RESEARCH ARTICLE • OPEN ACCESS

M_e^v-polynomial and K Banhatti Indices of Some Hat-graphs

Akar H. Karim and Ayad M. Ramadan

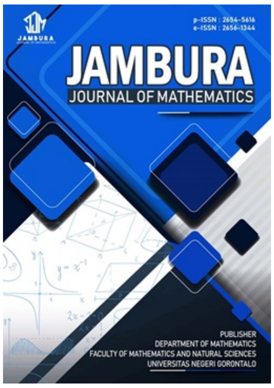
JAMBURA
JOURNAL OF MATHEMATICS

Volume 8, Issue 1, Pages 67–74, February 2026

Received 17 November 2025, Revised 26 January 2026, Accepted 30 January 2026, Published 8 February 2026

To Cite this Article : A. H. Karim and A. M. Ramadan,"M_e^v-polynomial and K Banhatti Indices of Some Hat-graphs", *Jambura J. Math*, vol. 8, no. 1, pp. 67–74, 2026, *https://doi.org/10.37905/jjom.v8i1.35382*

JOURNAL INFO • JAMBURA JOURNAL OF MATHEMATICS



Homepage : http://ejurnal.ung.ac.id/index.php/jjom/index
Journal Abbreviation : Jambura J. Math.
Frequency : Biannual (February and August)
Publication Language : English (preferable), Indonesia
DOI : https://doi.org/10.37905/jjom
Online ISSN : 2656-1344
Editor-in-Chief : Hasan S. Panigoro
Publisher : Department of Mathematics, Universitas Negeri Gorontalo
Country : Indonesia
OAI Address : http://ejurnal.ung.ac.id/index.php/jjom/oai
Google Scholar ID : iWLjgaUAAAAJ
Email : info.jjom@ung.ac.id

Jambura Journal of Mathematics, Volume 8, Issue 1, Pages 67–74, February 2026 https://doi.org/10.37905/jjom.v8i1.35382

Research Article

M_e^v-polynomial and K Banhatti Indices of Some Hat-graphs

Akar H. Karim[1,*], **Ayad M. Ramadan**[1]

[1]*Mathematics Department, College of Science, Sulaimani University, Sulaimaniya, Kurdistan Region of Iraq, Iraq*

ARTICLE HISTORY
Received 17 November 2025
Revised 26 January 2026
Accepted 30 January 2026
Published 8 February 2026

KEYWORDS
Graph polynomial
M_e^v-polynomial
First K Banhatti index
Second K Banhatti index
Hat-graphs

ABSTRACT. *A graph polynomial that has taken several attentions is M-polynomial due to it's significant a considerable number of studies have been conducting on it, moreover some other versions of this polynomial have been defined. In this paper an new version of M-polynomial is presented that will be known as M_e^v-Polynomial which is an extension of the notation of M-polynomial for comparison between vertices and corresponding adjacent edges. Then we investigate the mathematical relationship between M_e^v-Polynomial and two resent defined topological indices: first and second K Banhatti indices. Next, we establish explicit formulas for the M_e^v-Polynomial of some graphs in the family of hat-graphs with it's plots for special number of vertices. From these results we further deduce the corresponding K Banhatti indices.*

1. Introduction

In graph theory, a simple graph G is a combination of two sets $V(G)$ and $E(G)$ that are called vertex and edge sets respectively, in which neither two vertices are combined by more than one edge nor one edge joins a vertex with itself. Order of G means number of vertices in $V(G)$, meanwhile size of G is number of edges in $E(G)$. For a graph G, the degree of a vertex $v \in V(G)$, denoted by $d_G(v)$, is the number of edges incident to v. Similarly, the degree of an edge $e = uv \in E(G)$ is defined as the sum of the degrees of its two endpoints, excluding the edge itself. Formally, $d_G(e) = d_G(u) + d_G(v) - 2$ [1–3].

A large number of graph polynomials have been introduced in the literature, some of them proved to be applicable in the field of mathematical chemistry. For example the well-known distance-based polynomial is Hosoya polynomial [4], M-polynomial and CoM-polynomial [5, 6]. The defined polynomials are based on different categories, we see hosoya polynomial based on distance between vertices while M-polynomial [5] based on degree of vertices. one of the significant of graph polynomials is to obtain some topological indices from them, as the Wiener index is the first derivative of hosoya polynomial at $x = 1$ [7], while the M-polynomial has become well-known as a unifying framework because by performing some algebraic operations, a large range of degree-based topological indices can be produced [8–11]. Many variants and uses of the M-polynomial have been investigated in the last ten years for special graphs with particular chemical structures and classical graph families, see [8–12]. These investigations have shed important light on the relationship between polynomial invariants and graph structure. However, the classical M-polynomial does not fully capture certain structural features, motivating the search for refined formulations, such as the CoM-polynomial is considered for non-adjacent vertices , that yields degree-based topological co-indices [6]. Neighborhood M-polynomial which considered as a neighborhood degree-based polynomial [13]. M_{ve}-polynomial which based on vertex-edge degree of vertices of the graph [14]. In this paper, a new version of M-polynomial will be proposed that depends on the comparison of the graph's vertices with their corresponding adjacent edges which we can call as vertex-edge version of M-polynomial, and examine its characteristics for a family of graphs called hat-graphs [15].

Hat-graphs are a special family of composite graphs that are constructed from basic graphs by assigning new vertices and edges in a special way. Because of their structural characteristics and natural emergence as extensions of the graph families, hat-graphs are good candidates for evaluating the robustness of polynomial invariants. For a number of hat-graph subclasses, such as the hat-cycle, hat-path, hat-star, and hat-wheel graphs, we calculate the suggested polynomial.

