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Muhammad Luthfi Setiarno Putera and Purhadi Purhadi

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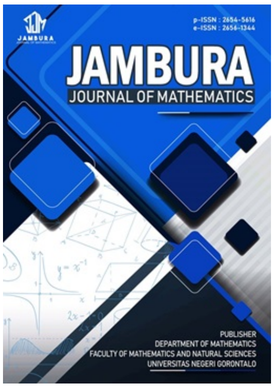

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Research Article

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Parameter Estimation of Generalized Modified Weibull Using the Maximum Likelihood on Simulation and Real-World Data

Muhammad Luthfi Setiarno Putera[1,*], **Purhadi Purhadi**[2]

[1]Department of Sharia, State Islamic University of Palangka Raya, Palangka Raya 73112, Indonesia
[2]Department of Statistics, Institut Teknologi Sepuluh Nopember, Surabaya 60111, Indonesia

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ABSTRACT. *This study estimates parameters of the generalized modified Weibull (GM Weibull) distribution using the Maximum Likelihood Estimation (MLE) method with the Broyden–Fletcher–Goldfarb–Shanno (BFGS) algorithm. The GM Weibull distribution, which includes four parameters ($\lambda, \theta, \phi, \tau$), offers greater flexibility than Weibull distribution in modeling data with monotonic and bathtub-shaped hazard patterns. Parameter estimation was conducted on three datasets: simulated data with sample sizes of 50, 200, and 500 observations; survival data from 45 heart transplant patients; and health indicator data from 27 districts/cities in Central and South Kalimantan provinces. The results demonstrate that while the standard Weibull remains a parsimonious choice for simple monotonic data, the GM Weibull produces parameter estimates closer to theoretical values in small-to-medium samples and significantly lower deviance in complex datasets. Specifically, for the heart transplant data, the GM Weibull offered better modeling long-term survival tails (800–1,000 days), while for the health indicator data, it effectively accommodated central tendencies within asymmetric distributions. Although AIC and BIC favor standard Weibull, the GM Weibull accurately identifies underlying structural fluctuations and non-monotonic failure characteristics. This study confirms that the MLE-based GM Weibull distribution is one of the robust tools for researchers requiring a more representative model for complex survival and health data.*

1. Introduction

The Weibull distribution is a continuous probability distribution commonly used in various fields. First introduced by Waloddi Weibull in 1939, it is known for its flexibility in modeling various types of data, such as component failure times and disease onset or deaths [1, 2]. Such a distribution is the primary choice for modeling events whose density equations exhibit skewness with a monotonic failure rate, meaning the risk of an event consistently increases or decreases over time [3, 4]. While traditionally associated with health and industrial fields, its application extends to any continuous variable where the rate of occurrence changes over a specific interval, including rainfall patterns and proportional indices [5, 6]. Therefore, the Weibull distribution plays a crucial role in modeling data with diverse characteristics.

However, the Weibull distribution parameter estimation results cannot be optimally applied to events with varying density equations, such as bathtub [7]. In survival analysis, this is seen in high infant mortality that stabilizes before rising in old age [8]. In percentage data, such as market penetration rates or chemical concentration levels, a similar pattern could occur where high initial or final frequencies are separated by a stable middle interval [9, 10]. Since the Weibull distribution is mathematically constrained to monotonic density equations, it cannot optimally address these complex, fluctuating patterns found in both time-to-event and proportional datasets.

Recent studies on parameter estimation have attempted to address these complexities, such as work by [11], who studied three Weibull parameters: the scale parameter, the shape parameter, and the location parameter. However, this estimation process assumed the shape parameter to be a fixed value. Thus, no estimation was performed on this parameter. Another study applied modifications to the parameter estimation method to address datasets with diverse failure characteristics [12]. The results implied that the parameter estimation was only intended for two parameters and did not accommodate additional parameters, such as the location parameter and others.

To cope with these issues, this study will apply parameter estimation to the generalized modified Weibull distribution. This distribution was initiated by Carrasco et al. by accommodating two additional parameters, in addition to the two parameters commonly found in the ordinary Weibull distribution [13]. These additional parameters are a shape parameter and a parameter associated with lifetime. The addition of these parameters can optimally accommodate various data characteristics, such as monotonic and bathtub. In addition, the GM Weibull distribution can also be a sub-distribution, such as the exponential Weibull, generalized Rayleigh, exponential, and so on [14]. The resulting parameter estimates are also closer to empirical values than the ordinary Weibull [4, 13].

