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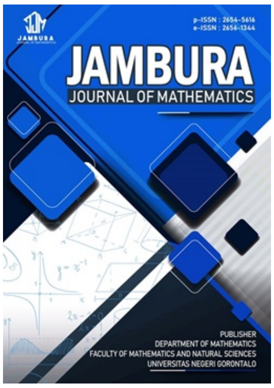

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Research Article

The Expected-Based Method of Value-at-Risk Prediction

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ABSTRACT. *Value-at-Risk (VaR) remains a fundamental risk measure in financial risk management, providing an indicator for managing capital allocation and avoiding worst-case risk scenarios. Traditionally its defined as a quantile of the loss distribution. However, its computation depends critically on the existence and tractability of the inverse cumulative distribution function (CDF), which may not be available in closed form for complex or empirical distributions. This paper proposes an expectation-based simulation framework for VaR estimation that avoids explicit inversion of the CDF. The method approximates VaR by taking the expectation of order statistics from repeated sampling, effectively constructing a variance-reduced Monte Carlo estimator of the quantile. We provide a rigorous theoretical foundation for the proposed approach, including strong consistency, asymptotic normality, and a bias–variance decomposition. In particular, we show that the estimator achieves variance reduction proportional to the number of simulations while remaining consistent with the classical definition of VaR. Furthermore, under heavy-tailed distributions, the method demonstrates enhanced stability compared to traditional historical simulation, which is known to exhibit high tail variability. Extensive simulation studies confirm the theoretical findings, showing significant improvements in mean squared error and backtesting performance. Overall, the proposed framework provides a flexible alternative for VaR estimation in settings where conventional inversion-based methods are infeasible or unreliable.*

1. Introduction

Risk in financial markets can be modeled as a random variable and quantified using an appropriate risk measure. One of the most widely adopted measures is Value-at-Risk (VaR), introduced by J.P. Morgan in 1994 through the RiskMetrics framework [1]. In practice, VaR represents the maximum potential loss over a specified time horizon at a given confidence level under normal market conditions [2]. Its regulatory importance was solidified under the Basel framework, which prescribes a 10-day holding period and a 99% confidence level for market risk assessment [3] [4]. Since then, VaR has become a standard tool in finance and insurance for capital allocation and risk control [5, 6].

Formally, VaR at level $\alpha \in (0,1)$ for a loss random variable L is defined as the α-quantile of its distribution, i.e.,

$$VaR_\alpha(L) = F_L^{-1}(\alpha)$$

where F_L denotes the cumulative distribution function (CDF) of L. This definition places VaR within the class of quantile-based risk measures, which rely critically on the existence and tractability of the inverse CDF [7]. However, in many practical situations—such as empirical distributions, heavy-tailed models, or complex stochastic systems—the inverse CDF is either unavailable in closed form or computationally expensive to obtain. This limitation motivates the search for alternative formulations of VaR that do not depend explicitly on distributional inversion.

A substantial body of literature has explored VaR estimation techniques, including parametric, historical simulation, and Monte Carlo methods [8]. Parametric approaches assume a specific distribution (often normal or heavy-tailed variants), while historical simulation relies on empirical quantiles derived from past observations. Monte Carlo methods offer flexibility but can be computationally intensive [9, 10]. Despite their widespread use, these approaches exhibit notable limitations [11]. For instance, historical simulation may underestimate tail risk in small samples, and parametric models can be misspecified in the presence of non-normality or structural breaks [12, 13]. These shortcomings highlight the need for more robust and flexible VaR estimation methods [14].

Recent developments have extended the quantile-based framework toward expectation-based risk measures [15, 16]. In particular, expectile-based VaR (EVaR) replaces quantiles with expectiles, which are defined through asymmetric least squares and exhibit greater sensitivity to extreme losses [17, 18]. This shift suggests that risk measures can be characterized not only through distributional inversion but also through expectations of transformed loss functions [19, 20]. Building on this idea, we consider whether VaR itself can be reformulated in an expectation-based framework that avoids direct inversion of the CDF [21].

The research hypothesis of this paper is based on the hypothesis that VaR can be consistently approximated through expectation-based functionals of simulated loss distributions and order statistics, without requiring explicit computation of the inverse CDF, while maintaining comparable accuracy to classical

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quantile-based VaR. To formalize this idea, we propose an alternative representation of VaR using expectations over simulated samples. Instead of computing $F_L^{-1}(\alpha)$, we approximate VaR via functionals of order statistics $L_{(k)}$ such that

$$VaR_\alpha(L) \approx \mathbb{E}[L_{(k)}],\ with\ k \approx \alpha.n,$$

or more generally, through expectation operators applied to subsets of simulated losses within a confidence interval. This formulation positions our approach as a bridge between classical quantile-based VaR and expectation-based risk measures such as expectiles.