In terms of topological indices, depending on the comparison of vertices with their adjacent edges some indices have been introduced, such as first and second K Banhatti Indices by Kulli [16]. For a simple graph G having order n and size m, these two indices are defined as follows:
First K Banhatti index:

$$B_1(G) = \sum_{ve}[d_G(v) + d_G(e)] = \sum_{i=1}^{n} \sum_{j=1}^{d_G(v_i)} [d_G(v_i) + d_G(e_j)]$$

Second K Banhatti index:

$$B_2(G) = \sum_{ve} d_G(v) d_G(e) = \sum_{i=1}^{n} \sum_{j=1}^{d_G(v_i)} [d_G(v_i) d_G(e_j)]$$

From their formulas, these indices can be seen as vertex-edge versions of first and second Zagreb indices respectively [17]. Hence, by this extension in the concept of M-polynomial all topo-

---

[*] Corresponding Author.

Email : *akar.karim@univsul.edu.iq* (A. H. Karim)

Homepage : http://ejurnal.ung.ac.id/index.php/jjom/index / E-ISSN : 2656-1344

logical indices depending on vertex to adjacent edges comparison could be underling as vertex-edge version of their original degree-based indices.

2. Methods

2.1. Conceptual Framework

This study is grounded in the theory of degree-based graph polynomials and their applications to topological indices. Building upon the classical M-polynomial, we introduce a vertex-edge version, denoted as M_e^v-polynomial, which incorporates both vertex and edge degrees through incidence relations. This extension enables the systematic derivation of vertex-edge topological indices, including the first and second K Banhatti indices.

2.2. Definition of M_e^v-Polynomial

For a simple graph G with n vertices and m edges the the vertex to edge version of M-polynomial is a graph polynomial based on the comparison between vertices and edges of the graph G, which is defined as:

$$M_e^v(G,x,y) = \sum_{i=1}^{n}\sum_{j=1}^{d_G(v_i)} x^{d_G(v_i)}y^{d_G(e_j)}$$

where $d_G(v)$ is the degree of the vertex v and $d_G(e)$ is the degree of the edge e. This formulation generalizes the M-polynomial to capture vertex-edge interactions, thereby providing a foundation for computing new invariants such as, first and second K Banhatti indices.

2.3. Application to Hat-Graphs

The proposed methodology was applied to the following hat-graphs:

1. Hat-complete graph (K_n^H),
2. Hat-cycle graph (C_n^H),
3. Hat-star graph (S_n^H),
4. Hat-wheel graph (W_n^H),
5. Hat-complete bipartite graph ($K_{n,m}^H$).

For each graph, the following steps were systematically followed:

1. Vertices and edges were classified according to their degree values.
2. The M_e^v-polynomial was explicitly computed by summing over all vertex-edge incidences.
3. Exact formulas for $B_1(G)$ and $B_2(G)$ were derived using the differentiation operators.
4. Generate plots of M_e^v-polynomials for selected hat-graphs.

3. Results and Discussion

A vertex-edge topological index (resp. $ve-$topological index) of a given graph $G=(V,E)$ is a graph invariant of the form

$$I_e^v(G) = \sum_{ve} f(d_G(v), d_G(e)), \tag{1}$$

where ve indicates the vertex v and edge e are incident in G. If the general form of degree-based topological indices is given by

$$I(G) = \sum_{vw\in E(G)} f(d_G(v), d_G(w)),$$

then eq. (1) represents it's corresponding vertex-edge version.

Next, the direct formula for driving the first and second K Banhatti indices from the M_e^v-polynomial of a certain simple graph are given with applying on hat-graphs.

Theorem 1. *Let $M_e^v(G,x,y)$ be the M_e^v-polynomial of a graph G, then the first K Banhatti index of G can be obtained from $M_e^v(G,x,y)$ by*

$$B_1(G) = [(D_x + D_y)M_e^v(G,x,y)]_{x=y=1}.$$

where

$$D_x M_e^v(G,x,y) = x\frac{\partial M_e^v(G,x,y)}{\partial x},$$
$$D_y M_e^v(G,x,y) = y\frac{\partial M_e^v(G,x,y)}{\partial y}.$$

Proof. We have $M_e^v(G,x,y) = \sum_{ve} x^{d_G(v)}y^{d_G(e)}$, then we obtain

$$\begin{aligned} D_x M_e^v(G,x,y) &= \sum_{ve} d_G(v)x^{d_G(v)}y^{d_G(e)} \\ [D_x M_e^v(G,x,y)]_{x=y=1} &= \sum_{ve} d_G(v) \\ D_y M_e^v(G,x,y) &= \sum_{ve} d_G(e)x^{d_G(v)}y^{d_G(e)} \\ [D_y M_e^v(G,x,y)]_{x=y=1} &= \sum_{ve} d_G(e) \\ [(D_x + D_y)M_e^v(G,x,y)]_{x=y=1} &= \sum_{ve}[d_G(v) + d_G(e)] \\ &= B_1(G). \end{aligned}$$

□

Theorem 2. *Let $M_e^v(G,x,y)$ be the M_e^v-polynomial of a graph G, then the second K Banhatti index of G can be obtained from $M_e^v(G,x,y)$ by*

$$B_2(G) = [(D_x D_y)M_e^v(G,x,y)]_{x=y=1}.$$

Proof. Similar to the proof of Theorem 1 □

Theorem 3. *Let K_n^H be a hat-complete graph then it's M_e^v-polynomial is*

$$M_e^v(K_n^H,x,y) = n[x^{n+1}((n-1)y^{2n}+2y^{n+1})+2x^2(y^{n+1}+y^2)].$$

Table 1. vertex, edge distribution with degrees of K_n^H

v_i	$d_{K_n^H}(v_i)$	e_j	$d_{K_n^H}(e_j)$
$u_i, 1 \leq i \leq n$	$n+1$	$e_j = u_i u_j, 1 \leq j \leq n$ and $j \neq i$	$2n$
		$e_n = u_i v_{2i-1}$	$n+1$
		$e_{n+1} = u_i v_{2(i-1)}, v_0 = v_{2n}$	$n+1$
$v_i, 1 \leq i \leq 2n$	2	$e_1 = v_{2i-1} v_{2i}, 1 \leq i \leq n$	2
		$e_2 = \begin{cases} v_i u_{\frac{i+1}{2}}, & 1 \leq i \leq 2n \text{ and } i \text{ is odd} \\ v_i u_{\frac{i+2}{2}}, & 1 \leq i \leq 2n, i \text{ is even and } u_{n+1} = u_1 \end{cases}$	$n+1$

