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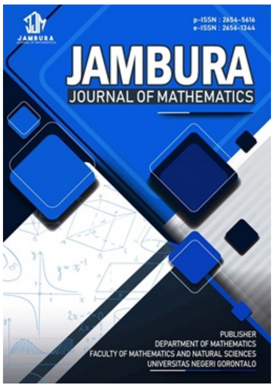


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Research Article



# Elegant Labeling of the Lotus Graph and Its Variations

**Faliza Hanifatul Husna**[1], **Desi Rahmadani**[1,*]

[1]*Department of Mathematics, Universitas Negeri Malang, Malang, 65145, Indonesia*

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**ABSTRACT.** *An elegant labeling $f$ of a graph $G(V,E)$ with $|E(G)| = q$ is an injective function $f : V(G) \to \{0,1,2,\ldots,q\}$ such that each edge $xy$ in $E(G)$ is assigned the induced label $f^*(xy) = (f(x)+f(y)) \bmod (q+1)$, where all resulting edge labels are distinct and non-zero. A graph that admits an elegant labeling is called an elegant graph. Research on elegant labelings has been extensively conducted on simple graph classes such as cycle graphs and star graphs. However, for more complex graph structures like the lotus graph, studies on elegant labelings remain unavailable. The lotus graph possesses a unique symmetrical characteristic, systematically constructed from a combination of cycles and a high-degree central vertex. This symmetrical structure presents a distinct mathematical challenge to ensure that the labeling criteria are satisfied. A lotus graph visually resembles a blooming lotus petal, a distinctive structure that mathematically can only be fully realized if it meets the lower bound of $p \geq 3$. This study aims to demonstrate that the lotus graph $L(p)$ and its two variations, namely the graph $L^*(p)$ and the graph $L(p,p)$, are elegant graphs for any $p \geq 3$. These two variations of the lotus graph are obtained through the deletion of pendant vertices for $L^*(p)$ and the addition of pendant vertices for $L(p,p)$. An exploration patterns is conducted on several simpler lotus graphs, which are subsequently generalized for $p \geq 3$, while the analytical method is utilized for formal mathematical verification. The results show that the lotus graph $L(p)$ and its two structural variations, $L^*(p)$ and $L(p,p)$, are elegant graphs for any $p \geq 3$. The findings of this research are expected to expand the scope of study on graph classes that admit elegant labelings and serve as a foundation for future advanced studies regarding other structural graph variations.*



## 1. Introduction

A graph $G$, commonly denoted by $G(V,E)$, is defined as an ordered pair of sets $(V,E)$, where $V$ denotes the vertex set and $E$ denotes the edge set [1]. The shell graph, denoted by $C(p,p-3)$ for $p \geq 3$, is constructed from the cycle $C_p$ by adding $(p-3)$ chords, that is, edges connecting two vertices of the cycle but not belonging to the cycle itself, that share a common endpoint called the apex. The fan graph $F_n$ is isomorphic to the shell graph $C(n+1,n-2)$. The lotus graph $L(p)$ is constructed from the shell graph $C(p+1,p-2)$ for $p \geq 3$ by adding one new vertex between each pair of adjacent vertices on its cycle and adding one edge incident to the apex [2]. The structure of the lotus graph $L(p)$ is shown in Figure 1. An edge incident to the apex whose other endpoint has degree one is called a pendant edge, while the vertex of degree one is called a pendant vertex.

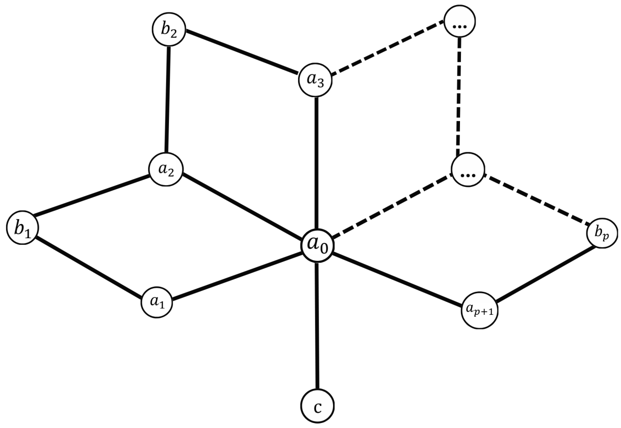


**Figure 1.** The lotus graph $L(p)$.

Graph labeling is one of the interesting topics in graph theory. Research on graph labeling has developed rapidly since various types of labeling were introduced in the 1960s. Graph labeling is an assignment of labels, which are elements of a set, to the elements of $G$, usually to the vertices or edges, or both, of $G$ [1]. Several types of graph labeling have been introduced, including prime, harmonious, elegant, and graceful labeling. Research on graph labeling, particularly harmonious labeling and its variations, has been conducted on various classes of graphs, including wheel graphs $W_n$ [3], snake graphs [4], helm graphs [5], sehati graphs [6], triangular ribbon ladder graphs [7], alternating heart graphs [8], and unions of double string graphs [9].