*Corresponding Author.

Email : m.luthfi@uin-palangkaraya.ac.id (M. L. S. Putera)
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Based on previous studies, it is indicated that parameter estimation for data with diverse characteristics is better approached by the generalized modified Weibull distribution [4, 13]. This is as evidenced by the log-likelihood, deviance, and other comparative indicators. Accordingly, our study relies on the hypothesis that the GM Weibull distribution provides a better fit for non-monotonic patterns compared to traditional Weibull models across diverse data formats. To achieve this, simulation data and application data are required. It is expected that the results of parameter estimation approximated by the generalized modified Weibull distribution can be compared with similar distributions, such as Weibull, for datasets in the health sector and other fields. By performing the two models on various datasets, we aim to obtain parameter estimates that align more closely with empirical reality.

2. Methods

2.1. Generalized Modified Weibull Distribution

The Weibull distribution generally comprises two parameters, scale parameter λ and shape parameter θ [15]. Let z be a Weibull random variable, then its density function is given by

$$g(z|\lambda,\theta) = \lambda\theta z^{\theta-1}\exp(-\lambda z^{\theta}). \tag{1}$$

There are several modified forms of the Weibull distribution, including the generalized modified Weibull distribution [13]. A random variable Z has a generalized modified Weibull distribution with a density function $g(z)$ given by

$$g(z|\lambda,\theta,\tau,\phi) = \lambda\phi z^{\theta-1}(\theta+\tau z)\exp\left\{\tau z-\lambda z^{\theta}\exp(\tau z)\right\} \left[1-\exp\left\{-\lambda z^{\theta}\exp(\tau z)\right\}\right]^{\phi-1}. \tag{2}$$

The scale parameter in eq. (2) is denoted by λ, while the shape parameters are denoted by θ and ϕ. The characteristics regarding the indicators of the sustainability of individual life over time are denoted by the parameter τ [4]. The cumulative distribution function of the generalized modified Weibull is expressed as

$$G(z) = \left(1-\exp\left\{-\lambda z^{\theta}\exp(\tau z)\right\}\right)^{\phi}. \tag{3}$$

If m is a positive integer, the m-th moment of the generalized modified Weibull distribution is

$$\mu_m = \lambda\phi\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(\phi)}{\Gamma(\phi-j)j!\{\lambda(j+1)\}^{m/\theta+1}} \sum_{i_1,\dots,i_m=0}^{\infty}\frac{\Gamma\left(\frac{i_1+\cdots+i_m+m}{\theta}+1\right)}{(i_1+1)^{1-i_1}\cdots(i_m+1)^{1-i_m}}\frac{x_1^{i_1}\cdots x_m^{i_m}}{i_1!\cdots i_m!}, \tag{4}$$

with $x_i = \left(-\frac{\tau}{\theta}\right)\{\lambda(j+1)\}^{-1/\theta}$ and the parameters τ and θ are both positive, along with $\lambda>0$ and $\phi\neq 1$ [13]. The moments in eq. (4) can be used to determine the mean, variance, and so on, according to the order of the moments.

2.2. Maximum Likelihood Estimation

Maximum Likelihood Estimation (MLE) is a method for estimating the parameters of a probability distribution. This is carried out by discovering the parameter values that enable the observed data most likely to occur [16]. Suppose there are random samples $z_1, z_2, \dots, z_n$ originating from the generalized modified Weibull distribution. Then, the joint probability of all observed data, known as the likelihood function $L(\gamma)$, as shown in eq. (5), where $\gamma = [\lambda,\theta,\tau,\phi]^T$ is an unknown parameter vector of the generalized modified Weibull distribution.

$$L(\gamma|z_1,z_2,\dots,z_n) = \prod_{i=1}^{n} g(z_i|\gamma). \tag{5}$$

In order to simplify the calculations, it is necessary to apply the natural logarithm to the likelihood function, known as the log-likelihood function $l(\gamma)$, as formulated in eq. (6)

$$\begin{aligned} l(\gamma) &= \ln L(\gamma) \\ &= \ln\left(\prod_{i=1}^{n} g(z_i|\gamma)\right) \\ &= \sum_{i=1}^{n}\ln g(z_i|\gamma). \end{aligned} \tag{6}$$

Due to the property of logarithms satisfying the principle of monotonicity, maximizing $l(\gamma)$ will produce the same estimate of $\hat{\gamma}$ as maximizing $L(\gamma)$ [17].