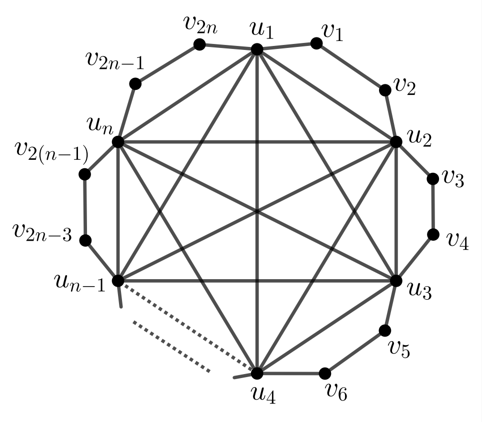


Figure 1. K_n^H

Proof. K_n^H has $3n$ vertices with degrees $n+1, 2$ and $\frac{n(n+5)}{2}$ edges with degrees $2n, n+1, 2$ as explained in Table 1.

$$\begin{aligned} M_e^v(K_n^H, x, y) &= \sum_{ve} x^{d_{K_n^H}(v)} y^{d_{K_n^H}(e)} \\ &= \sum_{i=1}^{3n} \sum_{j=1}^{d_{K_n^H}(v_i)} x^{d_{K_n^H}(v_i)} y^{d_{K_n^H}(e_j)} \\ &= n x^{n+1} \big((n-1) y^{2n} + 2 y^{n+1} \big) \\ &\quad + 2n x^2 \big(y^{n+1} + y^2 \big) \\ &= n \big[x^{n+1} \big((n-1) y^{2n} + 2 y^{n+1} \big) \\ &\quad + 2 x^2 \big(y^{n+1} + y^2 \big) \big]. \end{aligned}$$

□

Corollary 1. *The K Banhatti indices of* K_n^H *are*

$$\begin{aligned} B_1(K_n^H) &= n[3n^2 + 4n + 17], \\ B_2(K_n^H) &= 2n[n^3 + n^2 + 3n + 7]. \end{aligned}$$

Proof. From Theorem 3, we have

$$\begin{aligned} M_e^v(K_n^H, x, y) &= n(n-1) x^{n+1} y^{2n} \\ &\quad + 2n x^{n+1} y^{n+1} + 2n x^2 y^{n+1} \\ &\quad + 2n x^2 y^2 \end{aligned}$$

$$\begin{aligned} D_x M_e^v(K_n^H, x, y) &= n(n^2 - 1) x^{n+1} y^{2n} \\ &\quad + 2n(n+1) x^{n+1} y^{n+1} \\ &\quad + 4n x^2 y^{n+1} + 4n x^2 y^2 \\ D_y M_e^v(K_n^H, x, y) &= 2n^2(n-1) x^{n+1} y^{2n} \\ &\quad + 2n(n+1) x^{n+1} y^{n+1} \\ &\quad + 2n(n+1) x^2 y^{n+1} + 4n x^2 y^2 \\ D_x D_y M_e^v(K_n^H, x, y) &= 2n^2(n^2 - 1) x^{n+1} y^{2n} \\ &\quad + 2n(n+1)^2 x^{n+1} y^{n+1} \\ &\quad + 4n(n+1) x^2 y^{n+1} + 8n x^2 y^2 \\ [(D_x + D_y) M_e^v(K_n^H, x, y)]_{x=y=1} &= n[3n^2 + 4n + 17] \\ [D_x D_y M_c^v(K_n^H, x, y)]_{x=y=1} &= 2n[n^3 + n^2 + 3n + 7]. \end{aligned}$$

It follows from Theorem 1 and Theorem 2,

$$\begin{aligned} B_1(K_n^H) &= n[3n^2 + 4n + 17], \\ B_2(K_n^H) &= 2n[n^3 + n^2 + 3n + 7]. \end{aligned}$$

□

Theorem 4. *Let* C_n^H *be a hat-cycle graph then it's* M_e^v*-polynomial is*

$$M_e^v(C_n^H, x, y) = 2n(xy)^2 \big((xy)^2 + 1 \big) \big(y^2 + 1 \big).$$

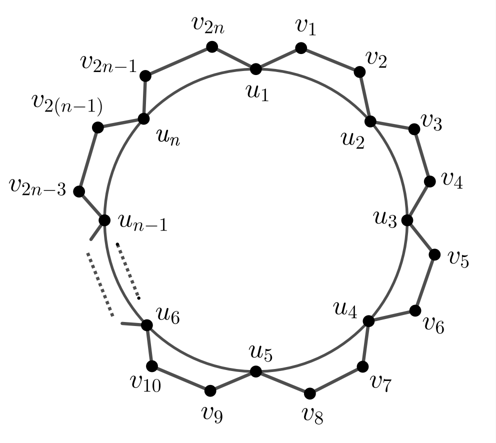


Figure 2. C_n^H

Proof. C_n^H has $3n$ vertices with degrees $4, 2$ and $4n$ edges with

Table 2. vertex, edge distribution with degrees of C_n^H

v_i	$d_{C_n^H}(v_i)$	e_j	$d_{C_n^H}(e_j)$
$u_i, 1 \leq i \leq n$	4	$e_1 = u_i u_{i+1}, u_{n+1} = u_1$	6
		$e_2 = u_i u_{i-1}, u_0 = u_n$	6
		$e_3 = u_i v_{2(i-1)}, v_0 = v_{2n}$	4
		$e_4 = u_i v_{2i-1}$	4
$v_i, 1 \leq i \leq 2n$	2	$e_1 = v_{2i-1} v_{2i}, 1 \leq i \leq n$	2
		$e_2 = \begin{cases} v_i u_{\frac{i+1}{2}}, & 1 \leq i \leq 2n \text{ and } i \text{ is odd} \\ v_i u_{\frac{i+2}{2}}, & 1 \leq i \leq 2n, i \text{ is even and } u_{n+1} = u_1 \end{cases}$	4