In 1980, Graham and Sloane introduced harmonious labeling, which subsequently provided a basis for the development of various new labeling schemes, including elegant labeling. An elegant labeling $f$ of a graph $G$ with $|E(G)| = q$ is an injective function $f$ from the vertex set $V(G)$ to the set $\{0,1,2,\ldots,q\}$

*Corresponding Author.

Email : *desi.rahmadani.fmipa@um.ac.id* (D. Rahmadani)
Homepage : http://ejurnal.ung.ac.id/index.php/jjom/index / E-ISSN : 2656-1344

such that every edge $xy \in E(G)$ is assigned the label

$$f(xy) = (f(x) + f(y)) \pmod{q+1},$$

and all resulting edge labels are distinct and nonzero [10]. The injectivity condition on the vertex labeling ensures that distinct vertices receive unique labels from $\{0, 1, \ldots, q\}$. The condition on the edge labeling ensures that no two edges receive the same label and that no edge is assigned the label zero. A graph that admits an elegant labeling is called an elegant graph.

Gallian [10] reported that elegant labeling has been established for several basic graphs, including path graphs $P_n$, cycle graphs $C_n$, fan graphs $F_n$, and bistar graphs $B_{n,n}$. The elegant property has also been established for graphs obtained from the join operation $S_m + S_n$ and the product $C_3 \times P_m$ [11], as well as for other graph classes such as fuzzy star graphs [12], diamond ladder graphs [13], sunlet graphs $S_n$, and helm graphs $H_n$ [14]. The exploration of elegant labeling was extended to classes of line graphs by Shigehalli et al. [15] and Lakshmi et al. [16], and was further developed by Su et al. [17] for large-scale tree graphs, as well as through the concept of elegant fuzzy labeling by Nithya Sree et al. [18].

The lotus graph has a unique symmetric structure that is systematically formed from a combination of a cycle and a high-degree central vertex. Previous studies on lotus graphs have been limited to the analysis of automorphism properties [2], the rainbow connection number [19], and other types of labeling, including odd harmonious, graceful, and felicitous labeling [20]. Although elegant labeling has been investigated for various classes of graphs, its application to lotus graphs remains an open problem. Therefore, this study focuses on elegant labeling of lotus graphs. Visually, the lotus graph resembles the petals of a blooming lotus flower, and this characteristic structure is fully formed mathematically when the lower bound $p \geq 3$ is satisfied.

This study investigates not only the lotus graph $L(p)$ but also two classes of its variations, namely $L^*(p)$ and $L(p,p)$. Lotus graph variation I, denoted by $L^*(p)$, is obtained from the lotus graph $L(p)$ by removing the pendant vertex. The structure of lotus graph variation I $L^*(p)$ is shown in Figure 2. Meanwhile, lotus graph variation II, denoted by $L(p,p)$, is obtained from the lotus graph $L(p)$ by connecting the pendant vertex to $p$ new vertices. The structure of lotus graph variation II $L(p,p)$ is illustrated in Figure 3.

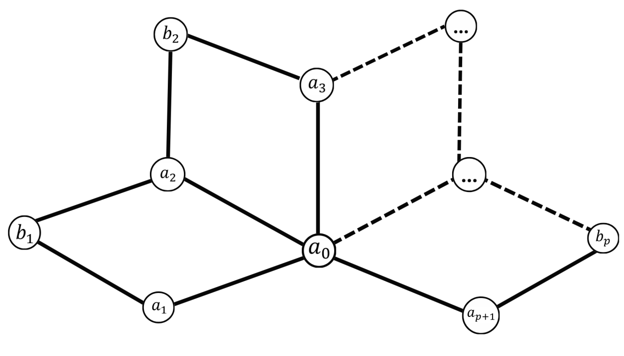


**Figure 2.** Lotus graph variation I $L^*(p)$.

The mathematical challenge in this study arises from the structure of the lotus graph, which is more complex than that of the shell graph from which it is constructed. Consequently, satisfying the elegant labeling requirement that the modulo-based edge labels must be unique and nonzero becomes more difficult. Therefore, this study aims to analytically prove that the lotus graph $L(p)$ and its two structural variations, $L^*(p)$ and $L(p,p)$, are elegant graphs for every $p \geq 3$.