The main objective of MLE is to estimate the value of γ that maximizes the log-likelihood function in eq. (6). To estimate such value of γ, a derivative operation of the log-likelihood function with respect to γ is required and the result is equated to zero to then solve the equation [17]. In this case, eq. (7) is derived each with respect to the 4 (four) parameters of the GM Weibull distribution.

$$\frac{\partial l(\gamma)}{\partial\lambda}=0;\quad \frac{\partial l(\gamma)}{\partial\theta}=0; \quad \frac{\partial l(\gamma)}{\partial\phi}=0;\quad \frac{\partial l(\gamma)}{\partial\tau}=0. \tag{7}$$

The derivative process in eq. (7) generally produces an analytical solution, but a numerical method is needed to estimate the optimal $\hat{\gamma}$ in the context of the generalized modified Weibull distribution [13]. By applying mathematical rules to eq. (7), eq. (8) presents the derivative results for all parameters in the i-th observation.

$$\begin{aligned} \frac{\partial l(\gamma)}{\partial\lambda} &= \sum_{i=1}^{n}\left[\frac{1}{\lambda}-z_i^{\theta}\exp(\tau z_i)+(\phi-1) \frac{z_i^{\theta}\exp(\tau z_i)\exp(-\lambda z_i^{\theta}\exp(\tau z_i))}{1-\exp(-\lambda z_i^{\theta}\exp(\tau z_i))}\right] \\ \frac{\partial l(\gamma)}{\partial\theta} &= \sum_{i=1}^{n}\left[\ln z_i+\frac{1}{\theta+\tau z_i}-\lambda z_i^{\theta}\exp(\tau z_i)\ln z_i +(\phi-1)\frac{\lambda z_i^{\theta}\ln z_i\exp(\tau z_i)\exp(-\lambda z_i^{\theta}\exp(\tau z_i))}{1-\exp(-\lambda z_i^{\theta}\exp(\tau z_i))}\right] \\ \frac{\partial l(\gamma)}{\partial\phi} &= \sum_{i=1}^{n}\left[\frac{1}{\phi}+\ln\left(1-\exp(-\lambda z_i^{\theta}\exp(\tau z_i))\right)\right] \\ \frac{\partial l(\gamma)}{\partial\tau} &= \sum_{i=1}^{n}\left[\frac{z_i}{\theta+\tau z_i}+z_i-\lambda z_i^{\theta+1}\exp(\tau z_i) +(\phi-1)\frac{\lambda z_i^{\theta+1}\exp(\tau z_i)\exp(-\lambda z_i^{\theta}\exp(\tau z_i))}{1-\exp(-\lambda z_i^{\theta}\exp(\tau z_i))}\right]. \end{aligned} \tag{8}$$

2.3. Broyden–Fletcher–Goldfarb–Shanno Algorithm

The BFGS algorithm can be used for iterative solution of non-linear equations, both constrained and unconstrained [18]. In the context of MLE estimation, BFGS maximizes the log-likelihood equation $l(\gamma)$ by minimizing the objective function $f(\gamma) = -l(\gamma)$ where γ is a parameter vector. The following are the stages of the BFGS algorithm used in this study [19].

1. Determining a random sample of n observations for 1 random variable $(z_1, z_2, \ldots, z_i, \ldots, z_n)$ which has a generalized modified Weibull distribution.
2. Determining the initial value of the parameter vector γ_0. The value used can be a random value with a positive value. To avoid local optima, initial parameters for Weibull were set to $\beta = 1$ and $\lambda = 1$, while initial parameters for GM Weibull were set to values near the standard Weibull characteristics, e.g. $\beta = 0.9$, $\lambda = 0.1$, $\phi = 0.8$, and $\tau = 0.0005$.
3. Determining the approximate Hessian matrix H_0, exactly in the form of the identity matrix I.
4. Determining the convergence criteria where $\|\nabla f(\gamma_m)\| \leq 10^{-5}$.
5. Starting the BFGS computation from $m = 0$ with the initial gradient $s_0 = \nabla f(\gamma_m) = -\nabla l(\gamma_m)$ with the following steps, where $m = 0, 1, 2, \ldots$.
   (a) Determine the direction of decrease $p_m = -H_m s_m$, with $s_m = \nabla f(\gamma_m)$.
   (b) Find the step size $\alpha_m > 0$ that minimizes $f(\gamma_m + \alpha_m p_m)$ or satisfies Wolfe's condition.
   (c) Update the parameters by calculating $w_m = \alpha_m p_m$ and updating $\gamma_{m+1} = \gamma_m + w_m$.
   (d) Calculate the new gradient, namely $s_{m+1} = \nabla f(\gamma_{m+1})$. Then, calculate the difference value of the gradient $y_m = s_{m+1} - s_m$.
   (e) Update Hessian matrix approximation based on