Table 3. vertex, edge distribution with degrees of S_n^H

v_i	$d_{S_n^H}(v_i)$	e_j	$d_{S_n^H}(e_j)$
u_{n+1}	n	$e_j = u_{n+1} u_j, 1 \leq j \leq n$	$n+1$
$u_i, 1 \leq i \leq n$	3	$e_1 = u_i u_{n+1}$	$n+1$
		$e_2 = u_i v_{2i-1}$	3
		$e_3 = u_i v_{2(i-1)}, v_0 = v_{2n}$	3
$v_i, 1 \leq i \leq 2n$	2	$e_1 = v_{2i-1} v_{2i}$, for $1 \leq i \leq n$	2
		$e_2 = \begin{cases} v_i u_{\frac{i+1}{2}}, & 1 \leq i \leq 2n \text{ and } i \text{ is odd} \\ v_i u_{\frac{i+2}{2}}, & 1 \leq i \leq 2n \text{ and } i \text{ is even} \\ & and (u_{n+1} = u_1) \end{cases}$	3

degrees $2, 4, 6$ as explained in Table 2.

$$\begin{aligned} M_e^v(C_n^H, x, y) &= \sum_{ve} x^{d_{C_n^H}(v)} y^{d_{C_n^H}(e)} \\ &= \sum_{i=1}^{3n} \sum_{j=1}^{d_{C_n^H}(v_i)} x^{d_{C_n^H}(v_i)} y^{d_{C_n^H}(e_j)} \\ &= nx^4(2y^6 + 2y^4) + 2nx^2(y^4 + y^2) \\ &= 2n(xy)^2\big((xy)^2 + 1\big)\big(y^2 + 1\big). \end{aligned}$$

□

Corollary 2. *The K Banhatti indices of C_n^H are*

$$B_1(C_n^H) = 56n,$$
$$B_2(C_n^H) = 104n.$$

Proof. From Theorem 4, we have

$$\begin{aligned} M_e^v(C_n^H, x, y) &= 2nx^4y^6 + 2n(xy)^4 \\ &\quad + 2nx^2y^4 + 2n(xy)^2 \\ D_x M_e^v(C_n^H, x, y) &= 8nx^4y^6 + 8n(xy)^4 \\ &\quad + 4nx^2y^4 + 4n(xy)^2 \\ D_y M_e^v(C_n^H, x, y) &= 12nx^4y^6 + 8n(xy)^4 \\ &\quad + 8nx^2y^4 + 4n(xy)^2 \\ D_x D_y M_e^v(C_n^H, x, y) &= 48nx^4y^6 + 32n(xy)^4 \\ &\quad + 16nx^2y^4 + 8n(xy)^2 \\ [(D_x + D_y) M_e^v(C_n^H, x, y)]_{x=y=1} &= 56n \\ [D_x D_y M_c^v(C_n^H, x, y)]_{x=y=1} &= 104n. \end{aligned}$$

It follows from Theorem 1 and Theorem 2,

$$B_1(C_n^H) = 56n,$$
$$B_2(C_n^H) = 104n.$$

□

Theorem 5. *Let S_n^H be a hat-star graph then it's M_e^v-polynomial is*

$$M_e^v(S_n^H, x, y) = n(xy)^2\big[x^{n-2}y^{n-1} + xy^{n-1} + 2xy + 2y + 2\big].$$

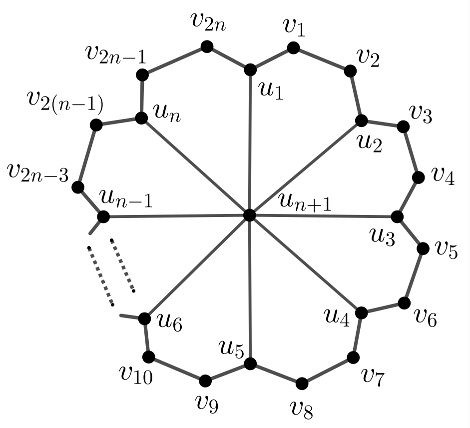


Figure 3. S_n^H

Proof. S_n^H has $3n+1$ vertices with degrees $2, 3, n$ and $4n$ edges with degrees $2, 3, n+1$ as explained in Table 3.

$$M_e^v(S_n^H, x, y) = \sum_{i=1}^{3n+1} \sum_{j=1}^{d_{S_n^H}(v_i)} x^{d_{S_n^H}(v_i)} y^{d_{S_n^H}(e_j)}$$

Table 4. vertex, edge distribution with degrees of W_n^H

v_i	$d_{W_n^H}(v_i)$	e_j	$d_{W_n^H}(e_j)$
u_{n+1}	n	$e_j = u_{n+1}u_i, 1 \le i \le n$	$n+3$
$u_i,\ 1 \le i \le n$	5	$e_1 = u_i u_{n+1}$	$n+3$
		$e_2 = u_i v_{2i-1}$	5
		$e_3 = u_i v_{2(i-1)},\ v_0 = v_{2n}$	5
		$e_4 = u_i u_{i+1},\ u_{n+1} = u_1$	8
		$e_5 = u_i u_{i-1},\ u_0 = u_n$	8
$v_i,\ 1 \le i \le 2n$	2	$e_1 = v_{2i-1}v_{2i},\ 1 \le i \le n$	2
		$e_2 = \begin{cases} v_i u_{\frac{i+1}{2}}, & i \text{ is odd} \\ v_i u_{\frac{i}{2}}, & i \text{ is even } (u_{n+1} = u_1) \end{cases}$	5

$$\begin{aligned} &= x^n(ny^{n+1}) + nx^3(y^{n+1} + 2y^3) \\ &\quad + 2nx^2(y^3 + y^2) \\ &= n(xy)^2\left[x^{n-2}y^{n-1} + xy^{n-1} + 2xy + 2y + 2\right]. \end{aligned}$$