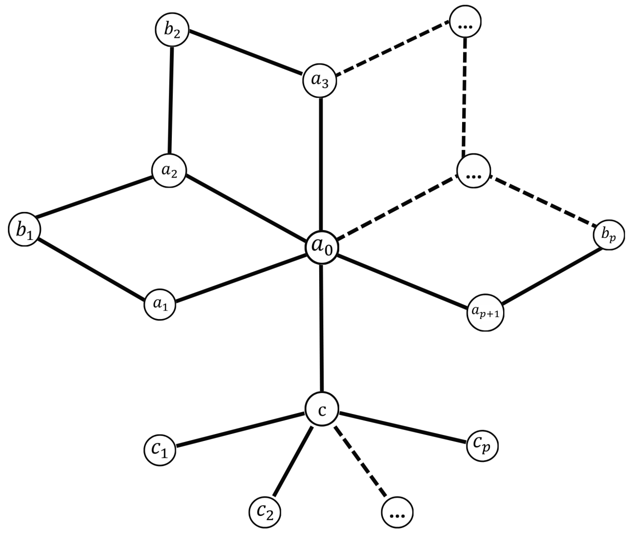


**Figure 3.** Lotus graph variation II $L(p,p)$.

## 2. Method

This study investigates elegant labeling on the lotus graph $L(p)$ and two classes of its variations, namely $L^*(p)$ and $L(p,p)$, using pattern exploration and analytical methods. The pattern exploration method is used to obtain valid labeling patterns. Meanwhile, the analytical method is selected as the main approach because the sizes of the lotus graph and its variations form infinite graph families. Therefore, the analytical approach is used to mathematically prove that the proposed labeling satisfies the conditions of elegant labeling for all parameter values $p \geq 3$.

The research procedures are carried out systematically through the following steps:

1. Conduct a literature review on the concept of elegant labeling and the characteristics of lotus graphs based on relevant previous studies.
2. Explore labeling patterns on several specific lotus graphs $L_p$ for selected values of $p$.
3. Identify the labeling patterns obtained from the previous exploration and generalize them for every $p \geq 3$ using the parameter variable $p$ and the index $i$.
4. Validate the results through mathematical proofs to ensure that the obtained labeling satisfies all conditions of an elegant graph. The injectivity of the vertex-labeling function and the uniqueness of the edge labels are verified using direct proof to guarantee distinct labels for every element. In particular, for the edge-labeling verification, odd–even parity analysis is applied by separating the cases for odd and even values of $p$.

## 3. Results and Discussion

This section presents the construction and proof that the lotus graph and its variations are elegant graphs. The results are presented in Theorem 1, Theorem 2, and Theorem 3.

**Theorem 1.** *The lotus graph $L(p)$ is an elegant graph for $p \geq 3$.*

*Proof.* Let the lotus graph $L(p)$ have the vertex set

$$V(L(p)) = \{a_0, a_1, a_2, \dots, a_{p+1}\} \cup \{b_1, b_2, \dots, b_p\} \cup \{c\},$$

and the edge set

$$\begin{aligned} E(L(p)) = &\{a_0 a_i \mid i = 1, 2, \dots, p+1\} \\ &\cup \{a_i b_i, a_{i+1} b_i \mid i = 1, 2, \dots, p\} \\ &\cup \{a_0 c\}, \end{aligned}$$

with the number of edges

$$q = 3p + 2.$$

Define a vertex labeling

$$f : V(L(p)) \to \{0, 1, 2, \dots, 3p + 2\}.$$

Through the pattern exploration method, a vertex labeling satisfying the conditions of elegant labeling on $L(p)$ is defined as follows:

$$f(v) = \begin{cases} 2p - i + 3, & \text{for } v = a_i, \quad 0 \leq i \leq p+1, \\ p - i + 1, & \text{for } v = b_i, \quad 1 \leq i \leq p, \\ p + 1, & \text{for } v = c. \end{cases} \tag{1}$$

First, it is proved that $f$ is injective, that is, every distinct vertex has a unique label. Let $u, v \in V(L(p))$ be any two distinct vertices. The proof is carried out by considering the following cases:

- **Case 1:** If $u = a_i$ and $v = a_j$, then
$$2p - i + 3 \neq 2p - j + 3.$$
Since $a_i \neq a_j$, it follows that $f(a_i) \neq f(a_j)$.
- **Case 2:** If $u = b_i$ and $v = b_j$, then
$$p - i + 1 \neq p - j + 1,$$
and hence $f(b_i) \neq f(b_j)$.
- **Case 3:** If $u$ and $v$ belong to different vertex groups, the labels of the vertices $b_i$ lie in the range
$$1 \leq f(b_i) \leq p,$$
the central vertex has label
$$f(c) = p + 1,$$
and the labels of the vertices $a_i$ lie in the range
$$p + 2 \leq f(a_i) \leq 2p + 3.$$
Therefore, the vertex labels from different groups are distinct.