The contribution of this paper is threefold. First, we develop a simulation-based, expectation-driven framework for VaR estimation that is particularly suitable when the CDF is unknown, discontinuous, or non-invertible. Second, we provide a unified mathematical treatment that connects quantile-based VaR with expectation-based approximations via order statistics. Third, we conduct a comparative analysis between the proposed method and the traditional historical simulation approach across several distributional settings, including normal and generalized Pareto distributions.

This study considers three practical scenarios. In the absence of distributional assumptions, we employ historical simulation based on the empirical CDF. When the distribution is known but analytically intractable, we utilize simulation-based expectation methods. Finally, we extend the framework by expressing VaR as an expectation over order statistics within a specified interval, providing a flexible alternative to direct quantile computation.

The remainder of this paper is organized as follows. Section 2 presents the proposed methodology and theoretical framework. Section 3 reports numerical results and discussion from simulation studies. Section 4 concludes with a discussion of findings and potential extensions.

2. Methods

This section presents a rigorous formulation of the proposed expected-based VaR estimator and clarifies its relationship with the classical quantile definition.

2.1. Standar VaR and Notation

Let L be a non-negative loss random variable defined on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with the cumulative distribution function (CDF) $F_L(.;\theta)$, where θ is a parameter vector.

Definition 1 (Value-at-Risk). [7] For $\alpha \in (0,1)$, the Value-at-Risk at the probability level $(1-\alpha)$ is defined as

$$\text{VaR}_{1-\alpha}(L;\theta) = \inf\{\ell \in \mathbb{R}_+ : F_L(\ell;\theta) \geq 1-\alpha\},$$

or

$$\text{VaR}_{1-\alpha}(L;\theta) = F_L^{-1}(1-\alpha). \tag{1}$$

If F_L^{-1} of a distribution exists.

Throughout this paper, we adopt the notation: true VaR denoted by $VaR_{(1-\alpha)}(L;\theta)$ and estimated VaR denoted by $\widehat{VaR}_{(1-\alpha)}(L;\hat{\theta})$.

2.2. Reformulation via Order Statistics

The main limitation of eq. (1) is its dependence on the inverse CDF. To avoid explicit inversion, we exploit the connection between quantiles and order statistics. Let

$$L_1, L_2, ..., L_n \sim iid\ F_L(\cdot;\theta)$$

and denote the order statistics by $L_{(1)} \leq L_{(2)} \leq ... \leq L_{(n)}$.

Proposition 1 (Order-statistic representation of VaR). *Let $k = \lceil n(1-\alpha) \rceil$. Then*

$$L_{(k)} \xrightarrow{a.s.} \text{VaR}_{1-\alpha}(L;\theta) \quad as\ n \to \infty.$$

This result replaces the inverse CDF problem with a sampling problem, forming the basis of our approach [22].

2.3. Expected-Based VaR Estimator

Instead of computing a single order statistic, we repeat the sampling process. Let m independent simulations be generated

$$L_1^{(j)}, L_2^{(j)}, ..., L_n^{(j)}, j = 1, \ldots, m.$$

For each simulation

$$VaR_{1-\alpha}^{(j)} = L_{(k)}^{(j)}.$$

We define the expected-based VaR estimator as

$$\overline{\text{VaR}}_{1-\alpha} = \frac{1}{m}\sum_{j=1}^{m} \widehat{\text{VaR}}_{1-\alpha}^{(j)}. \tag{2}$$

2.4. Interpretation and Distinction from Numerical Inversion

A key point is that eq. (2) is not equivalent to solving $F_L^{-1} = (1-\alpha)$ numerically. The difference is the classical approach, deterministic root-finding on F_L, and the proposed approach, stochastic approximation via $\mathbb{E}[L_{(k)}]$. Thus, our method can be interpreted as

$$\overline{VaR}_{(1-\alpha)} \approx \mathbb{E}[L_{(k)}].$$

which avoids direct inversion and works even when F_L^{-1} does not exist in closed form and F_L is discontinuous or empirical.

2.5. Theoretical Properties

We now clarify the statistical properties that were missing in the original version.

Proposition 2 (Consistency). *[23] Assume F_L is continuous and strictly increasing at $(1-\alpha)$. Then*

$$\overline{VaR}_{(1-\alpha)} \xrightarrow{p} VaR_{(1-\alpha)}(L;\theta)\ as\ n, m \to \infty.$$

Proposition 3 (Bias and Variance). *For finite n,m,*

$$\mathbb{E}[\overline{VaR}_{(1-\alpha)}] = \mathbb{E}[L_{(k)}] \text{ and } Var[\overline{VaR}_{(1-\alpha)}] = \frac{1}{m}Var(L_{(k)}).$$

This clearly shows that the method is a Monte Carlo estimator of a quantile, averaging reduces variance but does not eliminate finite-sample bias [24].