□

Corollary 3. *The K Banhatti indices of S_n^H are*

$$\begin{aligned} B_1(S_n^H) &= n(3n + 35), \\ B_2(S_n^H) &= n[n(n + 4) + 41]. \end{aligned}$$

Proof. From Theorem 5, we have

$$\begin{aligned} M_e^v(S_n^H, x, y) &= nx^ny^{n+1} + nx^3y^{n+1} \\ &\quad + 2n(xy)^3 + 2nx^2y^3 \\ &\quad + 2n(xy)^2 \\ D_xM_e^v(S_n^H, x, y) &= n^2x^ny^{n+1} + 3nx^3y^{n+1} \\ &\quad + 6n(xy)^3 + 4nx^2y^3 \\ &\quad + 4n(xy)^2 \\ D_yM_e^v(S_n^H, x, y) &= n(n+1)x^ny^{n+1} \\ &\quad + n(n+1)x^3y^{n+1} \\ &\quad + 6n(xy)^3 + 6nx^2y^3 \\ &\quad + 4n(xy)^2 \\ D_xD_yM_e^v(S_n^H, x, y) &= n^2(n+1)x^ny^{n+1} \\ &\quad + 3n(n+1)x^3y^{n+1} \\ &\quad + 18n(xy)^3 + 12nx^2y^3 \\ &\quad + 8n(xy)^2 \\ [(D_x + D_y)M_e^v(S_n^H, x, y)]_{x=y=1} &= n(3n + 35) \\ [D_xD_yM_c^v(S_n^H, x, y)]_{x=y=1} &= n[n(n+4) + 41]. \end{aligned}$$

It follows from Theorem 1 and Theorem 2,

$$\begin{aligned} B_1(S_n^H) &= n(3n + 35), \\ B_2(S_n^H) &= n[n(n + 4) + 41]. \end{aligned}$$

□

Theorem 6. *Let W_n^H be a hat-wheel graph then it's M_e^v-polynomial is*

$$\begin{aligned} M_e^v(W_n^H, x, y) = n(xy)^2\Big[&x^{n-2}y^{n+1} + x^3y^{n+1} + 2x^3y^6 \\ &+ 2(xy)^3 + 2y^3 + 2\Big]. \end{aligned}$$

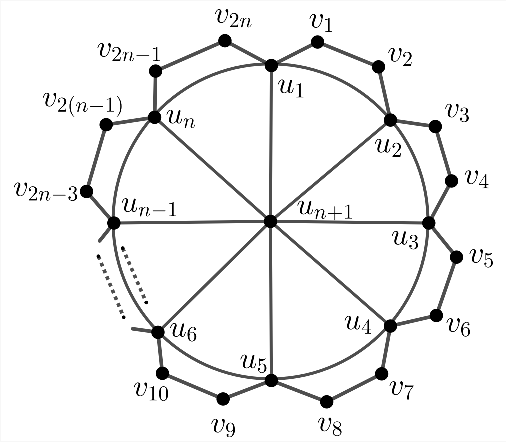


Figure 4. W_n^H

Proof. W_n^H has $3n+1$ vertices with degrees $2, 5, n$ and $5n$ edges with degrees $2, 5, 8, n+3$ as explained in Table 4.

$$\begin{aligned} M_e^v(W_n^H, x, y) &= \sum_{i=1}^{3n+1} \sum_{j=1}^{d_{W_n^H}(v_i)} x^{d_{W_n^H}(v_i)} y^{d_{W_n^H}(e_j)} \\ &= x^n(ny^{n+3}) + nx^5(y^{n+3} + 2y^8 + 2y^5) \\ &\quad + 2nx^2(y^5 + y^2) \\ &= n(xy)^2\left[x^{n-2}y^{n+1} + x^3y^{n+1} + 2x^3y^6 \right. \\ &\quad \left. + 2(xy)^3 + 2y^3 + 2\right]. \end{aligned}$$

□

Table 5. Vertex and edge distribution with degrees of $K_{n,m}^H$

v_i	$d_{K_{n,m}^H}(v_i)$	e_j	$d_{K_{n,m}^H}(e_j)$
$u_i,\ 1 \le i \le n$	$m+2$	$e_j = u_i u_j',\ 1 \le j \le m,$ $e_{m+1} = u_i v_{2i-1},$ $e_{m+2} = u_i v_{2i}$	$n+m+2,$ $m+2,$ $m+2$
$u_i',\ 1 \le i \le m$	$n+2$	$e_j = u_i' u_j,\ 1 \le j \le n,$ $e_{n+1} = u_i' v_{2(n+i)-1},$ $e_{n+2} = u_i' v_{2(n+i)}$	$n+m+2,$ $n+2,$ $n+2$
$v_i,\ 1 \le i \le 2(n+m)$	2	$e_1 = \begin{cases} v_i v_{i+1}, & i \text{ even},\ v_{2(n+m)+1} = v_1, \\ v_i v_{i-1}, & i \text{ odd},\ v_0 = v_{2(n+m)} \end{cases}$ $e_2 = \begin{cases} v_i u_{\frac{i+1}{2}}, & 1 \le i \le 2n,\ i \text{ odd}, \\ v_i u_{\frac{i}{2}}, & 1 \le i \le 2n,\ i \text{ even}, \\ v_i u'_{\frac{i+1-2n}{2}}, & 2n+1 \le i \le 2(n+m),\ i \text{ odd}, \\ v_i u'_{\frac{i-2n}{2}}, & 2n+1 \le i \le 2(n+m),\ i \text{ even} \end{cases}$	$2,$ $m+2,$ $n+2$