Based on all these cases, for every $u \neq v$, we have $f(u) \neq f(v)$. Thus, the vertex-labeling function $f$ is injective.

From the defined vertex labeling, the induced edge labeling

$$f^* : E(L(p)) \to \{1, 2, \dots, 3p + 2\}$$

is given by

$$f^*(xy) = (f(x) + f(y)) \pmod{3p + 3},$$

for every $xy \in E(L(p))$. Based on algebraic simplification of the modulo operation, the edge-labeling function can be expressed as

$$f^*(e) = \begin{cases} 1, & \text{for } e = a_0 c, \\ (p - i + 3) \pmod{3p + 3}, & \text{for } e = a_0 a_i, \quad 1 \leq i \leq p+1, \\ (3p - 2i + 4) \pmod{3p + 3}, & \text{for } e = a_i b_i, \quad 1 \leq i \leq p, \\ (3p - 2i + 3) \pmod{3p + 3}, & \text{for } e = a_{i+1} b_i, \quad 1 \leq i \leq p. \end{cases} \tag{2}$$

Next, it is proved that the edge labeling $f^*$ is injective. Let $e_1, e_2 \in E(L(p))$ be any two distinct edges. The proof is carried out by considering the following cases:

- **Case 1:** Both edges belong to the same component group.
  - For $e_1 = a_0 a_i$ and $e_2 = a_0 a_j$, we have
  $$p - i + 3 \neq p - j + 3,$$
  and therefore
  $$f^*(a_0 a_i) \neq f^*(a_0 a_j).$$
  - For $e_1 = a_i b_i$ and $e_2 = a_j b_j$, we have
  $$3p - 2i + 4 \neq 3p - 2j + 4,$$
  and therefore
  $$f^*(a_i b_i) \neq f^*(a_j b_j).$$
  - For $e_1 = a_{i+1} b_i$ and $e_2 = a_{j+1} b_j$, we have
  $$3p - 2i + 3 \neq 3p - 2j + 3,$$
  and therefore
  $$f^*(a_{i+1} b_i) \neq f^*(a_{j+1} b_j).$$
- **Case 2:** The two edges belong to different component groups.
  - The edge $a_0 c$ has the unique label 1. Furthermore, the labels $f^*(a_0 a_i)$ lie in the range
  $$2 \leq f^*(a_0 a_i) \leq p + 2,$$
  whereas the edge group $\{a_j b_j, a_{j+1} b_j\}$ has labels in the range
  $$p + 3 \leq f^*(e_2) \leq 3p + 2.$$
  Thus, $f^*(e_1) \neq f^*(e_2)$, and the edge labels are distinct.

– Consider $e_1 = a_i b_i$ and $e_2 = a_{j+1} b_j$. Both edge groups have labels in the same range,

$$p + 3 \leq f^*(e) \leq 3p + 2.$$

Since these groups share an overlapping range, parity analysis is used to verify label uniqueness based on whether $p$ is odd or even. If $p$ is even, then $3p$ is even, so

$$f^*(a_i b_i) = 3p - 2i + 4$$

is always even, whereas

$$f^*(a_{j+1} b_j) = 3p - 2j + 3$$

is always odd. If $p$ is odd, these parity conditions are reversed. The difference in parity guarantees that the values of the two component functions can never be equal within this range. Therefore,

$$f^*(a_i b_i) \neq f^*(a_{j+1} b_j).$$

Based on all these cases, for every $e_1 \neq e_2$, we have

$$f^*(e_1) \neq f^*(e_2).$$

Thus, the edge-labeling function $f^*$ is injective. Since the smallest edge-label value is $1$, no edge label is equal to zero. This completes the proof that the lotus graph $L(p)$ is an elegant graph. □

An elegant labeling of the lotus graph $L(5)$ is shown in Figure 4.

