f(x) - \lambda\int_0^x (x-t)z_{10}(t)\,dt \\ &= \frac{d^2}{dx^2}\left[\sum_{k=0}^{10} a_kL_k(x) + \sum_{r=1}^{3}\tau_rT_{3-r+1}^*(x)\right] - f(x) \\ &\quad - \lambda\int_0^x (x-t)\left[\sum_{k=0}^{10} a_kL_k(t) + \sum_{r=1}^{3}\tau_rT_{3-r+1}^*(t)\right]dt \end{aligned} \tag{16}$$

The Galerkin conditions are imposed by requiring:

$$\int_0^1 R(z,x)\cdot L_k(x)\,dx = 0, \quad k = 0,1,\ldots,10, \tag{17}$$

Solving the resulting system of equations gives the constants a_k ($k = 0,\ldots,10$), which are then substituted back into eq. (15) to obtain the approximate solution $z_{10}(x)$.

Example 2. Consider the third order linear Fredholm integro-differential equation

$$z'''(x) = 6 + x - \int_0^1 xz''(t)dt, \tag{18}$$

where $z(0) = -1$, $z'(0) = 1$, $z''(0) = -2$. The exact solution is $z(x) = x^3 - x^2 + x - 1$. Source: Ogunrinde et al. [16].

Solution:

To solve eq. (18), we consider eq. (13) when $N = 10, n = 4$ as a trial solution:

$$z_{10}(x) = \sum_{k=0}^{10} a_kL_k(x) + \sum_{r=1}^{4}\tau_rT_{(4-r+1)}^*(x) \tag{19}$$

On utilizing the basis functions:

$$\begin{aligned} z_{10}(x) &= a_0 + a_1(1-x) + \frac{1}{2}a_2(x^2-4x+2) + \frac{1}{6}a_3(-x^3 \\ &+ 9x^2 - 18x + 6) + \frac{1}{24}a_4(x^4 - 16x^3 + 72x^2 \\ &- 96x + 24) + \frac{1}{120}a_5(-x^5 + 24x^4 - 180x^3 + 600x^2 \\ &- 1000x + 600) + \frac{1}{720}a_6(x^6 - 36x^5 + 450x^4 \\ &- 3600x^3 + 18000x^2 - 54000x + 24000) \\ &+ \frac{1}{5040}a_7(-x^7 + 49x^6 - 1008x^5 + 12600x^4 \\ &- 90090x^3 + 420420x^2 - 1008000x + 1008000) \\ &+ \frac{1}{40320}a_8(x^8 - 64x^7 + 1680x^6 - 26880x^5 \\ &+ 282240x^4 - 1814400x^3 + 7257600x^2 - 17280000x \\ &+ 10080000) + \frac{1}{362880}a_9(-x^9 + 81x^8 - 2520x^7 \\ &+ 53130x^6 - 756756x^5 + 7567560x^4 - 45405400x^3 \\ &+ 181440000x^2 - 403200000x + 362880000) \\ &+ \tau_1(128x^4 - 256x^3 + 160x^2 - 32x + 1) - \tau_2(32x^3 \\ &- 48x^2 + 18x - 1) - \tau_3(8x^2 - 8x + 1) - \tau_4(2x - 1) \end{aligned} \tag{20}$$

On solving eq. (20) we get:

$$\begin{aligned} a_0 &= 4, & a_1 &= -15, & a_2 &= 16, \\ a_3 &= -6, & a_4 &= 0, & a_5 &= 0, \\ a_6 &= 0, & a_7 &= 0, & a_8 &= 0, \\ a_9 &= 0, & \tau_1 &= 0, & \tau_2 &= 0, \\ \tau_3 &= 0, & \tau_4 &= 0. \end{aligned}$$

On substituting the values of the constants into eq. (20) and simplifying, we obtain:

$$z_{10}(x) = x^3 - x^2 + x - 1.$$

Example 3. Consider the fourth order linear Fredholm integro-differential equation

$$z^{(4)}(x) = 5e^x - \int_0^x z(x)dt, \tag{21}$$

where $0 < x < 1$, $z(0) = 0$, $z'(0) = 1$, $z(1) = e$, $z'(1) = 2e$. The exact solution is $z(x) = xe^x$. Source: [16].

