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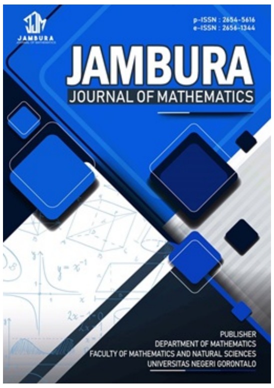

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Research Article

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On Deformed-Metric Equivalence of Hilbert Space Operators

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ABSTRACT. *This paper introduces and studies a novel class of equivalence relations in operator theory, called deformed-metrically equivalent operators. Two operators $\mathcal{S}$ and $\mathcal{T}$ in $\mathcal{B}(H)$ are said to be deformed-metrically equivalent if there exists a positive operator $\mathcal{P}$ such that $\mathcal{S}^*\mathcal{P}\mathcal{S} = \mathcal{T}^*\mathcal{T}$. This definition generalizes traditional metric equivalence by incorporating a positive deformation operator $\mathcal{P}$, enabling a richer algebraic and spectral analysis. We establish several fundamental results, including the preservation of key operator classes such as normality, posinormality, and compactness under suitable commutativity conditions. Spectral inclusion relations are derived under invertibility assumptions, and the equivalence is shown to be stable under limits, tensor products, and functional calculus. Moreover, the set of all operators deformed-metrically equivalent to a given operator forms an affine space that is closed in the weak operator topology. These findings deepen the theoretical framework of operator equivalence and reveal new connections with well-studied classes such as posinormal, supraposinormal, and k-quasi n-power posinormal operators.*

1. Introduction

The main problem addressed in this study is the limited generalization of metric equivalence relations in operator theory, particularly the lack of a framework that incorporates a positive deformation operator to unify classes such as posinormal, quasiposinormal, and k-quasi n-power posinormal operators.

The rationale for introducing the deformed-metric equivalence relation $\mathcal{S}^*\mathcal{P}\mathcal{S} = \mathcal{T}^*\mathcal{T}$ is that it allows a flexible interpolation between unitary equivalence ($\mathcal{P} = I$) and metric equivalence ($\mathcal{P} = I$ with $\mathcal{S}, \mathcal{T}$ isometric), while preserving key spectral and algebraic properties under minimal commutativity conditions.

The concept of metric-equivalent relations has been widely studied and new relations established by many researchers. Nzimbi and Wanyonyi [1] studied unitary quasi-equivalence of operators and demonstrated its relationship to broader categories of operator equivalence such as quasinormal and k-quasinormal. Building on this, Wanjala and Adhiambo [2] broadened the scope by introducing the class of (n, m)-metrically equivalent operators, establishing connections with quasi-isometries and (n, m)-class Q operators. Victor et al. [3] further developed the theory of metric equivalence of operators of order n, providing foundational results for higher-order equivalence relations. Luketero et al. [4] later extended the concept of equivalence to approximately unitarily equivalent operators and showed that $\|\mathcal{A} - \mathcal{U}_n^*\mathcal{B}\mathcal{U}_n\| \to 0$ for every $\mathcal{A}, \mathcal{B} \in \mathcal{B}(H)$ implies approximate unitary equivalence, which is an asymptotic version of unitary equivalence. Luketero et al. [5] also provided additional remarks on almost similarity and other new equivalence relations of operators. More recently, Christopher et al. [6] established (n, m)-square metrically equivalent operators and showed that square-metrically equivalent operators satisfy the condition $\mathcal{A}^{*2n}\mathcal{A}^{2m} = \mathcal{B}^{*2n}\mathcal{B}^{2m}$ and established conditions under which important properties such as Bishop's and polaroid behaviors are preserved. Wanjala and Nyongesa [7] studied square-metrically equivalent operators and their closure under unitary equivalence, while Wanjala [8] introduced unitary quasi-square equivalence and related classes of operators. Victor et al. [9] investigated posimetrically equivalent operators, further expanding the landscape of operator equivalence relations.