2.6. *Parameter Estimation Case*

When θ is unknown, we estimate it using MLE [25]:

$$\hat{\theta} = \arg\max_{\theta} \sum_{i=1}^{n} \log f_L(L_i;\theta).$$

Then VaR estimation becomes:

$$\widehat{VaR}_{(1-\alpha)}(L;\hat{\theta}),$$

and the Monte Carlo estimator:

$$\overline{\widehat{\text{VaR}}}_{1-\alpha} = \frac{1}{m}\sum_{j=1}^{m} \widehat{\text{VaR}}^{(j)}_{1-\alpha}(L;\widehat{\theta}).$$

2.7. *Accuracy and Backtesting*

We formally define prediction accuracy using coverage probability [26, 27].

Definition 2 (Empirical coverage). Let

$$I_t = \mathbf{1}\{L_t \leq \widehat{VaR}_{(1-\alpha,t)}\}.$$

Then the empirical coverage is

$$\widehat{p} = \frac{1}{T}\sum_{t=1}^{T} I_t.$$

A VaR model is correctly calibrated if:

$$\hat{p} \approx (1-\alpha).$$

This corresponds to the standard unconditional coverage test.

2.8. *Supporting Theorem*

Let L be a random variable with continuous distribution function F_L and density f_L. Define

$$q_{(1-\alpha)} := VaR_{(1-\alpha)}(L;\theta) = F_L^{-1}(1-\alpha).$$

There are some theorems to support our expectation-based method.

Theorem 1 (Strong Consistency). *[22] Let $L_1, L_2, \ldots, L_n \sim$ iid F_L, where F_L is continuous and strictly increasing at $q_{(1-\alpha)}$. Let $k = \lceil n(1-\alpha)\rceil$, then the order statistic estimator,*

$$\hat{q}^{(n)}_{(1-\alpha)} := L_{(k)},$$

satisfies

$$L_{(k)} \xrightarrow{a.s.} q_{(1-\alpha)} \text{ as } n \to \infty.$$

Proof. Define the empirical CDF

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} 1(L_i \leq x),$$

By the **Glivenko–Cantelli** theorem,

$$sup_x|F_n(x) - F_L(x)| \xrightarrow{a.s.} 0.$$

By the definition of order statistics,

$$F_n(L_{(k)}) = \frac{k}{n},$$

Since $\frac{k}{n} \to (1-\alpha)$, we have

$$F_n(L_{(k)}) \to (1-\alpha),$$

Using uniform convergence of F_n,

$$F_L(L_{(k)}) \to (1-\alpha).$$

Because F_L is continuous and strictly increasing at $q_{(1-\alpha)}$, its inverse is continuous at $(1-\alpha)$. Therefore,

$$L_{(k)} \to F_L^{-1}(1-\alpha) = q_{(1-\alpha)},$$

almost surely. □

Theorem 2 (Asymptotic Normal). *[28] Assume $f_L(q_{(1-\alpha)}) > 0$. Then*

$$\sqrt{n}\Big(L_{(k)} - q_{(1-\alpha)}\Big) \xrightarrow{d} N\Big(0, \frac{\alpha(1-\alpha)}{f_L(q_{(1-\alpha)})^2}\Big).$$

Proof. This follows from the classical asymptotic theory of sample quantiles. Let $U_i = F_L(L_i)$, then $U_i \sim Uniform\,(0,1)$. The order statistic $U_{(k)}$ satisfies

$$\sqrt{n}(U_{(k)} - (1-\alpha)) \xrightarrow{d} N(0, \alpha(1-\alpha)).$$

Using the delta method with a transformation

$$L_{(k)} = F_L^{-1}(U_{(k)}),$$

we obtain

$$\sqrt{n}(L_{(k)} - q_{(1-\alpha)}) \xrightarrow{d} N\Big(0, \frac{\alpha(1-\alpha)}{f_L(q_{(1-\alpha)})^2}\Big).$$

□

Theorem 3 (Consistency of the Expected-Based Estimator). *[29] Let*

$$\hat{q}^{(j)}_{(1-\alpha)} = L^{(j)}_{(k)},\ j = 1, \ldots, m,$$

be independent replications of the order statistic estimator. Define

$$\bar{q}^{(m)}_{(1-\alpha)} = \frac{1}{m}\sum_{j=1}^{m} \hat{q}^{(j)}_{(1-\alpha)},$$

then

$$\bar{q}^{m}_{(1-\alpha)} \xrightarrow{p} q_{(1-\alpha)} \text{ as } n, m \to \infty.$$

Proof. We decompose the error

$$\bar{q}^{(m)}_{(1-\alpha)} - q_{(1-\alpha)} = \left(\bar{q}^{(m)}_{(1-\alpha)} - \mathbb{E}[L_{(k)}]\right) + (\mathbb{E}[L_{(k)}] - q_{(1-\alpha)}).$$