Corollary 4. *The K Banhatti indices of W_n^H are*

$$B_1(W_n^H) = n(3n+79),$$
$$B_2(W_n^H) = n(n^2+8n+173).$$

Proof. From Theorem 6, we have

$$\begin{aligned} M_e^v(W_n^H, x, y) &= n x^n y^{n+3} + n x^5 y^{n+3} \\ &\quad + 2n x^5 y^8 + 2n(xy)^5 \\ &\quad + 2n x^2 y^5 + 2n(xy)^2 \\ D_x M_e^v(W_n^H, x, y) &= n^2 x^n y^{n+3} + 5n x^5 y^{n+3} \\ &\quad + 10n x^5 y^8 + 10n(xy)^5 \\ &\quad + 4n x^2 y^5 + 4n(xy)^2 \\ D_y M_e^v(W_n^H, x, y) &= n(n+3) x^n y^{n+3} \\ &\quad + n(n+3) x^5 y^{n+3} \\ &\quad + 16n x^5 y^8 + 10n(xy)^5 \\ &\quad + 10n x^2 y^5 + 4n(xy)^2 \\ D_x D_y M_e^v(W_n^H, x, y) &= n^2(n+3) x^n y^{n+3} \\ &\quad + 5n(n+3) x^5 y^{n+3} \\ &\quad + 80n x^5 y^8 + 50n(xy)^5 \\ &\quad + 20n x^2 y^5 + 8n(xy)^2 \\ [(D_x + D_y) M_e^v(W_n^H, x, y)]_{x=y=1} &= n(3n+79) \\ [D_x D_y M_c^v(W_n^H, x, y)]_{x=y=1} &= n(n^2+8n+173). \end{aligned}$$

It follows from Theorem 1 and Theorem 2,

$$B_1(W_n^H) = n(3n+79),$$
$$B_2(W_n^H) = n(n^2+8n+173).$$

□

Theorem 7. *Let $K_{n,m}^H$ be a hat-complete bipartite graph then it's M_e^v-polynomial is*

$$\begin{aligned} M_e^v(K_{n,m}^H, x, y) &= nm y^{n+m+2}[x^{n+2} + x^{m+2}] \\ &\quad + 2n(xy)^2[(xy)^m + y^m + 1] \\ &\quad + 2m(xy)^2[(xy)^n + y^n + 1]. \end{aligned}$$

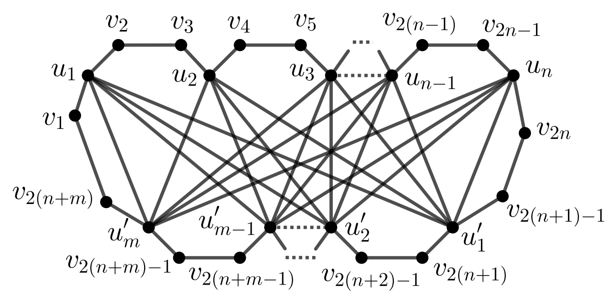


Figure 5. $K_{n,m}^H$

Proof. $K_{n,m}^H$ has $3(n+m)$ vertices with degrees $2, n+2, m+2$ and $3(n+m)+nm$ edges with degrees $2, n+2, m+2, n+m+2$ as explained in Table 5.

$$\begin{aligned} M_e^v(K_{n,m}^H, x, y) &= \sum_{i=1}^{3(n+m)} \sum_{j=1}^{d_{K_{n,m}^H}(v_i)} x^{d_{K_{n,m}^H}(v_i)} y^{d_{K_{n,m}^H}(e_j)} \\ &= n x^{m+2}[m y^{n+m+2} + 2y^{m+2}] \\ &\quad + m x^{n+2}[n y^{n+m+2} + 2y^{n+2}] \\ &\quad + 2n x^2[y^{m+2} + y^2] + 2m x^2[y^{n+2} + y^2] \\ &= nm y^{n+m+2}[x^{n+2} + x^{m+2}] \\ &\quad + 2n(xy)^2[(xy)^m + y^m + 1] \\ &\quad + 2m(xy)^2[(xy)^n + y^n + 1]. \end{aligned}$$

□

Corollary 5. *The K Banhatti indices of $K_{n,m}^H$ are*

$$B_1(K_{n,m}^H) = 3(nm+8)(n+m) + 8nm,$$
$$B_2(K_{n,m}^H) = nm[(n+m)(n+m+8)+32] + 24(n+m).$$