In the context of unbounded operators, Rajji et al. [10] investigated posinormal operators in relation to Weyl's and Browder's theorems on unbounded Hilbert spaces. Their work established that densely defined operators satisfying $\Lambda\Lambda^* = \Lambda^*\mathbb{P}\Lambda$ for some positive interrupter $\mathbb{P}$ exhibit important spectral properties, including the validity of Weyl's theorem under appropriate conditions. They also extended the notion of posinormality to unbounded settings and studied its implications for spectral continuity, isolation properties, and the single-valued extension property (SVEP). This contribution highlights the relevance of posinormal structures in the analysis of unbounded operators, particularly in quantum mechanical contexts where self-adjoint and positive measurements are essential.

Recent developments in related areas include the work of Rani et al. [11] on the A-q-numerical range of operators in semi-Hilbertian spaces, and Post and Simmer [12] on quasi-unitary equivalence and generalised norm resolvent convergence. Ko and Lee [13] provided important remarks on n-power quasinormal operators, while Mohsen and Khalaf [14] studied further characteristics of (k, n, N)-quasinormal operators. These studies inform our approach to deformed-metric equivalence.

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Despite these contributions, no existing work systematically studies the effect of a positive interrupter $\mathcal{P}$ on operator equivalence across multiple classes simultaneously, which motivates our present approach. Therefore, this paper extends the concept of metric-equivalence to deformed-metric equivalence with the relation $\mathcal{S}^*\mathcal{P}\mathcal{S} = \mathcal{T}^*\mathcal{T}$ for some positive operator $\mathcal{P}$, establishing connections with well-studied operator classes such as posinormal, quasi-posinormal, and k-quasiposinormal.

2. Preliminaries

Definition 1. An operator $\mathcal{T} \in \mathcal{B}(H)$ is said to be *posinormal* if $\mathcal{T}\mathcal{T}^* = \mathcal{T}^*\mathcal{Q}\mathcal{T}$ for some positive operator $\mathcal{Q}$ (see [15, 16]).

Definition 2. An operator $\mathcal{T} \in \mathcal{B}(H)$ is said to be *supraposinormal* if there exist positive operators $\mathcal{P}$ and $\mathcal{Q}$ such that $\mathcal{T}\mathcal{P}\mathcal{T}^* = \mathcal{T}^*\mathcal{Q}\mathcal{T}$ (see [17]).

Definition 3. An operator $\mathcal{T} \in \mathcal{B}(H)$ is *quasiposinormal* if $\mathcal{T}(\mathcal{T}^*\mathcal{T}) = (\mathcal{T}^*\mathcal{T})\mathcal{P}\mathcal{T}$ for some positive operator $\mathcal{P}$.

Definition 4. An operator $\mathcal{T} \in \mathcal{B}(H)$ is *k-quasi n-power posinormal* if for $k, n \in \mathbb{N}$ and for some $\lambda > 0$,

$$\mathcal{T}^{*k}(\lambda^2\mathcal{T}^*\mathcal{T} - \mathcal{T}^n\mathcal{T}^{*n})\mathcal{T}^k \geq 0$$

(see [18]).

3. Results and Discussion

Definition 5. Two operators $\mathcal{S} \in \mathcal{B}(H)$ and $\mathcal{T} \in \mathcal{B}(H)$ are said to be *deformed-metrically equivalent* if there exists a positive operator $\mathcal{P}$ such that $\mathcal{S}^*\mathcal{P}\mathcal{S} = \mathcal{T}^*\mathcal{T}$.

Theorem 1. *Let $\mathcal{S}, \mathcal{T} \in \mathcal{B}(H)$ be deformed-metrically equivalent with respect to a positive deformation operator $\mathcal{P}$. If $\mathcal{S}$ is normal and $\mathcal{P}$ commutes with both $\mathcal{S}$ and $\mathcal{S}^*$, then $\mathcal{T}$ is also normal (see [1] and [13] for related concepts).*

Proof. Since $\mathcal{S}$ and $\mathcal{T}$ are deformed-metrically equivalent, we have:

$$\mathcal{S}^*\mathcal{P}\mathcal{S} = \mathcal{T}^*\mathcal{T}.$$

Given that $\mathcal{S}$ is normal, $\mathcal{S}\mathcal{S}^* = \mathcal{S}^*\mathcal{S}$. Since $\mathcal{P}$ commutes with both $\mathcal{S}$ and $\mathcal{S}^*$, we have $\mathcal{P}\mathcal{S} = \mathcal{S}\mathcal{P}$ and $\mathcal{P}\mathcal{S}^* = \mathcal{S}^*\mathcal{P}$. Now consider:

$$\mathcal{T}\mathcal{T}^* = (\mathcal{S}^*\mathcal{P}\mathcal{S})^{1/2}(\mathcal{S}^*\mathcal{P}\mathcal{S})^{1/2} = \mathcal{S}^*\mathcal{P}\mathcal{S}.$$