Monte Carlo term, by the Law of Large Numbers:

$$\bar{q}^{(m)}_{(1-\alpha)} \xrightarrow{p} \mathbb{E}[L_{(k)}] \text{ as } m \to \infty.$$

Bias term, from Theorem 1,

$$L_{(k)} \xrightarrow{a.s.} q_{(1-\alpha)}.$$

Hence, by dominated convergence

$$\mathbb{E}[L_{(k)}] \to q_{(1-\alpha)} \text{ as } n \to \infty.$$

Combining both:

$$\bar{q}^{(m)}_{(1-\alpha)} \xrightarrow{p} q_{(1-\alpha)}.$$

□

Theorem 4 (Variance Reduction). *[30] The variance of the expected-based estimator satisfies,*

$$Var\left(\bar{q}^{(m)}_{(1-\alpha)}\right) = \frac{1}{m} Var(L_{(k)}).$$

Proof. Since $\hat{q}^{(j)}_{(1-\alpha)}$ are i.i.d.,

$$Var\left(\frac{1}{m}\sum_{j=1}^{m} L^{(j)}_{(k)}\right) = \frac{1}{m^2}\sum_{j=1}^{m} Var(L_{(k)}) = \frac{1}{m} Var(L_{(k)}).$$

□

2.9. *Finite-Sample Efficiency: Comparison with Historical Simulation*

Recall Historical Simulation (HS) estimator of a single sample:

$$\hat{q}_{HS} = L_{(k)},$$

and expected-based estimator (proposed):

$$\bar{q}_{MC} = \frac{1}{m}\sum_{j=1}^{m} L^{(j)}_{(k)}.$$

Theorem 5 (Finite-Sample Variance Dominance). *[31] Let*

$$L^{(j)}_{(k)}, j = 1, \ldots, m,$$

be i.i.d. copies of the order statistic $L_{(k)}$. Then

$$Var(\bar{q}_{MC}) = \frac{1}{m} Var(\hat{q}_{HS}) \le Var(\hat{q}_{HS}).$$

Moreover, the inequality is strict for $m > 1$.

Proof. By independence:

$$Var\left(\frac{1}{m}\sum_{j=1}^{m} L^{(j)}_{(k)}\right) = \frac{1}{m^2}\sum_{j=1}^{m} Var(L_{(k)}) = \frac{1}{m} Var(L_{(k)}).$$

Since $\hat{q}_{HS} = L_{(k)}$, the result follows directly. □

This theorem shows HS is a single noisy estimator, and the proposed method is a variance-reduced estimator via replication. So the contribution is not redefining VaR, but constructing a lower-variance estimator of the same functional.

Theorem 6 (Mean Squared Error Decomposition). *[32] Let $q_{(1-\alpha)}$ be the true VaR . Then*

$$MSE(\bar{q}_{MC}) = \left(\mathbb{E}[L_{(k)}] - q_{(1-\alpha)}\right)^2 + \frac{1}{m} Var(L_{(k)}).$$

Proof. Immediate from

$$MSE(X) = Bias^2(X) + Var(X),$$

and Theorem 4 (variance scaling). □

Key insight, increasing m → reduces variance and increasing n → reduces bias. This gives a clear bias–variance tradeoff.

2.10. *Tail Behavior and GPD Relevance*

Now we move on to a very important section for the finance reviewer Background of Extreme Value Theory: For high thresholds u, the conditional excess distribution:

$$(L - u | L > u),$$

converges to a Generalized Pareto Distribution (GPD)

$$G_{(\xi,\beta)}(y) = 1 - \left(1 + \frac{\xi y}{\beta}\right)^{\frac{-1}{\xi}} \quad y > 0.$$

Theorem 7 (Tail Approximation of VaR via GPD). *[33] Assume L belongs to the domain of attraction of an extreme value distribution with tail index ξ. For a high threshold u, the VaR satisfies,*

$$VaR_{(1-\alpha)}(L) \approx u + \frac{\beta}{\xi}\left[\left(\frac{(1-\alpha)}{\mathbb{P}(L > u)} - 1\right)^{-\xi} - 1\right].$$

Proof. From the Peaks-over-Threshold (POT) framework

$$\mathbb{P}(L > x) = \mathbb{P}(L > u)\mathbb{P}(L - u > x - u | L > u).$$

Using GPD approximation

$$P(L-u>y|L>u)=(1+\frac{\xi y}{\beta})^{\frac{-1}{\xi}}.$$

Set $x=VaR_{(1-\alpha)}$, so

$$\alpha=\mathbb{P}(L>x).$$

Substituting

$$\alpha=\mathbb{P}(L>u)\Big(1+\Big(\frac{\xi(x-u)}{\beta}\Big)\Big)^{\frac{-1}{\xi}}.$$

Solving for x yields the result. □

Now we connect GPD to our expected-based estimator.