$$H_{m+1} = H_m + \frac{(w_m^T y_m + y_m^T H_m y_m) w_m w_m^T}{(w_m^T y_m)^2} - \frac{H_m y_m w_m^T + w_m y_m^T H_m}{w_m^T y_m}.$$

   (f) Check for convergence. If $\|\nabla f(\gamma_m)\| \leq 10^{-5}$, then the iteration is stopped. If the convergence condition is not met, then the iteration continues from step 5) with $m = m + 1$. To handle potential numerical instabilities during iteration, log-likelihood values resulting in NaN or infinity were capped at a lower bound of $\ln(10^{-6})$.

2.4. The Step of Parameter Estimation

The parameter estimation of the generalized modified Weibull distribution is carried out using the Maximum Likelihood Estimation (MLE) method. In general, the likelihood function is constructed based on the probability density function in eq. (1) of the Weibull distribution and eq. (2) of the generalized modified Weibull distribution involving several parameters, namely λ, θ, ϕ, and τ. To produce a parameter estimate, the steps taken are to show the process of obtaining the gradient of all GM Weibull distribution parameters, which starts from deriving the log-likelihood function as in eq. (7).

Parameter estimation was first performed on simulated data obtained from a Weibull distribution generated via R software. The simulation utilized a scale parameter $\lambda = 5$ and a shape parameter $\theta = 1$. Such configuration was selected since it would allow the Weibull and GM Weibull distributions to characterize other distributions' behavior, e.g. an exponential distribution which has constant failure rate. Simulated data were generated for three sample sizes: 50, 200, and 500 observations. This was carried out to observe the accuracy of the MLE estimator at varying sample sizes.

After applying estimation to the simulated data, the parameter estimates were then employed on the real dataset. The first dataset was sourced from 69 heart transplant recipients [20]. Of the 69 individuals, 45 were not censored and became the focus of observation, with the data shown in Table 1.

Table 1. Data of heart transplant survival time

Survival time (in days)		
5	68	207
16	72	219
16	72	285
17	77	285
28	78	308
30	80	334
39	81	342
43	90	583
45	96	675
51	100	733
53	110	852
58	153	979
61	165	995
66	186	1032
68	188	1386

Source: [20]

The second dataset in Table 2 is sourced from the Indonesian Statistics regarding the Percentage of the Population with Health Complaints in the Past Month and Not Seeking Outpatient Treatment Due to Self-Management, by Regency/City in Central Kalimantan and South Kalimantan Provinces. The total observations were 27 regencies/cities in 2023. For this dataset, we shifted our perspective from time-to-failure data to percentage data. This study was going to confirm the versatility of the Weibull and GM Weibull model to estimate the parameter scales.

While the GM Weibull distribution is defined over the support $z \in [0, \infty)$, it could serve as a robust approximation for proportional or percentage data. In such contexts, the distribution is theoretically justified when the probability mass beyond the upper bound (e.g., $z > 100$) is negligible. Furthermore, the GM Weibull's four-parameter flexibility allows it to mimic the behavior of bounded distributions, such as the Beta or unit-Weibull, without the symmetry constraints of the Gaussian model [21, 22]. This makes it particularly effective for modeling rates and proportions that exhibit skewness or non-monotonic density characteristics.

A series of analyses was conducted to assess the ability of the GM Weibull distribution to estimate actual parameters based on empirical data. The parameters of the Weibull distribution and GM Weibull were estimated using the MLE method and the