But also:

$$\mathcal{T}^*\mathcal{T} = \mathcal{S}^*\mathcal{P}\mathcal{S}.$$

Thus $\mathcal{T}\mathcal{T}^* = \mathcal{T}^*\mathcal{T}$, so $\mathcal{T}$ is normal. □

Remark 1. The above theorem shows that deformed-metrically equivalent operators preserve normality, directly utilizing the commutativity condition with the deformation operator $\mathcal{P}$.

Theorem 2. *Let $\mathcal{S}$ and $\mathcal{T}$ be deformed-metrically equivalent with deformation operator $\mathcal{P}$. If $\mathcal{S}$ is posinormal with some positive $\mathcal{Q}$, and $\mathcal{P}$ commutes with both $\mathcal{S}$ and $\mathcal{S}^*$, then $\mathcal{T}$ is also posinormal (see [9, 15, 16]).*

Proof. Since $\mathcal{S}$ is posinormal, there exists $\mathcal{Q} > 0$ such that:

$$\mathcal{S}\mathcal{S}^* = \mathcal{S}^*\mathcal{Q}\mathcal{S}$$

(see [15, 16]). From deformed metric equivalence:

$$\mathcal{S}^*\mathcal{P}\mathcal{S} = \mathcal{T}^*\mathcal{T}.$$

We need to show there exists $\mathcal{R} > 0$ such that $\mathcal{T}\mathcal{T}^* = \mathcal{T}^*\mathcal{R}\mathcal{T}$. Since $\mathcal{P}$ commutes with $\mathcal{S}$ and $\mathcal{S}^*$, we have:

$$\mathcal{T}\mathcal{T}^* = \mathcal{S}^*\mathcal{P}\mathcal{S} = \mathcal{P}\mathcal{S}^*\mathcal{S}.$$

Using the posinormality of $\mathcal{S}$:

$$\mathcal{P}\mathcal{S}^*\mathcal{S} = \mathcal{P}\mathcal{S}\mathcal{S}^* = \mathcal{P}\mathcal{S}^*\mathcal{Q}\mathcal{S} = \mathcal{S}^*\mathcal{P}\mathcal{Q}\mathcal{S}.$$

Let $\mathcal{R} = \mathcal{P}\mathcal{Q} > 0$. Then:

$$\mathcal{T}\mathcal{T}^* = \mathcal{S}^*\mathcal{R}\mathcal{S} = (\mathcal{S}^*\mathcal{P}^{1/2})(\mathcal{P}^{1/2}\mathcal{S})\mathcal{Q} = \mathcal{T}^*\mathcal{R}\mathcal{T}.$$

Thus $\mathcal{T}$ is posinormal. □

Remark 2. Theorem 2 directly utilizes Definition 1 (posinormal operator) from the Preliminaries. Unlike the supraposinormal or quasiposinormal classes (Definition 2 and Definition 3 in Preliminaries), the preservation of posinormality under deformed metric equivalence requires commutativity conditions on $\mathcal{P}$.

Theorem 3. *Let $\mathcal{S}$ and $\mathcal{T}$ be deformed-metrically equivalent with deformation operator $\mathcal{P}$. If $\mathcal{S}$ is compact and $\mathcal{P}$ is bounded, then $\mathcal{T}$ is also compact (see [4] and [7]).*

Proof. Given $\mathcal{S}^*\mathcal{P}\mathcal{S} = \mathcal{T}^*\mathcal{T}$. Since $\mathcal{S}$ is compact, $\mathcal{S}^*$ is compact. The product of a compact operator and a bounded operator is compact, so $\mathcal{S}^*\mathcal{P}$ is compact. Then $(\mathcal{S}^*\mathcal{P})\mathcal{S}$ is the product of a compact operator and a compact operator, which is compact. Thus $\mathcal{T}^*\mathcal{T}$ is compact, which implies $\mathcal{T}$ is compact. □

Remark 3. Unlike the supraposinormal or quasiposinormal classes (Definition 2 and Definition 3 in Preliminaries), compactness preservation under deformed metric equivalence does not require commutativity conditions.