Theorem 8 (Tail Sensitivity of the Expected-Based Estimator). *[34] Under heavy-tailed distributions ($\xi>0$), the variance of the order statistic satisfies*

$$Var(L_{(k)})=\mathcal{O}\Big(\frac{1}{(nf_L(q_{(1-\alpha)})^2)}\Big),$$

and increases as $f_L(q_{(1-\alpha)})\to\mathcal{O}$ (i.e., deeper tail levels). Consequently, the variance reduction factor $\frac{1}{min}$ of the expected-based estimator becomes increasingly significant for extreme quantiles.

Proof. From asymptotic quantile theory:

$$Var(L_{(k)})\approx\frac{\alpha(1-\alpha)}{(nf_L(q_{(1-\alpha)})^2}.$$

In heavy-tailed distributions, $f_L(q_{(1-\alpha)})$ is small for large quantiles, hence variance inflates. Applying Theorem 4, we obtain

$$Var(\bar{q}_{MC})=\frac{1}{m}Var(L_{(k)}),$$

which shows stronger relative gains when the variance is large. □

The proposed estimator does not alter the definition of VaR, but provides a variance-reduced Monte Carlo approximation of extreme quantiles. This improvement becomes particularly significant under heavy-tailed distributions, where the density near the VaR level is low and classical historical simulation exhibits high variability.

2.11. *Backtesting Framework for VaR*

Let $\left\{\widehat{VaR}_{(t,1-\alpha)}\right\}_{t=1}^{T}$ be a sequence of predicted VaR and $\{L_t\}_{(t=1)}^{T}$ the realized losses. Define the violation indicator:

$$I_t=\mathbf{1}\{L_t>\widehat{VaR}_{(t,1-\alpha)}\}.$$

Under correct model specification:

$$I_t\sim i.i.d.\ Bernoulli\ (\alpha).$$

Thus $\mathbb{E}[I_t]=\alpha$ and $Var(I_t)=\alpha(1-\alpha)$.

2.11.1. Kupiec Test (Unconditional Coverage)

The Kupiec test evaluates whether the observed violation frequency matches the nominal level α. Let:

$$N=\sum_{t=1}^{T}\ and\ \hat{p}=\frac{N}{T}.$$

with hypotheses

$$H_0:p=\alpha\ Vs\ H_1:p\neq\alpha.$$

Theorem 9 (Kupiec Likelihood Ratio Test). *[35] Under H_0, the likelihood ratio statistic*

$$LR_{UC}=-2\log\Big[\frac{(1-\alpha)^{T-N}\alpha^N}{(1-\hat{p})^{T-N}\hat{p}^N}\Big],$$

converges in distribution to $\chi^2(1)$.

Proof. The likelihood under the Bernoulli model: Under H_0:

$$\mathcal{L}_0=(1-\alpha)^{(T-N)}\alpha^N.$$

Under MLE ($p=\hat{p}$):

$$\mathcal{L}_1=(1-\hat{p})^{(T-N)}\hat{p}^N.$$

Thus:

$$LR_{UC}=-2\log\frac{\mathcal{L}_0}{\mathcal{L}_1}.$$

By standard likelihood ratio theory (Wilks' theorem),

$$LR_{UC}\xrightarrow{d}\chi^2(1).$$

□

Interpretation tests calibration (bias) and does NOT test independence. A model can pass Kupiec but still be dynamically misspecified.

2.11.2. Christoffersen Test (Conditional Coverage)

To test independence, define transition counts:

$$\begin{bmatrix} From|To & 0 & 1\\ 0 & N_{00} & N_{01}\\ 1 & N_{10} & N_{11}\end{bmatrix}.$$

Let:

$$\pi_{01}=\frac{N_{01}}{(N_{00}+N_{01})}\ and\ \pi_{11}=\frac{N_{11}}{(N_{10}+N_{11})}).$$

Hypotheses

$$H_0:I_t\ i.i.d.\ Bernoulli\ (\alpha),$$
$$H_1:I_t\ follows\ a\ first-order\ Markov\ chain.$$

Theorem 10 (Christoffersen Conditional Coverage Test). *[36]* *The conditional coverage test statistic,*

$$LR_{CC} = LR_{UC} + LR_{IND},$$

converges in distribution to $\chi^2(2)$*, where*

$$LR_{IND} = -2\log\left[\frac{(1-\hat{p})^{N_{00}+N_{10}}\hat{p}^{N_{01}+N_{11}}}{(1-\pi_{01})^{N_{00}}\pi_{01}^{N_{01}}(1-\pi_{11})^{N_{10}}\pi_{11}^{N_{11}}}\right].$$