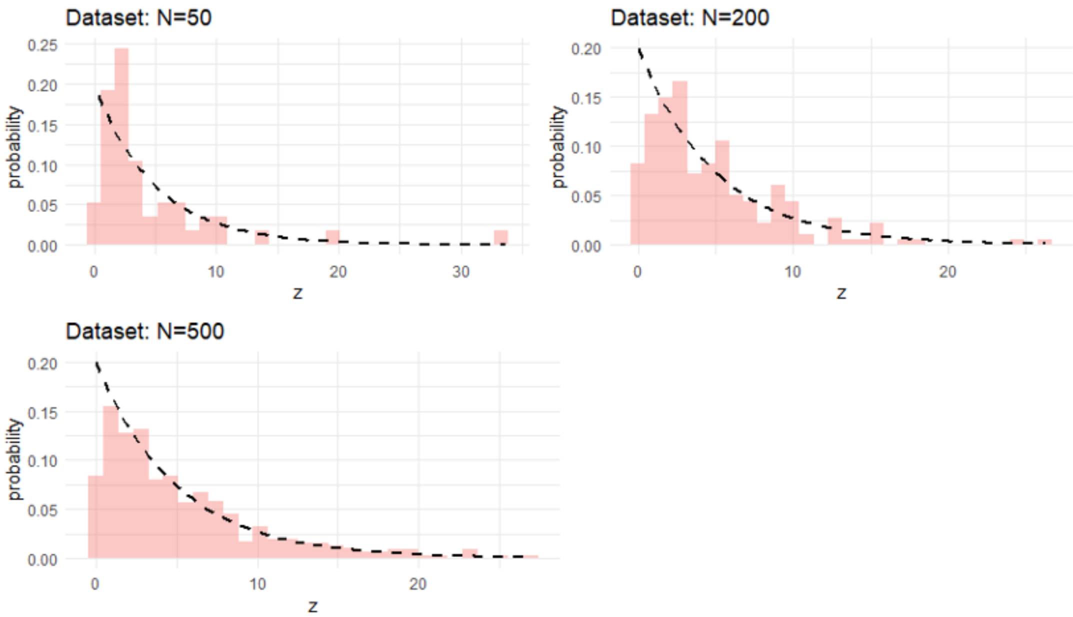

Figure 1. Histogram and probability density curve of the simulation datasets

Table 2. The percentage of the population who do not seek outpatient treatment due to self-management

Percentage (%)		
49.5	64.49	77.69
53.06	65.27	79.02
56.54	66.94	79.21
59.73	67.44	80.39
60.86	67.74	82.99
60.95	69.39	83.26
62.89	72.62	85.01
63.74	74.58	86.9
63.8	76.33	88.79

Source: [23, 24]

solution was approached using the BFGS algorithm as described in section 2.3. The parameter estimation on the simulated and real datasets was performed using the R application by means of the `maxLik` package. The optimization utilized the constraints argument to implement an Inequality-Constrained BFGS, ensuring that the shape and scale parameters did not converge to non-negative values. Data generation for simulation was performed using the `stats` package with `set.seed(234)` to ensure the exact datasets can be reconstructed for verification.

In addition to comparing the empirical and theoretical parameter estimates, a comparison was made between two distributions, namely Weibull and GM Weibull, based on the calculation of log-likelihood and deviance [25]. The log-likelihood value in question can be obtained from eq. (9)

$$l(\gamma|z_1, z_2, \ldots, z_n) = \sum_{i=1}^{n} \Big(\ln(\phi) + \ln(\lambda) + (\theta - 1)\ln z_i + \ln(\theta + \tau z_i) + \tau z_i - \lambda z_i^{\theta} \exp(\tau z_i) + (\phi - 1)\ln\left[1 - \exp\{-\lambda z_i^{\theta}\exp(\tau z_i)\}\right] \Big). \quad (9)$$

In the case of the Weibull distribution, the values of ϕ and τ in eq. (9) can be expressed as constants with values of 1 and 0, respectively [13]. The deviance is a measure that indicates the closeness of the estimated value to the actual value. The lower the deviance value, the closer the parameter estimate is to the actual value [25, 26]. The following is the equation for calculating deviance.

$$d^2(\gamma|z_1, z_2, \ldots, z_n) = -2\, l(\gamma|z_1, z_2, \ldots, z_n). \quad (10)$$

Thus, the deviance in eq. (10) is a negative multiple of 2 of the log-likelihood value.

To account for the difference in the number of parameters (k) between the Weibull ($k = 2$) and the GM Weibull ($k = 4$) distributions, the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC) are employed. These criteria ensure a fair comparison by penalizing the model for its complexity, thereby reducing the risk of overfitting [27]. The formulas are defined as follows:

$$\text{AIC} = 2k - 2l(\hat{\gamma}), \quad (11)$$
$$\text{BIC} = k\ln(n) - 2l(\hat{\gamma}), \quad (12)$$

where k is the number of estimated parameters, n is the sample size, and $l(\hat{\gamma})$ is the log-likelihood value. A lower value in eq. (11) and eq. (12) signifies a more parsimonious model that achieves an optimal balance between goodness-of-fit and mathematical simplicity.