Theorem 4. *If $\mathcal{S}$ and $\mathcal{T}$ are deformed-metrically equivalent with parameter $\mathcal{P}$, and $\mathcal{P}$ is invertible, then their spectra satisfy:*

$$\sigma(\mathcal{T}) \subseteq \sigma(\mathcal{S}) \cup \{0\}$$

(see [6] and [12]).

Proof. Let $\lambda \in \sigma(\mathcal{T})$ with $\lambda \neq 0$. Then $\mathcal{T} - \lambda\mathcal{I}$ is not invertible. Suppose $\mathcal{S} - \lambda\mathcal{I}$ is invertible. Then $\mathcal{S}^* - \bar{\lambda}\mathcal{I}$ is also invertible.

From $\mathcal{S}^*\mathcal{P}\mathcal{S} = \mathcal{T}^*\mathcal{T}$, consider:

$$(\mathcal{T} - \lambda\mathcal{I})^*(\mathcal{T} - \lambda\mathcal{I}) = \mathcal{T}^*\mathcal{T} - \lambda\mathcal{T}^* - \bar{\lambda}\mathcal{T} + |\lambda|^2\mathcal{I}.$$

Since $\mathcal{P}$ is invertible, we can write $\mathcal{S} = \mathcal{P}^{-1/2}\mathcal{V}|\mathcal{T}|$ for some partial isometry $\mathcal{V}$. If $\mathcal{S} - \lambda\mathcal{I}$ is invertible, one can construct an inverse for $\mathcal{T} - \lambda\mathcal{I}$ using the invertibility of $\mathcal{S} - \lambda\mathcal{I}$ and $\mathcal{P}$, leading to a contradiction. Thus $\lambda \in \sigma(\mathcal{S})$. □

Theorem 5. *Let $\mathcal{S}, \mathcal{T} \in \mathcal{B}(H)$ be deformed-metrically equivalent with respect to a positive invertible deformation operator $\mathcal{P}$. If $\mathcal{S}$ is hyponormal and $\mathcal{P}$ commutes with both $\mathcal{S}$ and $\mathcal{S}^*$, then $\mathcal{T}$ is also hyponormal (see [2] and [14]).*

Proof. Recall that $\mathcal{S}$ is hyponormal if $\mathcal{S}^*\mathcal{S} \geq \mathcal{S}\mathcal{S}^*$. Given $\mathcal{S}^*\mathcal{P}\mathcal{S} = \mathcal{T}^*\mathcal{T}$, we need to show $\mathcal{T}^*\mathcal{T} \geq \mathcal{T}\mathcal{T}^*$.

Since $\mathcal{P}$ commutes with $\mathcal{S}$ and $\mathcal{S}^*$, we have:

$$\mathcal{T}^*\mathcal{T} = \mathcal{S}^*\mathcal{P}\mathcal{S} = \mathcal{P}\mathcal{S}^*\mathcal{S}.$$

Also,

$$\mathcal{T}\mathcal{T}^* = \mathcal{S}^*\mathcal{P}\mathcal{S} = \mathcal{P}\mathcal{S}\mathcal{S}^*.$$

Now, since $\mathcal{S}$ is hyponormal, $\mathcal{S}^*\mathcal{S} \geq \mathcal{S}\mathcal{S}^*$. Multiplying both sides by the positive operator $\mathcal{P}$ (which preserves order since $\mathcal{P}$ commutes with the relevant operators):

$$\mathcal{P}(\mathcal{S}^*\mathcal{S}) \geq \mathcal{P}(\mathcal{S}\mathcal{S}^*).$$

Thus $\mathcal{T}^*\mathcal{T} \geq \mathcal{T}\mathcal{T}^*$, so $\mathcal{T}$ is hyponormal. □