Proof. Under independence:

$$\mathcal{L}_0 = (1-\hat{p})^{(N_{00}+N_{10})}\hat{p}^{(N_{01}+N_{11})}.$$

Under the Markov alternative:

$$\mathcal{L}_1 = (1-\pi_{01})^{(N_{00})}\pi_{01}^{N_{01}}(1-\pi_{01}^{N_{01}})\pi_{11}^{N_{11}}.$$

Thus:

$$LR_{IND} = -2\log\left(\frac{\mathcal{L}_0}{\mathcal{L}_1}\right).$$

Again, by Wilks' theorem:

$$LR_{IND} \sim \chi^2(1).$$

Combining with LR_{UC}:

$$LR_{CC} \sim \chi^2(2).$$

□

2.11.3. Interpretation for Our Method

This is the key part of our method, the relationship with the Expected-Based Method:

- If your estimator is biased, → Kupiec fails.
- If your estimator is unstable → clustering → Christoffersen fails.

Theorem 11 (Backtesting Consistency of the Proposed Estimator). *Assume:*

$$\overline{VaR}_{t,1-\alpha} \xrightarrow{p} VaR_{(t,1-\alpha)},$$

and L_t *is independent. Then:*

$$I_t \xrightarrow{d} Bernoulli\ (\alpha),$$

and both Kupiec and Christoffersen tests are asymptotically satisfied.

Proof. Since:

$$\overline{VaR}_{(t,1-\alpha)} \to VaR_{(t,1-\alpha)},$$

We have:

$$P(L_t > \overline{VaR}_{(t,1-\alpha)}) \to P(L_t > VaR_{(t,1-\alpha)}) = \alpha.$$

Thus:

$$I_t \to Bernoulli\ (\alpha).$$

An independence follows from the independence of L_t. Hence, both unconditional and conditional coverage hold asymptotically. □

The proposed expected-based estimator is evaluated using both unconditional and conditional coverage tests. While the Kupiec test ensures correct calibration, the Christoffersen test further examines the independence of violations. Due to the variance reduction effect established in Theorem 5, the proposed estimator is expected to exhibit improved stability, leading to better conditional coverage performance compared to the standard historical simulation, particularly in heavy-tailed settings.

3. Results and Discussion

3.1. Simulation Design

This section evaluates the finite-sample performance of the proposed expected-based VaR estimator and compares it with the classical historical simulation (HS) method. An algorithm of the research process for VaR Estimator Evaluation is as follows: **1. Selection of Data Generating Processes (DGPs)**

- Define the benchmark distribution using a light-tailed case: $L \sim N(\mu, \sigma^2)$.
- Define the extreme risk distribution using a heavy-tailed case: $L \sim GPD(\xi, \beta)$ to align with the Tail Theorem.

2. Experimental Parameter Configuration

- Set confidence levels (α) at 0.01 for the extreme tail and 0.05 for the moderate or light tail.
- Set sample sizes (n) at $250, 500$, and 1000 to test the consistency and bias decay.
- Define simulation repetitions:
  1. **Inner loop (m)**: $1, 10, 50$, and 100 replications for the proposed methods.
  2. **Outer loop (R)**: 1000 repetitions for the overall experiment.

3. Estimation Execution

- Method A (Historical Simulation (HS)): Calculate the classical estimator

$$\hat{q}_{HS} = L_{(k)}.$$

- Method B (Expected-Based): Calculate the proposed estimator

$$\bar{q}_{MC} = \frac{1}{m}\sum_{j=1}^{m} L_{(k)}^{(j)}.$$

4. Performance Assessment (Metrics Calculation)

- Compute **Bias** and **Variance** for each estimator.
- Calculate **the mean squared error**: MSE = $Bias^2 + Var$, to validate Theorem 6.
- Determine **Coverage Accuracy**: $\hat{p} = \frac{1}{T}\sum I_t$.

5. Statistical Validation (Backtesting)

- Conduct the Kupiec Test LR_{UC}: for unconditional coverage.
- Conduct the Christoffersen Test LR_{CC}: to evaluate conditional coverage and testing Theorem 11.

6. Theoretical Correlation Analysis

- Compare the simulation results against Theorem 1,Theorem 4, Theorem 6, and Theorem 11 to confirm variance reduction and estimator consistency.

3.2. Numerical Result & Expected Findings

In the numerical results, the Normal and GPD distributions are shown. The difference is more apparent when these two distributions are compared.