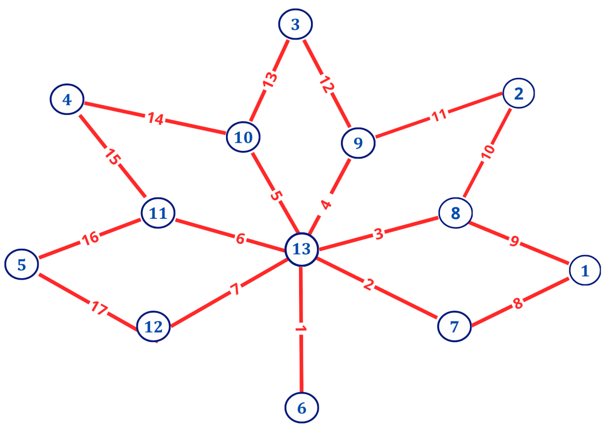


**Figure 4.** Elegant labeling of the lotus graph $L(5)$.

**Theorem 2.** *Lotus graph variation I $L^*(p)$ is an elegant graph for $p \geq 3$.*

*Proof.* Let lotus graph variation I $L^*(p)$ have the vertex set

$$V(L^*(p)) = \{a_0, a_1, a_2, \ldots, a_{p+1}\} \cup \{b_1, b_2, \ldots, b_p\}$$

and the edge set

$$E(L^*(p)) = \{a_0 a_i \mid i = 1, 2, \ldots, p+1\}$$
$$\cup \{a_i b_i, a_{i+1} b_i \mid i = 1, 2, \ldots, p\},$$

with the number of edges

$$q = 3p + 1.$$

Define a vertex labeling

$$f : V(L^*(p)) \to \{0, 1, 2, \ldots, 3p+1\}.$$

Through the pattern exploration method, a vertex labeling satisfying the conditions of elegant labeling on $L^*(p)$ is defined as follows:

$$f(v) = \begin{cases} 2p - i + 2, & \text{for } v = a_i, \quad 0 \leq i \leq p+1, \\ p - i + 1, & \text{for } v = b_i, \quad 1 \leq i \leq p. \end{cases} \tag{3}$$

First, it is proved that $f$ is injective, that is, every distinct vertex has a unique label. Let $u, v \in V(L^*(p))$ be any two distinct vertices. The proof is carried out by considering the following cases:

- **Case 1:** If $u = a_i$ and $v = a_j$, then

$$2p - i + 2 \neq 2p - j + 2.$$

Since $a_i \neq a_j$, it follows that $f(a_i) \neq f(a_j)$.
- **Case 2:** If $u = b_i$ and $v = b_j$, then

$$p - i + 1 \neq p - j + 1,$$

and hence $f(b_i) \neq f(b_j)$.
- **Case 3:** If $u$ and $v$ belong to different vertex groups, the labels of the vertices $b_i$ lie in the range

$$1 \leq f(b_i) \leq p,$$

whereas the labels of the vertices $a_i$ lie in the range

$$p + 1 \leq f(a_i) \leq 2p + 2.$$

Therefore, the vertex labels from different groups are distinct.

Based on all these cases, for every $u \neq v$, we have $f(u) \neq f(v)$. Thus, the vertex-labeling function $f$ is injective.

From the defined vertex labeling, the induced edge labeling

$$f^* : E(L^*(p)) \to \{1, 2, \ldots, 3p+1\}$$

is given by

$$f^*(xy) = (f(x) + f(y)) \pmod{3p+2},$$

for every $xy \in E(L^*(p))$. Based on algebraic simplification of the modulo operation, the edge-labeling function can be expressed as

$$f^*(e) = \begin{cases} (p - i + 2) \pmod{3p+2}, & \text{for } e = a_0 a_i, \quad 1 \leq i \leq p+1, \\ (3p - 2i + 3) \pmod{3p+2}, & \text{for } e = a_i b_i, \quad 1 \leq i \leq p, \\ (3p - 2i + 2) \pmod{3p+2}, & \text{for } e = a_{i+1} b_i, \quad 1 \leq i \leq p. \end{cases} \tag{4}$$

Next, it is proved that the edge labeling $f^*$ is injective. Let $e_1, e_2 \in E(L^*(p))$ be any two distinct edges. The proof is carried out by considering the following cases:

- **Case 1:** Both edges belong to the same component group.
  - For $e_1 = a_0a_i$ and $e_2 = a_0a_j$, we have
  $$p - i + 2 \neq p - j + 2,$$
  and therefore
  $$f^*(a_0a_i) \neq f^*(a_0a_j).$$
  - For $e_1 = a_ib_i$ and $e_2 = a_jb_j$, we have
  $$3p - 2i + 3 \neq 3p - 2j + 3,$$
  and therefore
  $$f^*(a_ib_i) \neq f^*(a_jb_j).$$
  - For $e_1 = a_{i+1}b_i$ and $e_2 = a_{j+1}b_j$, we have
  $$3p - 2i + 2 \neq 3p - 2j + 2,$$
  and therefore
  $$f^*(a_{i+1}b_i) \neq f^*(a_{j+1}b_j).$$
- **Case 2:** The two edges belong to different component groups.
  - The labels $f^*(a_0a_i)$ lie in the range
  $$1 \leq f^*(a_0a_i) \leq p + 1,$$
  whereas the edge group $\{a_jb_j, a_{j+1}b_j\}$ has labels in the range
  $$p + 2 \leq f^*(e_2) \leq 3p + 1.$$
  Thus, $f^*(e_1) \neq f^*(e_2)$, and the edge labels are distinct.
  - Consider $e_1 = a_ib_i$ and $e_2 = a_{j+1}b_j$. Both edge groups have labels in the same range,
  $$p + 2 \leq f^*(e) \leq 3p + 1.$$
  Since these groups share an overlapping range, parity analysis is used to verify label uniqueness based on whether $p$ is odd or even. If $p$ is even, then $3p$ is even, so
  $$f^*(a_ib_i) = 3p - 2i + 3$$
  is always odd, whereas
  $$f^*(a_{j+1}b_j) = 3p - 2j + 2$$
  is always even. If $p$ is odd, these parity conditions are reversed. The difference in parity guarantees that the values of the two component functions can never be equal within this range. Therefore,
  $$f^*(a_ib_i) \neq f^*(a_{j+1}b_j).$$

Based on all these cases, for every $e_1 \neq e_2$, we have
$$f^*(e_1) \neq f^*(e_2).$$

Thus, the edge-labeling function $f^*$ is injective. Since the smallest edge-label value is 1, no edge label is equal to zero. This completes the proof that lotus graph variation I $L^*(p)$ is an elegant graph. □

An elegant labeling of lotus graph variation I $L^*(5)$ is shown in Figure 5.

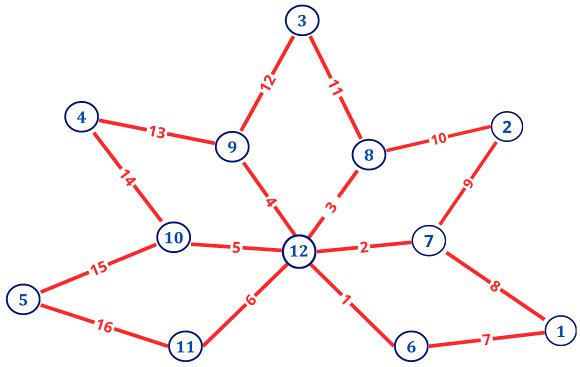


Figure 5. Elegant labeling of lotus graph variation I $L^*(5)$.

**Theorem 3.** *Lotus graph variation II $L(p, p)$ is an elegant graph for $p \geq 3$.*

*Proof.* Let lotus graph variation II $L(p, p)$ have the vertex set
$$\begin{aligned} V(L(p,p)) = \{&a_0, a_1, a_2, \ldots, a_{p+1}\} \\ &\cup \{b_1, b_2, \ldots, b_p\} \\ &\cup \{c_0, c_1, c_2, \ldots, c_p\} \end{aligned}$$
and the edge set
$$\begin{aligned} E(L(p,p)) = &\{a_0a_i \mid i = 1, 2, \ldots, p+1\} \\ &\cup \{a_ib_i, a_{i+1}b_i, c_0c_i \mid i = 1, 2, \ldots, p\} \\ &\cup \{a_0c_0\}, \end{aligned}$$
with the number of edges
$$q = 4p + 2.$$
Define a vertex labeling
$$f : V(L(p,p)) \to \{0, 1, 2, \ldots, 4p + 2\}.$$
Through the pattern exploration method, a vertex labeling satisfying the conditions of elegant labeling on $L(p, p)$ is defined as follows:
$$f(v) = \begin{cases} 3p - i + 3, & \text{for } v = a_i, \quad 0 \leq i \leq p + 1, \\ p - i + 1, & \text{for } v = b_i, \quad 1 \leq i \leq p, \\ p + 1, & \text{for } v = c_0, \\ (3p + i + 3) \pmod{4p + 3}, & \text{for } v = c_i, \quad 1 \leq i \leq p. \end{cases} \tag{5}$$

First, it is proved that $f$ is injective, that is, every distinct vertex has a unique label. Let $u, v \in V(L(p,p))$ be any two distinct vertices. The proof is carried out by considering the following cases:

- **Case 1:** If $u = a_i$ and $v = a_j$, then
$$3p - i + 3 \neq 3p - j + 3.$$
Since $a_i \neq a_j$, it follows that $f(a_i) \neq f(a_j)$.
- **Case 2:** If $u = b_i$ and $v = b_j$, then
$$p - i + 1 \neq p - j + 1,$$
and hence $f(b_i) \neq f(b_j)$.