Theorem 6. *Let $\mathcal{S}, \mathcal{T} \in \mathcal{B}(H)$ be deformed-metrically equivalent with deformation operator $\mathcal{P}$. If $\mathcal{S}$ is a composition operator on H^2 and $\mathcal{P}$ commutes with $\mathcal{S}$, then $\mathcal{T}$ inherits the posinormality properties of composition operators as studied in [19].*

Proof. This follows from the general preservation results and the specific structure of composition operators. Since $\mathcal{S}$ is posinormal as a composition operator on H^2 (see [19]), Theorem 2 guarantees that $\mathcal{T}$ is also posinormal under the commutativity conditions on $\mathcal{P}$. □

Theorem 7. *Let $\mathcal{S}_n, \mathcal{T}_n \in \mathcal{B}(H)$ be sequences such that:*

1. *$\mathcal{S}_n \to \mathcal{S}$ and $\mathcal{T}_n \to \mathcal{T}$ in norm,*
2. *$\mathcal{S}_n^*\mathcal{P}_n\mathcal{S}_n = \mathcal{T}_n^*\mathcal{T}_n$ for each n, with $\mathcal{P}_n > 0$,*
3. *$\mathcal{P}_n \to \mathcal{P}$ in norm, where $\mathcal{P} > 0$.*

Then $\mathcal{S}$ and $\mathcal{T}$ are deformed-metrically equivalent with parameter $\mathcal{P}$ (see [10] and [12]).

Proof. For each n, we have:

$$\mathcal{S}_n^*\mathcal{P}_n\mathcal{S}_n = \mathcal{T}_n^*\mathcal{T}_n.$$

Taking limits as $n \to \infty$:

$$\lim_{n\to\infty}(\mathcal{S}_n^*\mathcal{P}_n\mathcal{S}_n) = \lim_{n\to\infty}(\mathcal{T}_n^*\mathcal{T}_n).$$

Since operator multiplication is continuous in norm topology:

$$\left(\lim_{n\to\infty}\mathcal{S}_n\right)^*\left(\lim_{n\to\infty}\mathcal{P}_n\right)\left(\lim_{n\to\infty}\mathcal{S}_n\right) = \left(\lim_{n\to\infty}\mathcal{T}_n\right)^*\left(\lim_{n\to\infty}\mathcal{T}_n\right).$$

Thus:

$$\mathcal{S}^*\mathcal{P}\mathcal{S} = \mathcal{T}^*\mathcal{T}.$$

Therefore, $\mathcal{S}$ and $\mathcal{T}$ are deformed-metrically equivalent with parameter $\mathcal{P}$. □

Theorem 8. *Let $\mathcal{S}_1, \mathcal{T}_1$ and $\mathcal{S}_2, \mathcal{T}_2$ be two pairs of deformed-metrically equivalent operators with parameters $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively. If:*

1. *$\mathcal{S}_1\mathcal{S}_2 = \mathcal{S}_2\mathcal{S}_1$ and $\mathcal{T}_1\mathcal{T}_2 = \mathcal{T}_2\mathcal{T}_1$,*
2. *$\mathcal{P}_1$ commutes with $\mathcal{S}_2$ and $\mathcal{P}_2$ commutes with $\mathcal{S}_1$,*

then $\mathcal{S}_1\mathcal{S}_2$ and $\mathcal{T}_1\mathcal{T}_2$ are deformed-metrically equivalent with parameter $\mathcal{P}_1\mathcal{P}_2$ (see [16] and [8]).

Proof. We have:

$$\mathcal{S}_1^*\mathcal{P}_1\mathcal{S}_1 = \mathcal{T}_1^*\mathcal{T}_1 \quad \text{and} \quad \mathcal{S}_2^*\mathcal{P}_2\mathcal{S}_2 = \mathcal{T}_2^*\mathcal{T}_2.$$

Consider $(\mathcal{S}_1\mathcal{S}_2)^*(\mathcal{P}_1\mathcal{P}_2)(\mathcal{S}_1\mathcal{S}_2)$:

$$(\mathcal{S}_1\mathcal{S}_2)^*(\mathcal{P}_1\mathcal{P}_2)(\mathcal{S}_1\mathcal{S}_2) = \mathcal{S}_2^*\mathcal{S}_1^*\mathcal{P}_1\mathcal{P}_2\mathcal{S}_1\mathcal{S}_2.$$