3. Results and Discussion

The discussion of the results of this study involves three datasets, namely (1) simulation data, (2) heart transplant data, and (3) health indicator data in Central Kalimantan and South Kalimantan. All data were estimated using Weibull and generalized modified Weibull (GM Weibull).

3.1. Simulation Study

The simulation was employed by generating three datasets containing single variables with Weibull distributions. The num-

Table 3. Summary of estimated parameters of simulation variables

Parameters	$N = 50$ obs.		$N = 200$ obs.		$N = 500$ obs.	
	Weibull	GMW	Weibull	GMW	Weibull	GMW
Shape (θ)	1.0048	1.0186	1.1070	1.0641	1.1061	1.0770
Scale 1 (λ)	4.5225	4.6728	4.7537	5.1092	5.2541	5.6794
Scale 2 (ϕ)	–	1.0305	–	1.0416	–	1.0430
Time-related (τ)	–	7.98×10^{-9}	–	9.52×10^{-8}	–	6.42×10^{-8}
Log-likelihood	-125.3323	-125.2426	-502.2366	-502.1561	-1306.679	-1306.686
Deviance	250.6646	250.4852	1004.473	1004.312	2613.357	2613.372
AIC	254.6646	258.4852	1008.473	1012.312	2617.357	2621.372
BIC	258.4886	266.1331	1015.070	1025.505	2625.786	2638.230

ber of observations (N) included 50, 200, and 500 observations. We conducted the simulation with the theoretical value of the shape parameter (θ) set to 1 and the scale parameter (λ) set to 5. Such configuration was selected as a fundamental consistency test. Since $\theta = 1$ reduces the model to an exponential distribution, it allows us to evaluate the GM Weibull's robustness in identifying a sub-case model. The histogram of the simulated data is shown in Figure 1.

Figure 1 shows three panels representing each dataset generated with a Weibull distribution, where the black dashed line represents the theoretical density of the Weibull distribution. Figure 1 indicates that the probability density is monotonically decreasing. This reflects a constant failure rate (random failure) and characterizes the exponential distribution properties as a sub-case of the Weibull and GM Weibull when the shape parameter (θ) is unity. As the sample size increases, the empirical histograms exhibit asymptotic convergence toward the theoretical density.

The parameter estimation of the 3 (three) dataset scenarios was continued by referring to the MLE method on the Weibull probability density function and GM Weibull. Table 3 shows the estimation results produced by BFGS algorithm.

Table 3 shows that the estimated Weibull shape parameter (θ) generated in the three datasets' scenarios ranges from 1.0048 to 1.1070. The Weibull scale parameter (λ) ranges from 4.5225 to 5.2541. This indicates that the empirical estimates generated are not much different from the theoretical parameters. Table 3 also presents the parameter estimates for the generalized modified Weibull distribution, which has two additional parameters. The estimated shape parameter (θ) ranges from 1.0186 to 1.0770, and the estimated scale parameter (λ) ranges from 4.6728 to 5.6794. The GM Weibull distribution estimates yield values close to the theoretical parameters. The two parameters, ϕ and τ, yielded estimates close to 1 and 0, respectively. This also reflects their similarity to the theoretical parameters.

Based on the log-likelihood and deviance indicators, the parameter estimates generated by GM Weibull are better than those of regular Weibull, especially in the case of small ($N = 50$) and medium ($N = 200$) sized data sets. This is indicated by the log-likelihood value being closer to 0 and the lower deviance. Meanwhile, in the case of large sample sizes, for example $N = 500$, the parameter estimates of GM Weibull are not better than those of regular Weibull.

The finding in this study that GM Weibull is more representative of the state of empirical data is in line with existing literature showing that generalized or modified Weibull often outperform Weibull distribution in small to moderate samples by accommodating greater variability in shape [4, 13, 28]. However, as sample size increases, the advantage of GM Weibull diminishes, and simpler models converge to similar performance due to the consistency of maximum likelihood estimators. As shown in Table 3, while the GM Weibull achieves a lower deviance in the $N = 50$ and $N = 200$ scenarios, the AIC and BIC favor the standard Weibull. This suggests that for datasets with strictly monotonic characteristics, the ϕ and τ in the GM Weibull do not provide a statistically significant improvement over the parsimonious Weibull model.