Proof. From Theorem 7, we have

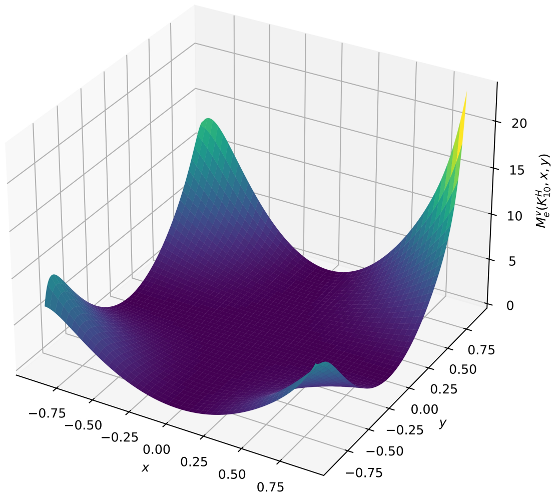
$$\begin{aligned}
M_e^v(K_{n,m}^H, x, y) &= nm[x^{n+2}y^{n+m+2} \\
&\quad + x^{m+2}y^{n+m+2}] \\
&\quad + 2n[(xy)^{m+2} + x^2y^{m+2} \\
&\quad + (xy)^2] + 2m[(xy)^{n+2} \\
&\quad + x^2y^{n+2} + (xy)^2] \\
D_x M_e^v(K_{n,m}^H, x, y) &= nm[(n+2)x^{n+2}y^{n+m+2} \\
&\quad + (m+2)x^{m+2}y^{n+m+2}] \\
&\quad + 2n[(m+2)(xy)^{m+2} \\
&\quad + 2x^2y^{m+2} + 2(xy)^2] \\
&\quad + 2m\big[(n+2)(xy)^{n+2} \\
&\quad + 2x^2y^{n+2} + 2(xy)^2\big] \\
D_y M_e^v(K_{n,m}^H, x, y) &= nm\big[(n+m+2)x^{n+2} \\
&\quad y^{n+m+2} + (n+m+2) \\
&\quad x^{m+2}y^{n+m+2}\big] \\
&\quad + 2n[(m+2)(xy)^{m+2} \\
&\quad + (m+2)x^2y^{m+2} + 2(xy)^2] \\
&\quad + 2m[(n+2)(xy)^{n+2} \\
&\quad + (n+2)x^2y^{n+2} + 2(xy)^2] \\
D_xD_y M_e^v(K_{n,m}^H, x, y) &= nm\big[(n+2)(n+m+2) \\
&\quad x^{n+2}y^{n+m+2} + (m+2) \\
&\quad (n+m+2)x^{m+2}y^{n+m+2}\big] \\
&\quad + 2n\big[(m+2)^2(xy)^{m+2} \\
&\quad + 2(m+2)x^2y^{m+2} + 4(xy)^2\big] \\
&\quad + 2m\big[(n+2)^2(xy)^{n+2} \\
&\quad + 2(n+2)x^2y^{n+2} + 4(xy)^2\big] \\
[(D_x + D_y)M_e^v(K_{n,m}^H, x, y)]_{x=y=1} &= 3(nm+8)(n+m) + 8nm \\
[D_xD_y M_c^v(K_{n,m}^H, x, y)]_{x=y=1} &= nm\big[(n+m)(n+m+8) \\
&\quad + 32\big] + 24(n+m).
\end{aligned}$$

It follows from Theorem 1 and Theorem 2,

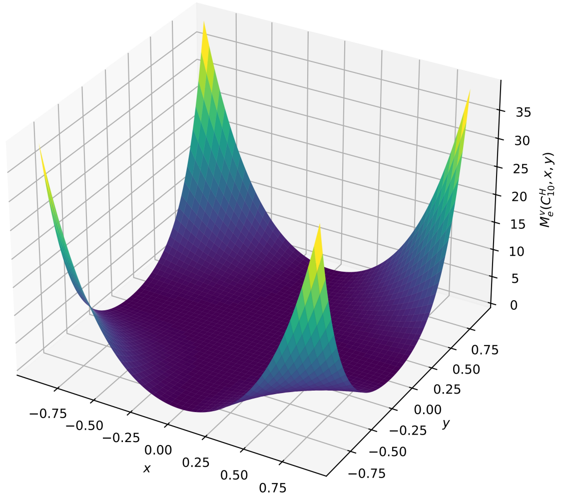
$$B_1(K_{n,m}^H) = 3(nm+8)(n+m) + 8nm,$$
$$B_2(K_{n,m}^H) = nm[(n+m)(n+m+8)+32] + 24(n+m).$$

□

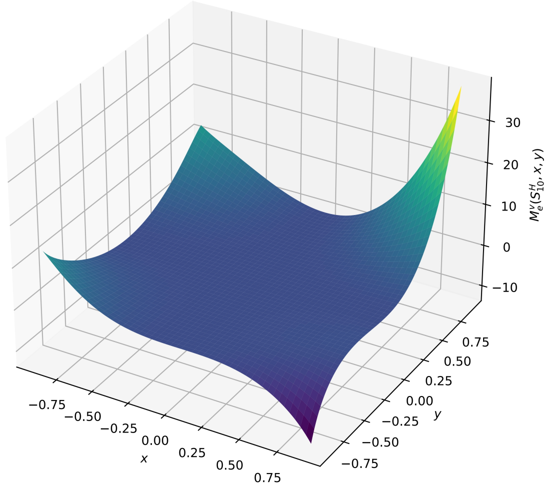
In the following figures, Python has been used to plot M_e^v-polynomials of the above mentioned graphs.



(a) $M_e^v(K_{10}^H, x, y)$

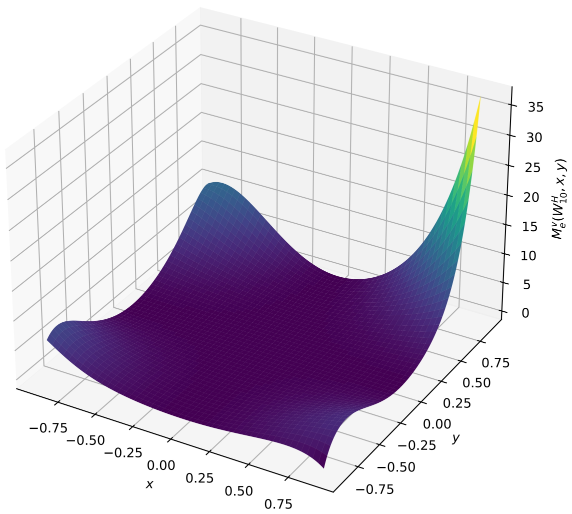


(b) $M_e^v(C_{10}^H, x, y)$

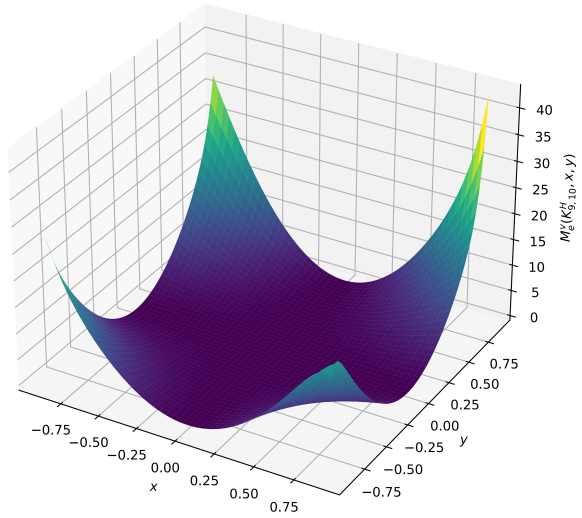


(c) $M_e^v(S_{10}^H, x, y)$

Figure 6. Plots of M_e^v-polynomial of hat-graphs K_{10}^H, C_{10}^H, and S_{10}^H