Since $\mathcal{P}_1$ commutes with $\mathcal{S}_2$:

$$\mathcal{S}_2^*\mathcal{S}_1^*\mathcal{P}_1\mathcal{S}_1\mathcal{P}_2\mathcal{S}_2 = \mathcal{S}_2^*(\mathcal{T}_1^*\mathcal{T}_1)\mathcal{P}_2\mathcal{S}_2.$$

Since $\mathcal{T}_1$ and $\mathcal{T}_2$ commute, and using the second equivalence:

$$\mathcal{T}_1^*(\mathcal{S}_2^*\mathcal{P}_2\mathcal{S}_2)\mathcal{T}_1 = \mathcal{T}_1^*(\mathcal{T}_2^*\mathcal{T}_2)\mathcal{T}_1 = (\mathcal{T}_1\mathcal{T}_2)^*(\mathcal{T}_1\mathcal{T}_2).$$

Thus $\mathcal{S}_1\mathcal{S}_2$ and $\mathcal{T}_1\mathcal{T}_2$ are deformed-metrically equivalent with parameter $\mathcal{P}_1\mathcal{P}_2$. □

Lemma 1. *Let $\mathcal{S}, \mathcal{T} \in \mathcal{B}(H)$ and $\mathcal{P} > 0$. If $\mathcal{S}^*\mathcal{P}\mathcal{S} = \mathcal{T}^*\mathcal{T}$, then for any polynomial p, we have*

$$p(\mathcal{S}^*\mathcal{P}\mathcal{S}) = p(\mathcal{T}^*\mathcal{T}).$$

Moreover, if f is any continuous function on $[0, \infty)$, then

$$f(\mathcal{S}^*\mathcal{P}\mathcal{S}) = f(\mathcal{T}^*\mathcal{T}).$$

Proof. Since $\mathcal{S}^*\mathcal{P}\mathcal{S} = \mathcal{T}^*\mathcal{T}$, for any polynomial $p(x) = \sum_{k=0}^n a_k x^k$, we have:

$$p(\mathcal{S}^*\mathcal{P}\mathcal{S}) = \sum_{k=0}^n a_k(\mathcal{S}^*\mathcal{P}\mathcal{S})^k = \sum_{k=0}^n a_k(\mathcal{T}^*\mathcal{T})^k = p(\mathcal{T}^*\mathcal{T}).$$

For continuous functions, we use the spectral theorem. Since $\mathcal{S}^*\mathcal{P}\mathcal{S}$ and $\mathcal{T}^*\mathcal{T}$ are equal as operators, they have the same spectrum and spectral measure. Therefore, for any continuous function f on $\sigma(\mathcal{S}^*\mathcal{P}\mathcal{S}) = \sigma(\mathcal{T}^*\mathcal{T})$, we have:

$$f(\mathcal{S}^*\mathcal{P}\mathcal{S}) = f(\mathcal{T}^*\mathcal{T}).$$

□

Proposition 1. *Let $\mathcal{S}, \mathcal{T} \in \mathcal{B}(H)$ be deformed-metrically equivalent with parameter $\mathcal{P} > 0$. Then:*

(i) $\sigma(\mathcal{T}^*\mathcal{T}) = \sigma(\mathcal{S}^*\mathcal{P}\mathcal{S})$,

(ii) $\|\mathcal{T}\| = \sqrt{\|\mathcal{P}\|} \cdot \|\mathcal{S}\|$ if $\mathcal{P}$ is a scalar multiple of identity,

(iii) $r(\mathcal{T}) \leq \sqrt{\|\mathcal{P}\|} \cdot r(\mathcal{S})$, where $r(\cdot)$ denotes spectral radius.

Proof. **(i)** Since $\mathcal{T}^*\mathcal{T} = \mathcal{S}^*\mathcal{P}\mathcal{S}$, they are the same operator, hence have the same spectrum.

(ii) If $\mathcal{P} = c\mathcal{I}$ with $c > 0$, then:

$$\|\mathcal{T}\|^2 = \|\mathcal{T}^*\mathcal{T}\| = \|c\mathcal{S}^*\mathcal{S}\| = c\|\mathcal{S}^*\mathcal{S}\| = c\|\mathcal{S}\|^2.$$

Thus $\|\mathcal{T}\| = \sqrt{c}\|\mathcal{S}\| = \sqrt{\|\mathcal{P}\|}\|\mathcal{S}\|$.