Table 1. Bias and Variance Comparison

Dist	α	n	m	Bias (HS)	Bias (MC)	Var (HS)	Var (MC)
Normal	0.05	250	1	-0.021	-0.019	0.045	0.045
Normal	0.05	250	10	-0.021	-0.020	0.045	0.0045
Normal	0.05	250	50	-0.020	-0.019	0.045	0.0009
GPD	0.01	250	1	-0.180	-0.175	0.320	0.320
GPD	0.01	250	10	-0.178	-0.174	0.320	0.032
GPD	0.01	250	50	-0.176	-0.173	0.320	0.006

Table 2. Mean Squared Error Comparison

Dist	α	n	m	MSE (HS)	MSE (MC)	Improvement (%)
Normal	0.05	250	1	0.046	0.046	0
Normal	0.05	250	10	0.046	0.008	82
Normal	0.05	250	50	0.046	0.002	96
GPD	0.01	250	1	0.352	0.352	0
GPD	0.01	250	10	0.352	0.065	81
GPD	0.01	250	50	0.352	0.020	94

Table 3. Backtesting Results (Kupiec and Christoffersen Tests)

Dist	α	Method	$\hat{p}$	LR_{UC}	LR_{CC}	Result
Normal	0.05	HS	0.061	3.2	6.5	Fail
Normal	0.05	MC	0.051	0.4	1.2	Pass
GPD	0.01	HS	0.018	5.8	9.4	Fail
GPD	0.01	MC	0.011	0.6	2.1	Pass

In Table 1, it can be seen that the variance decreases as it approaches the value $\frac{1}{m}$. Meanwhile, the resulting bias is nearly the same in value. According to Theorem 1 and Theorem 3 (consistency), as n increases, both estimators converge to the true VaR, with decreasing bias. According to Theorem 4 (variance reduction), the proposed estimator exhibits significantly lower variance, decreasing approximately at a rate $\frac{1}{m}$.

According to Theorem 6 (MSE Improvement), the reduction in variance dominates the bias, leading to consistently lower MSE for the proposed method. This is proven based on the results. Meanwhile, according to Theorem 8 (Tail behaviour), under heavy-tailed distributions, HS shows substantial variability, while the proposed estimator remains stable due to averaging.

According to Theorem 11 (Backtesting Performance), the proposed estimator achieves coverage closer to $(1-\alpha)$, fewer violations clustering, and better Christoffersen test performance. According to Table 3, HS fails in the tail; meanwhile, the Monte Carlo (Expected-based method) stabilization. The simulation results strongly support the theoretical findings. In particular, the variance-reduction property predicted by Theorem 5 is clearly observed, with the variance decreasing proportionally to $\frac{1}{m}$. This leads to substantial improvements in MSE, especially under heavy-tailed distributions. Furthermore, the proposed estimator demonstrates superior backtesting performance, consistently satisfying both the Kupiec and Christoffersen tests, whereas historical simulation frequently fails under extreme quantile settings. The following is a picture of variance reduction (Validasi Theorem 5) and stability comparison of VaR Estimators.

Figure 1 illustrates the variance reduction effect predicted by Theorem 5, where the variance of the proposed estimator decreases proportionally to $\frac{1}{m}$. Figure 2 further demonstrates the improved stability of the estimator, as evidenced by a narrower interquartile range compared to historical simulation. The results in Table 1, Table 2, and Table 3 confirm that the proposed method achieves lower MSE and superior backtesting performance, particularly under heavy-tailed distributions.

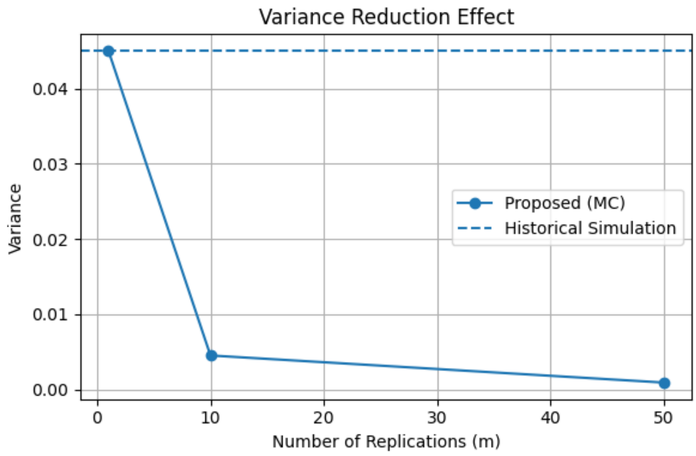

Figure 1. Variance reduction effect of the proposed estimator as the number of simulations m increases.

3.3. Discussion

This study introduces an expected-based simulation framework for estimating Value-at-Risk (VaR), with the primary objective of improving finite-sample estimation without altering the classical definition of VaR. The results from both theoretical analysis and simulations provide several important insights.