- **Case 3:** If $u = c_i$ and $v = c_j$ with $1 \leq i, j \leq p$, then

$$(3p+i+3) \pmod{4p+3} \neq (3p+j+3) \pmod{4p+3},$$

and hence $f(c_i) \neq f(c_j)$.
- **Case 4:** If $u$ and $v$ belong to different vertex groups, the labels of the vertices $b_i$ lie in the range

$$1 \leq f(b_i) \leq p,$$

the central vertex has label

$$f(c_0) = p + 1,$$

the labels of the vertices $a_i$ lie in the range

$$2p + 2 \leq f(a_i) \leq 3p + 3,$$

and the labels of the vertices $c_i$ lie in the set

$$\{3p + 4, 3p + 5, \ldots, 4p + 2\} \cup \{0\}.$$

Therefore, the vertex labels from different groups are distinct.

Based on all these cases, for every $u \neq v$, we have $f(u) \neq f(v)$. Thus, the vertex-labeling function $f$ is injective.

From the defined vertex labeling, the induced edge labeling

$$f^* : E(L(p,p)) \to \{1, 2, \ldots, 4p + 2\}$$

is given by

$$f^*(xy) = (f(x) + f(y)) \pmod{4p+3},$$

for every $xy \in E(L(p,p))$. Based on algebraic simplification of the modulo operation, the edge-labeling function can be expressed as

$$f^*(e) = \begin{cases} 1, & \text{for } e = a_0c_0, \\ (2p - i + 3) \pmod{4p+3}, & \text{for } e = a_0a_i, \quad 1 \leq i \leq p+1, \\ (4p - 2i + 4) \pmod{4p+3}, & \text{for } e = a_ib_i, \quad 1 \leq i \leq p, \\ (4p - 2i + 3) \pmod{4p+3}, & \text{for } e = a_{i+1}b_i, \quad 1 \leq i \leq p, \\ (i + 1) \pmod{4p+3}, & \text{for } e = c_0c_i, \quad 1 \leq i \leq p. \end{cases} \tag{6}$$

Next, it is proved that the edge labeling $f^*$ is injective. Let $e_1, e_2 \in E(L(p,p))$ be any two distinct edges. The proof is carried out by considering the following cases:

- **Case 1:** Both edges belong to the same component group.
  - For $e_1 = a_0a_i$ and $e_2 = a_0a_j$, we have

  $$2p - i + 3 \neq 2p - j + 3,$$

  and therefore

  $$f^*(a_0a_i) \neq f^*(a_0a_j).$$

  - For $e_1 = a_ib_i$ and $e_2 = a_jb_j$, we have

  $$4p - 2i + 4 \neq 4p - 2j + 4,$$

  and therefore

  $$f^*(a_ib_i) \neq f^*(a_jb_j).$$

  - For $e_1 = a_{i+1}b_i$ and $e_2 = a_{j+1}b_j$, we have

  $$4p - 2i + 3 \neq 4p - 2j + 3,$$

  and therefore

  $$f^*(a_{i+1}b_i) \neq f^*(a_{j+1}b_j).$$

  - For $e_1 = c_0c_i$ and $e_2 = c_0c_j$, we have

  $$i + 1 \neq j + 1,$$

  and therefore

  $$f^*(c_0c_i) \neq f^*(c_0c_j).$$

- **Case 2:** The two edges belong to different component groups.
  - The edge $a_0c_0$ has the unique label 1. Furthermore, the edge group $c_0c_i$ has labels in the range

  $$2 \leq f^*(c_0c_i) \leq p + 1,$$

  the edges $a_0a_i$ have labels in the range

  $$p + 2 \leq f^*(a_0a_i) \leq 2p + 2,$$

  whereas the edge group $\{a_jb_j, a_{j+1}b_j\}$ has labels in the range

  $$2p + 3 \leq f^*(e_2) \leq 4p + 2.$$

  Thus, $f^*(e_1) \neq f^*(e_2)$, and the edge labels are distinct.
  - Consider $e_1 = a_ib_i$ and $e_2 = a_{j+1}b_j$. Both edge groups have labels in the same range,

  $$2p + 3 \leq f^*(e) \leq 4p + 2.$$

  Since these groups share an overlapping range, parity analysis is used to verify label uniqueness. Since $4p$ is always even for every positive integer $p$,