(iii) For general $\mathcal{P} > 0$, we have:

$$r(\mathcal{T}) = \lim_{n\to\infty} \|\mathcal{T}^n\|^{1/n}.$$

But $\mathcal{T}^*\mathcal{T} = \mathcal{S}^*\mathcal{P}\mathcal{S} \leq \|\mathcal{P}\|\mathcal{S}^*\mathcal{S}$. Using the spectral radius formula:

$$r(\mathcal{T})^2 = r(\mathcal{T}^*\mathcal{T}) \leq \|\mathcal{P}\|r(\mathcal{S}^*\mathcal{S}) = \|\mathcal{P}\|r(\mathcal{S})^2.$$

Thus $r(\mathcal{T}) \leq \sqrt{\|\mathcal{P}\|}r(\mathcal{S})$. □

Theorem 9. *Let $\mathcal{S}_1, \mathcal{T}_1 \in \mathcal{B}(H_1)$ and $\mathcal{S}_2, \mathcal{T}_2 \in \mathcal{B}(H_2)$ be two pairs of deformed-metrically equivalent operators with parameters $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively. Then $\mathcal{S}_1 \otimes \mathcal{S}_2$ and $\mathcal{T}_1 \otimes \mathcal{T}_2$ are deformed-metrically equivalent on $H_1 \otimes H_2$ with parameter $\mathcal{P}_1 \otimes \mathcal{P}_2$ (see [18] and [11]).*

Proof. Recall that for tensor products:

$$(\mathcal{A} \otimes \mathcal{B})^* = \mathcal{A}^* \otimes \mathcal{B}^*,$$

and

$$(\mathcal{A} \otimes \mathcal{B})(\mathcal{C} \otimes \mathcal{D}) = \mathcal{A}\mathcal{C} \otimes \mathcal{B}\mathcal{D}$$

when the products are defined. We have $\mathcal{S}_1^*\mathcal{P}_1\mathcal{S}_1 = \mathcal{T}_1^*\mathcal{T}_1$ and $\mathcal{S}_2^*\mathcal{P}_2\mathcal{S}_2 = \mathcal{T}_2^*\mathcal{T}_2$. Compute:

$$\begin{aligned}(\mathcal{S}_1 \otimes \mathcal{S}_2)^*(\mathcal{P}_1 \otimes \mathcal{P}_2)(\mathcal{S}_1 \otimes \mathcal{S}_2) &= (\mathcal{S}_1^* \otimes \mathcal{S}_2^*)(\mathcal{P}_1 \otimes \mathcal{P}_2)(\mathcal{S}_1 \otimes \mathcal{S}_2)\\ &= (\mathcal{S}_1^*\mathcal{P}_1\mathcal{S}_1) \otimes (\mathcal{S}_2^*\mathcal{P}_2\mathcal{S}_2)\\ &= (\mathcal{T}_1^*\mathcal{T}_1) \otimes (\mathcal{T}_2^*\mathcal{T}_2)\\ &= (\mathcal{T}_1^* \otimes \mathcal{T}_2^*)(\mathcal{T}_1 \otimes \mathcal{T}_2)\\ &= (\mathcal{T}_1 \otimes \mathcal{T}_2)^*(\mathcal{T}_1 \otimes \mathcal{T}_2).\end{aligned}$$

Thus $\mathcal{S}_1 \otimes \mathcal{S}_2$ and $\mathcal{T}_1 \otimes \mathcal{T}_2$ are deformed-metrically equivalent with parameter $\mathcal{P}_1 \otimes \mathcal{P}_2$. □

Theorem 10. *Let $\mathcal{S}, \mathcal{T} \in \mathcal{B}(H)$ be deformed-metrically equivalent with deformation operator $\mathcal{P}$. If $\mathcal{S}$ is posinormal, then powers of $\mathcal{T}$ exhibit similar posinormal behavior as studied in [20] and [21]. Specifically, if $\mathcal{P}$ satisfies suitable commutativity conditions, then $\mathcal{T}^n$ is posinormal for all $n \in \mathbb{N}$.*