Table 2. Exact and Approximate Results of Example 3

x	Exact	PGM	R.S. Chandel et al. (LWM)	Error (PGM)	Error (LWM)
0.0	0.00000	0.00000	0.00000	0.000e+00	0.000e+00
0.2	0.24428	0.24428	0.24426	2.137e-07	4.0e-06
0.4	0.59673	0.59673	0.59673	2.453e-08	1.2e-06
0.6	1.09327	1.09327	1.09327	2.275e-08	2.28e-05
0.8	1.78043	1.78043	1.78043	7.895e-08	3.94e-06
1.0	2.71828	2.71828	2.71828	1.751e-07	7.0e-06

Table 3. Exact and Approximate Results of Example 4

x	Exact	PGM	Olayiwola et al. (LCM)	Error (PGM)	Error (LMQ)
0.1	0.19983	0.19983	0.19983	9.165e-08	3.3315e-07
0.2	0.39867	0.39867	0.39867	6.409e-08	1.0644e-05
0.3	0.59552	0.59552	0.59560	7.579e-08	8.0615e-05
0.4	0.78942	0.78942	0.78970	1.510e-07	3.3848e-04
0.5	0.97943	0.97943	0.98040	9.165e-08	1.0283e-03
0.6	1.16464	1.16464	1.16710	2.896e-08	2.5449e-03
0.7	1.34422	1.34422	1.34960	9.166e-08	5.4667e-03
0.8	1.51736	1.51736	1.34960	5.373e-08	1.0586e-02
0.9	1.68333	1.68333	1.70226	3.790e-09	1.8937e-02
1.0	1.84147	1.84147	1.87320	2.110e-09	3.1826e-02

Approximate solution at $N = 10$ gives:

$$\begin{aligned} z_{10}(x) = 1.000004x - 9.98 \times 10^{-7} + 0.5000082x^3 + 0.9999857x^2 \\ + 0.045013707x^5 + 0.165858433x^4 + 0.0032307307x^6 \\ + 0.00001252485071x^9 - 0.0006168343954x^8 \\ + 0.00478554199x^7. \end{aligned} \tag{22}$$

Example 4. Consider the third order linear Fredholm integro-differential equation

$$z''(x) = -x - \frac{x^3}{6} - \int_0^x (x-t)z(t)\,dt,\ z(0) = 0,\ z'(0) = 2. \tag{23}$$

The exact solution is $z(x) = x + \sin x$. Source: [9]. Approximate solution at $N = 5$ gives:

$$\begin{aligned} z_6(x) = 1.999999993x + 2.4 \times 10^{-9} + 0.007220251968x^5 \\ + 0.00157138216x^4 - 0.167443171x^3 + 0.000122523x^2. \end{aligned}$$

Approximate solution at $N = 10$ gives:

$$\begin{aligned} z_{10}(x) = 1.999999998x + 1.439054136 \times 10^{-8}x^9 \\ - 0.00001124269724x^8 + 0.00003602954926x^7 \\ + 0.009466753798x^5 - 0.0008183427688x^6 \\ - 0.00075386394x^4 - 0.0000261527x^2 \\ - 0.1664323299x^3 + 5.9 \times 10^{-10}. \end{aligned} \tag{24}$$

3.2. Discussion

The tables and figures below present a comparative analysis of three solutions: the proposed Perturbed Laguerre-Galerkin method, a method from existing literature, and the exact solution. These comparisons highlight the accuracy and effectiveness of the proposed approach in relation to established techniques. The results demonstrate how the proposed method aligns with or improves upon existing solutions. Through these tabular representations, a clear assessment of performance and reliability is provided.

4. Conclusion

The analytical solution of many integro-differential equations poses significant challenges, often making exact solutions unattainable. Consequently, numerical methods are essential for obtaining reliable approximations. In this study, we introduced the Laguerre Perturbed Galerkin (LPG) method, which integrates Laguerre polynomials as basis functions and Chebyshev polynomials as perturbation terms to enhance accuracy and efficiency. The numerical results demonstrate that the method performs exceptionally well, yielding exact solutions in certain cases and exhibiting superior accuracy compared to existing methods. Notably, the approach maintains its effectiveness even at lower degrees of approximant (N), achieving better results with fewer computational resources. This improvement underscores the efficiency of the LPG method, as it provides highly accurate solutions without requiring a high-degree polynomial approximation. Overall, the findings confirm the robustness and practicality of the LPG method for solving higher-order integro-differential equations. The enhanced accuracy, stability, and computational efficiency suggest that this technique holds significant promise for further applications in mathematical modeling across various scientific and engineering domains.

Author Contributions. **Ajimot Folasade Adebisi**: Conceptualization; Methodology; Formal analysis; Investigation; Writing—original draft

preparation. **Taiwo Adetola Ojurongbe**: Software; Validation; Data curation; Visualization; Writing—review and editing. **Kazeem Adekunle Okunola**: Resources; Project administration; Supervision; Writing—review and editing. All authors discussed the results and contributed to the final manuscript.

Acknowledgement. The authors are thankful the editors and reviewers who have supported us in improving this manuscript.

Funding. This research received no external funding.

Conflict of interest. The authors declare no conflict of interest regarding the publication of this paper.

Data availability. Not applicable.

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