3.2. *Study on Heart Transplant Patient Data*

This sub-section focused on 45 heart transplant patients, with the variable examined being patient survival time after receiving a transplant. The mean survival time was 253.9 days, with a median of 90 days. The shortest and longest survival times were recorded at 5 days and 1,386 days, respectively. Table 4 summarizes the results of parameter estimates for patient survival time.

Table 4. Summary of estimated parameters of survival time

Parameter	Weibull	GM Weibull
Shape (θ)	0.8252	0.7986
Scale 1 (λ)	2.2578	1.7409
Scale 2 (ϕ)	-	1.2491
Time-related (τ)	-	4.5053e-08
Log-likelihood	-85.2902	-85.1083
Deviance	170.5803	170.2166
AIC	174.5803	178.2166
BIC	178.1936	185.4432

Based on Table 4, the parameter estimates generated by Weibull and GM Weibull are relatively similar in terms of magnitude, particularly for parameters θ and λ. Since both θ values are less than 1.0, both models agree that the failure rate starts high and decreases as time progresses (monotonically decreasing). Meanwhile, the additional parameter estimates in GM Weibull, namely ϕ and τ are 1.2491 and 4.5053e-08, respectively.

While the GM Weibull distribution achieves a lower deviance (170.2166) compared to the standard Weibull (170.5803) as seen in Table 4, the AIC and BIC values both favor the standard Weibull model (174.58 and 178.19, respectively). For data that follows a simple monotonic trend, the additional flexibility provided by the parameters ϕ and τ in the GM Weibull model does not provide enough improvement to overcome the penalty for model complexity. However, the GM Weibull remains a ro-

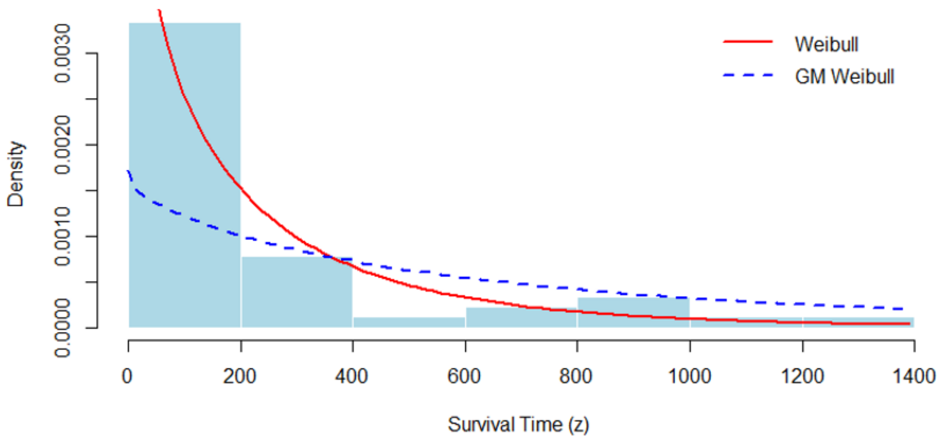

Figure 2. Histogram and probability density curve of the first real variable

bust alternative as it effectively reduces to the Weibull behavior when these parameters are near their identity values, ensuring that the model can handle both simple and complex failure patterns within a single framework [13, 28].

Figure 2 presents the histograms and probability density curves of both distributions based on the obtained parameter estimates. Figure 2 shows that both curves have a unique characteristic: the Weibull curve tends to accurately accommodate short-term patient frequency, while the GM Weibull curve tends to accurately accommodate relatively long-term patient durations, such as 800–1000 days. This reflects the findings that GM Weibull's additional parameters enhance its flexibility in modeling tail behavior and extreme values, making it more suitable for capturing both the central tendency and the longer time-to-event that regular Weibull may underestimate [4, 13].

3.3. *Study on Population with Medical Complaints and Choosing Self-Medication*

This sub-section focused on 27 districts/cities in Central Kalimantan and South Kalimantan provinces. The interest was the percentage of people with medical complaints who did not seek outpatient care due to self-medication. The average percentage was 70.34%, with a median of 67.74%. The lowest and the highest rates were 49.5% (Kota Baru Regency) and 88.79% (Balangan Regency), respectively. Parameter estimates for the percentage of people with medical complaints who did not seek outpatient care due to self-medication are shown in Table 5.