(a) $M_e^v(W_{10}^H, x, y)$



(b) $M_e^v(K_{9,10}^H, x, y)$

Figure 7. Plots of M_e^v-polynomial of hat-graphs W_{10}^H and $K_{9,10}^H$

4. Conclusion

In this paper, we introduced a new version of the M-polynomial based on vertex-edge comparison and applied it to some hat-graph: hat-complete, hat-cycle, hat-star, and hat-wheel and hat-complete bipartite graphs. Using this formulation, we obtained the proposed M_e^v-polynomials and derived closed formulas for their k Banhatti indices. These results extend existing work on hat-graphs and show that the proposed M_e^v-polynomial offers an effective tool for computing ve-indices. Future work may apply this approach to other graph classes or explore its potential applications in chemical and network structures.

Author Contributions. **Akar H. Karim**: Conceptualization, Methodology, Formal analysis, Writing–original draft. **Ayad M. Ramadan**: Visualization, Software. Both authors reviewed and approved the final version of the manuscript.

Acknowledgement. The author would like to thank the editors and reviewers for their insightful comments and helpful suggestions that significantly improved the quality of this manuscript.

Funding. The author received no funding for this study.

Conflict of interest. The authors declare no conflict of interest.

Data availability. Not applicable.

References

[1] G. Chartrand and P. Zhang, *Chromatic graph theory*. Chapman and Hall/CRC, 2019, doi: 10.1201/9780429438868.

[2] N. Biggs, E. K. Lloyd, and R. J. Wilson, *Graph Theory, 1736-1936*. Oxford University Press, 1986, iSBN: 978-0198539162.

[3] C. Vasudev, *Graph theory with applications*. New Age International, 2006, iSBN: 978-8122417371.

[4] H. Hosoya, "On some counting polynomials in chemistry," *Discrete applied mathematics*, vol. 19, no. 1-3, pp. 239–257, 1988, doi: 10.1016/0166-218X(88)90017-0.

[5] E. Deutsch and S. Klavžar, "M-polynomial and degree-based topological indices," *Iranian Journal of Mathematical Chemistry*, vol. 6, no. 2, pp. 93–102, 2015, doi: 10.22052/ijmc.2015.10106.

[6] S. A. K. Kirmani and P. Ali, "Com-polynomial and topological coindices of hyaluronic acid conjugates," *Arabian Journal of Chemistry*, vol. 15, no. 7, p. 103911, 2022, doi: 10.1016/j.arabjc.2022.103911.

[7] H. Wiener, "Structural determination of paraffin boiling points," *Journal of the American chemical society*, vol. 69, no. 1, pp. 17–20, 1947, doi: 10.1021/ja01193a005.

[8] A. J. M. Khalaf, S. Hussain, D. Afzal, F. Afzal, and A. Maqbool, "M-polynomial and topological indices of book graph," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 23, no. 6, pp. 1217–1237, 2020, doi: 10.1080/09720529.2020.1809115.

[9] B. Basavanagoud, A. P. Barangi, and P. Jakkannavar, "M-polynomial of some graph operations and cycle related graphs," *Iranian Journal of Mathematical Chemistry*, vol. 10, no. 2, pp. 127–150, 2019, doi: 10.22052/ijmc.2019.146761.1388.

[10] F. Afzal, S. Hussain, D. Afzal, and S. Razaq, "Some new degree based topological indices via m-polynomial," *Journal of Information and Optimization Sciences*, vol. 41, no. 4, pp. 1061–1076, 2020, doi: 10.1080/02522667.2020.1744307.

[11] M. Cancan, D. Afzal, S. Hussain, A. Maqbool, and F. Afzal, "Some new topological indices of silicate network via m-polynomial," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 23, no. 6, pp. 1157–1171, 2020, doi: 10.1080/09720529.2020.1809776.

[12] A. H. Karim, N. E. Arif, and A. M. Ramadan, "The m-polynomial and nirmala index of certain composite graphs," *Tikrit Journal of Pure Science*, vol. 27, no. 3, pp. 92–101, 2022, doi: 10.25130/tjps.v27i3.45.

[13] S. Mondal, M. K. Siddiqui, N. De, and A. Pal, "Neighborhood m-polynomial of crystallographic structures," *Biointerface Res. Appl. Chem*, vol. 11, no. 2, pp. 9372–9381, 2021, doi: 10.33263/BRIAC112.93729381.

[14] K. B. Rasool, P. A. Rashed, and A. M. Ali, "Relations between vertex-edge degree based topological indices and m_{ve} -polynomial of $r-$ regular simple graph," *European Journal of Pure and Applied Mathematics*, vol. 16, no. 2, pp. 773–783, 2023, doi: 10.29020/nybg.ejpam.v16i2.4698.

[15] A. H. Karim and N. E. Arif, "M-polynomial and com-polynomial of hat-graphs," *Basrah Journal of Sciences*, vol. 43, no. 1, pp. 1–17, 2025, doi: 10.29072/basjs.20250101.

[16] V. Kulli, "On k banhatti indices of graphs," *Journal of Computer and Mathematical Sciences*, vol. 7, no. 4, pp. 213–218, 2016.

[17] I. Gutman and N. Trinajstić, "Graph theory and molecular orbitals. total φ-electron energy of alternant hydrocarbons," *Chemical physics letters*, vol. 17, no. 4, pp. 535–538, 1972, doi: 10.1016/0009-2614(72)85099-1.