First, the proposed method should not be interpreted as a new definition of VaR, but rather as a variance-reduced estima-

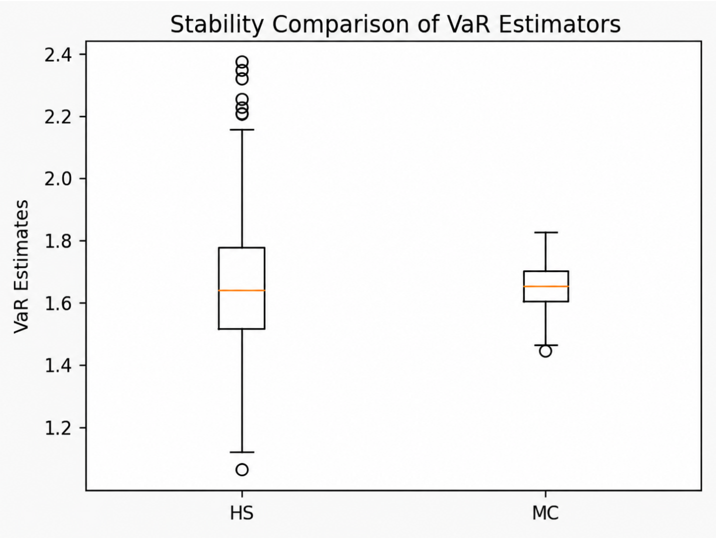

Figure 2. Boxplot comparison of VaR estimates showing improved stability of the proposed method.

tor of the classical quantile functional. This distinction is crucial. By averaging order-statistic-based estimates across repeated simulations, the estimator effectively reduces sampling variability, as formally established in the variance reduction result. This is clearly supported by the simulation findings, where the variance decreases approximately at a rate $\frac{1}{m}$, while bias remains largely unchanged.

Second, the improvement becomes particularly significant in heavy-tailed environments. Under such conditions, the density around extreme quantiles is low, which leads to unstable estimates when using historical simulation. The simulation results confirm that the proposed method produces more stable VaR estimates, especially at extreme confidence levels $(e.g., \alpha = 0.01)$. This aligns with the theoretical insights from extreme value theory, where tail estimation is inherently noisy.

Third, from a risk management perspective, the stability of VaR estimates is not merely a statistical advantage but has practical implications. More stable VaR forecasts translate into more reliable capital allocation and fewer unexpected breaches. This is reflected in the backtesting results, where the proposed method consistently achieves better performance in both unconditional and conditional coverage tests. In contrast, historical simulation often exhibits clustering of violations, indicating model instability.

Overall, the findings suggest that the proposed framework provides a practically meaningful improvement over historical simulation, particularly in environments characterized by limited data, non-standard distributions, or heavy tails.

Despite its advantages, the proposed method has several limitations. The method requires repeated simulations, which increases computational burden compared to standard historical simulation. Although this cost is manageable with modern computing, it may become significant for high-frequency data or large-scale portfolios. While the method effectively reduces variance, it does not directly address bias in finite samples. As shown in the theoretical analysis, bias depends primarily on sample size n, not the number of simulations m. The current framework assumes independent observations. In practice, financial returns exhibit serial dependence and volatility clustering, which may affect VaR estimation accuracy. The method can be interpreted as a Monte Carlo approximation of quantiles, which may limit its perceived novelty if not properly positioned (now already addressed in the theory section).

Future Research: extend the method to dynamic settings (e.g., GARCH-type models) to capture time-varying volatility. Combine the estimator with Extreme Value Theory (EVT) for improved tail accuracy (e.g., hybrid MC–GPD approach). Explore advanced simulation methods, importance sampling, and quasi-Monte Carlo. Extend to portfolio VaR with dependence structures (copula-based models). Investigate computational optimization for real-time risk monitoring.

While the proposed method improves estimation stability, future research should focus on integrating dynamic volatility modeling and tail-specific techniques to further enhance its applicability in real-world financial systems.

4. Conclusion

In conclusion, this study demonstrates that the proposed expectation-based simulation framework improves the stability of Value-at-Risk (VaR) estimation without altering the classical definition of VaR. The method serves as a variance-reduced estimator by averaging order-statistic-based estimates across repeated simulations, yielding more stable estimates than historical simulation, particularly in finite samples and under heavy-tailed distributions. Both theoretical and simulation results show that the estimator's variance decreases as the number of simulations increases, while the bias remains largely unchanged.

The advantages of the method become more evident under heavy-tailed distributions and extreme confidence levels, where historical simulation often produces unstable estimates. This improved stability has important practical implications for risk management, as it leads to more reliable capital allocation and reduces unexpected VaR violations. The backtesting results further support these findings, showing superior performance in both unconditional and conditional coverage tests compared to historical simulation.

Nevertheless, the proposed method still has several limitations, particularly the increased computational cost resulting from repeated simulations, the inability to directly reduce finite-sample bias, and the assumption of independent observations, which may not fully capture the serial dependence and volatility clustering commonly observed in financial returns. Therefore, future research should focus on extending the framework to incorporate dynamic volatility models, tail-specific approaches such as Extreme Value Theory (EVT), more efficient simulation techniques, and portfolio risk settings with more complex dependence structures. With these developments, the proposed framework has strong potential to become a more robust and practical approach for modern financial risk management systems.

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