Table 5. Summary of estimated parameters of percentage of population with medical complaints and choosing self-medication

Parameters	Weibull	GM Weibull
Shape (θ)	7.5659	0.1135
Scale 1 (λ)	74.8816	5.7389
Scale 2 (ϕ)	-	14.0107
Time-related (τ)	-	0.0338
Log-likelihood	-102.0043	-101.7841
Deviance	204.0085	203.5683
AIC	208.0085	211.5683
BIC	210.6001	216.7515

Unlike the first real dataset, the estimated parameters by Weibull and GM Weibull in Table 5 show that neither the shape parameter (θ) nor the scale parameter (λ) have similar values. The estimated values for ϕ and τ in GM Weibull are 14.0107 and 0.0338, respectively. The standard Weibull model yielded an increasing failure rate, indicated by a θ value exceeding 1, whereas the GM Weibull suggested a bathtub-shaped failure rate, as evidenced by its τ value.

While the GM Weibull achieves a lower deviance (203.5683), the AIC and BIC favor the standard Weibull model. This is largely attributed to the small sample size ($n = 27$), where the mathematical penalty for estimating four parameters is substantial. However, the standard Weibull characterizes an increasing failure rate ($\theta > 1$), whereas the GM Weibull identifies a bathtub-shaped pattern via the τ parameter (0.0338). This suggests that while the Weibull is more parsimonious, the GM Weibull could be more representative by empirical values, capturing structural fluctuations in the population that a two-parameter model is theoretically incapable of seeing [4, 13].

Figure 3 displays the histograms and probability density curves of both distributions based on the obtained parameter estimates. Figure 3 indicates a data distribution that resembles normality, yet is asymmetrical. Each curve has its own characteristics: the Weibull curve tends to accurately accommodate percentages in the left and right tails, while the GM Weibull curve tends to accurately accommodate percentages in the middle zone. This finding is relevant to the findings that model performance varies depending on the specific regions of the distribution being examined, with GM Weibull excelling at capturing central tendency and modal behavior while Weibull demonstrates superior performance in modeling extreme values and tail probabilities [4, 13, 28].

4. Conclusion

This study demonstrates that the generalized modified Weibull (GMW) distribution provides a robust parameter estimation across both simulated and real-world contexts. Through simulations, the GMW confirmed its stability by correctly converging to theoretical values in small and medium sample scenarios, while the standard Weibull maintained comparable performance in larger samples. In the application context, GM Weibull is better at addressing the survival of transplant patients, es-

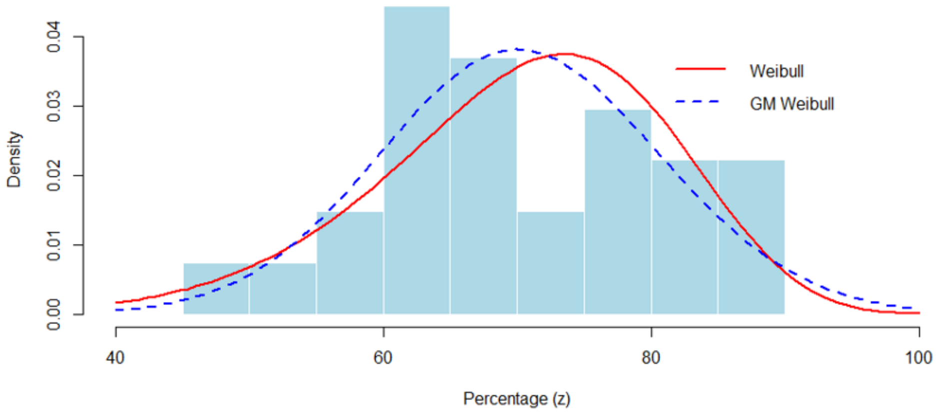

Figure 3. Histogram and probability density curve of the second real variable

pecially for longer survival periods (800–1000 days), while the Weibull distribution shows equality with GM Weibull for short survival. The application to bounded percentage data confirmed its ability to model asymmetric distributions and identify non-monotonic, bathtub-shaped failure rates. Although the addition of the parameters ϕ and τ introduces a mathematical penalty in information criteria like AIC and BIC, this trade-off is justified by the model's increased flexibility in accommodating the structural fluctuations and complex failure patterns inherent in real-world data. Further studies may consider GM Weibull regression modeling with several estimation methods, for example Markov Chain Monte Carlo (MCMC). Since the observation unit of the second dataset is area/region, instead of time, then it will be challenging task being left for us to develop the spatial or spatio-temporal framework based on GM Weibull in future research.

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