  $$f^*(a_ib_i) = 4p - 2i + 4$$

  is always even, whereas

  $$f^*(a_{j+1}b_j) = 4p - 2j + 3$$

  is always odd. The difference in parity guarantees that the values of the two component functions can never be equal within this range. Therefore,

  $$f^*(a_ib_i) \neq f^*(a_{j+1}b_j).$$

Based on all these cases, for every $e_1 \neq e_2$, we have

$$f^*(e_1) \neq f^*(e_2).$$

Thus, the edge-labeling function $f^*$ is injective. Since the smallest edge-label value is 1, no edge label is equal to zero. This completes the proof that lotus graph variation II $L(p,p)$ is an elegant graph. □

An elegant labeling of lotus graph variation II $L(5,5)$ is shown in Figure 6.

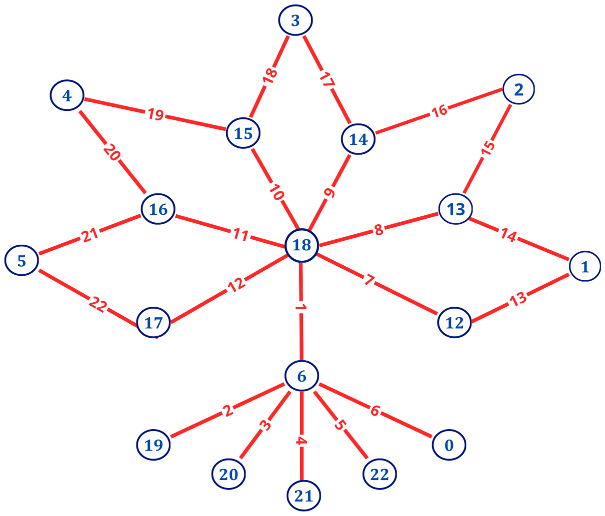

**Figure 6.** Elegant labeling of lotus graph variation II $L(5,5)$.

Based on the analytical proofs in Theorem 1, Theorem 2, and Theorem 3, there is a close structural and functional relationship in preserving the elegant property of the lotus graph $L(p)$ and its variation classes. The structures of $L^*(p)$ and $L(p,p)$ are obtained by modifying the lotus graph $L(p)$ through the removal and addition of pendant vertices, respectively. The relationship among the sizes $q$ of these three graphs can be expressed as

$$q(L^*(p)) = q(L(p)) - 1 = 3p + 1, \tag{7}$$

and

$$q(L(p,p)) = q(L(p)) + p = 4p + 2. \tag{8}$$

Although these structural modifications change the graph size $q$ and consequently the modulus $(q+1)$ used in each theorem, the elegant labeling property can still be preserved for the following mathematical reasons:

1. The removal of the pendant vertex in $L^*(p)$ and the addition of new pendant vertices in $L(p,p)$ are performed symmetrically without altering the main cycle structure of the lotus graph. Since the core cycle structure is preserved, the distribution pattern of labels among adjacent vertices in the main subgraph retains the regularity of a linear sequence.
2. Changes in the number of edges $q$ directly modify the upper bound of the allowable labels. As the graph structure expands from $L^*(p)$ to $L(p)$ and then to $L(p,p)$, the corresponding expansion of the label set is balanced by shifting the initial constants in the vertex-labeling formulas, from $2p+2$ in Theorem 2, to $2p+3$ in Theorem 1, and finally to $3p+3$ in Theorem 3.
3. The local structural modifications do not alter the odd–even parity characteristics of the edge groups $a_ib_i$ and $a_{i+1}b_i$. The shift in the labeling constants is adaptive to changes in the modulus $(q+1)$, so that the distribution of remainders over all edge groups continues to produce distinct and nonzero induced edge labels.

## 4. Conclusion

The results of this study show that the lotus graph $L(p)$ and its two variations, namely $L^*(p)$ and $L(p,p)$, are elegant graphs for every $p \geq 3$. The construction of the vertex-labeling function $f$ and the induced edge-labeling function $f^*$ has been analytically proven to be injective, producing distinct and nonzero values. Based on the proofs that have been established, the elegant property of the lotus graph and its two variations, $L^*(p)$ and $L(p,p)$, can be preserved even though the number of edges of the graphs changes.

Thus, the objective of this study to determine the existence of elegant labeling on the class of lotus graphs and their variations has been achieved. The results of this study are expected to extend research on graph classes that admit elegant labeling and may serve as a basis for further studies involving other variations of graph structures in the future.

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**Desi Rahmadani**: Writing—review and editing, supervision, project administration.

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