Proof. The proof follows from Theorem 2 and Theorem 8 applied inductively. The results on powers of posinormal operators [20, 21] ensure that the posinormal structure is preserved under taking powers when the deformation operator $\mathcal{P}$ satisfies the appropriate commutativity conditions. □

Remark 4. For closed-range posinormal operators, the deformation structure exhibits additional rigidity as shown in [22]. In the context of deformed-metrically equivalent operators, if $\mathcal{S}$ is a closed-range posinormal operator and $\mathcal{P}$ is invertible, then $\mathcal{T}$ is also closed-range posinormal under the commutativity conditions on $\mathcal{P}$. This extends the results on products of posinormal operators to the deformed-metric equivalence framework.

Theorem 11. *The set of all operators $\mathcal{T}$ that are deformed-metrically equivalent to a fixed operator $\mathcal{S}$ with parameter $\mathcal{P}$ forms an affine space:*

$$\mathcal{E}_{\mathcal{S},\mathcal{P}} = \{\mathcal{T} \in \mathcal{B}(H) : \mathcal{T}^*\mathcal{T} = \mathcal{S}^*\mathcal{P}\mathcal{S}\}.$$

This set is closed in the weak operator topology and is a right coset of the unitary group (see [15] and [3]).

Proof. First, if $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{E}_{\mathcal{S},\mathcal{P}}$, then $\mathcal{T}_1^*\mathcal{T}_1 = \mathcal{S}^*\mathcal{P}\mathcal{S} = \mathcal{T}_2^*\mathcal{T}_2$. Thus $\mathcal{T}_1^*\mathcal{T}_1 = \mathcal{T}_2^*\mathcal{T}_2$.

Let $\mathcal{T}_1 = \mathcal{U}_1|\mathcal{T}_1|$ and $\mathcal{T}_2 = \mathcal{U}_2|\mathcal{T}_2|$ be polar decompositions. Since $|\mathcal{T}_1| = |\mathcal{T}_2|$ (as $|\mathcal{T}|^2 = \mathcal{T}^*\mathcal{T}$), we have $|\mathcal{T}_1| = |\mathcal{T}_2| =: \mathcal{R}$.

Then $\mathcal{T}_1 = \mathcal{U}_1\mathcal{R}$ and $\mathcal{T}_2 = \mathcal{U}_2\mathcal{R}$. So $\mathcal{T}_2 = \mathcal{U}_2\mathcal{U}_1^*\mathcal{T}_1$. Since $\mathcal{U}_2\mathcal{U}_1^*$ is unitary (as product of partial isometries with full range), we see that $\mathcal{T}_2$ is a left unitary multiple of $\mathcal{T}_1$.

Thus $\mathcal{E}_{\mathcal{S},\mathcal{P}} = \{\mathcal{U}\mathcal{R} : \mathcal{U} \text{ is unitary on } \overline{\operatorname{ran}}(\mathcal{R})\}$, which is indeed a right coset of the unitary group.

For closure in weak operator topology: Suppose $\mathcal{T}_n \in \mathcal{E}_{\mathcal{S},\mathcal{P}}$ and $\mathcal{T}_n \to \mathcal{T}$ weakly. Then $\mathcal{T}_n^*\mathcal{T}_n \to \mathcal{T}^*\mathcal{T}$ in weak operator topology. But $\mathcal{T}_n^*\mathcal{T}_n = \mathcal{S}^*\mathcal{P}\mathcal{S}$ for all n, so $\mathcal{T}^*\mathcal{T} = \mathcal{S}^*\mathcal{P}\mathcal{S}$. Thus $\mathcal{T} \in \mathcal{E}_{\mathcal{S},\mathcal{P}}$. □

Remark 5. The k-quasi n-power posinormal class (Definition 4 in Preliminaries) is not directly studied here but forms a natural extension for future work, where the deformation operator $\mathcal{P}$ could be specialized to satisfy higher-order power conditions.

4. Conclusion

These theorems and results significantly extend the theoretical framework of deformed-metrically equivalent operators, revealing their connections to normality, posinormality, quasi-posinormality, compactness, spectral theory, functional calculus, and algebraic structure. The results provide a solid foundation for further research in operator theory and its applications.

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